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$$D(n_1=0) = \frac{u_1}{3n\pi\sigma^2} \sqrt{\frac{m_2}{m_1+m_2}} \frac{1}{1 - \frac{m_1 - 0.188m_2}{m_1+m_2}} =$$

$$= \frac{u_1}{3n\pi\sigma^2} \sqrt{\frac{m_1+m_2}{m_1}} \times \frac{1}{1.188} = \frac{2}{3\pi n\sigma^2} \frac{1}{1.188} \frac{1}{\sqrt{\pi h}} \sqrt{\frac{m_1+m_2}{m_1m_2}}.$$

The symmetry of this expression shows that exactly the same value holds for $n_2=0$. The form of D also agrees with STEFAN's expression: the coefficients are in the relation of 1:1.05; therefore, considering the approximate character of the deduction, there is practically complete agreement.

For intermediate compositions the difference between the two expressions for D becomes material only when m_1 and m_2 are very different. This is probably due to the method of calculation which compels us to work with averages from the beginning. Moreover JEANS's method of calculating the persistence is not rigorous: it might perhaps be found possible by applying more rigorous methods to reduce the remaining difference between MEYER's corrected formula and the other one. As a matter of fact the object of this paper was not so much to deduce a correct formula, considering that the near accuracy of LANGEVIN's method cannot well be doubted, as to remove the strong contradiction between the two results.

In conclusion it may be added, that the method which is indicated in this paper can immediately be used to deduce rational formulae for the viscosity and the conduction of heat for gas mixtures.

Mathematics. — "On bilinear null-systems." Communicated by Prof. JAN DE VRIES.

(Communicated in the meeting of January 25, 1913).

§ 1. In a *bilinear* null-system any point admits one *null-plane*, any plane one *null-point*. The lines incident with a point and its null-plane are called *null-rays*. If these lines form a linear complex, we have the generally known null-system, which is a special case of the correlation of two collocal spaces (null-system of MOBIUS). The null-rays of any other null-system (1,1) fill the entire space of rays; with R. STURM we denote by γ the number indicating how many times any line is null-ray.

In the first we suppose $\gamma=1$ and we examine the null-systems which may be called *trilinear* and which can be represented by (1,1,1).

§ 2. If a plane φ rotates around the line l its null-point F describes a conic $(l)^2$; for on account of $\gamma = 1$ there is one position of φ for which F lies on l .

The null-points of the planes φ passing through any point P lie on a quadratic surface $(P)^2$; evidently it contains P and on account of $\gamma = 1$ one point more on any line through P .

Evidently the null-plane of P touches $(P)^2$ in P and cuts it according to two lines g, g' . Any point F of one of these lines is null-point of a plane φ passing through F and also through P . Therefore these lines are null-rays of ∞^1 pencils (F, φ) , i.e. *singular*.

So the singular lines of a $(1,1,1)$ form a congruence $(2,2)$.

All the other lines of $(P)^2$ are characterized by the fact that the null-planes of their points concur in P ; otherwise: P is the vertex of the quadratic cone enveloped by these planes.

§ 3. Two surfaces $(P_1)^2$ and $(P_2)^2$ have in common the conic $(l)^2$ corresponding to the line $l = P_1P_2$. As any other common point S bears two and therefore ∞^1 null-planes, it is *singular*. The locus of this point S is a conic σ^2 meeting $(l)^2$ in two points.

The surfaces $(P)^2$ corresponding to the points P of l form a pencil; the surface passing through any point F is indicated by the point of intersection of l and the null-plane of F . The null-planes of any point S evidently form a pencil, the axis of which may be represented by s^* .

As σ^2 contains two points of $(l)^2$, the line l bears two null-planes the null-points of which lie on σ^2 ; therefore the locus of the axes s^* is a quadratic scroll or *regulus*.

According to the laws of duality there is a *quadratic cone* Σ_2 , any tangent plane of which is *singular*, as it contains ∞^1 null-points lying on a line s_k ; these lines generate a *second regulus*.

§ 4. We now consider three surfaces $(P)^2$. As any pencil of planes (s^*) admits a plane passing through a point P_3 , the surface $(P_3)^2$ also contains σ^2 . Amongst the points common to $(P_3)^2$ and the conic $(l_{12})^2$ we find in the first place the points of intersection of $(l_{12})^2$ and σ^2 . One of the two remaining points common to $(P_3)^2$ and $(l_{12})^2$ is the null-point of the plane $P_1P_2P_3$, the other which may be denoted by T lies in three null-planes which do not pass through a line, on account of the arbitrary position of the points P ; so T bears ∞^2 null-points, i.e. T is *principal point*.

Evidently all the surfaces $(P)^2$ form a *complex* with the *singular conic* σ^2 and the *principal point* T as common elements. This com-

plex is *linear*, for through any triplet of points F_k passes the surface corresponding to the point of intersection of the null-planes $\varphi_1, \varphi_2, \varphi_3$.

The *vertex* of the *singular cone* Σ_s bears ∞^1 *singular null-planes* σ not passing through a line; from this ensues that it coincides with the *principal point* T .

In an analogous way the *plane* τ of the *singular conic* σ^2 is *principal plane* of the null-system.

Let us consider the plane through T and one of the axes s^* ; it has for null-point the singular point S lying on s^* but at the same time the principal point T ; so it is singular and its null-points lie on the line $s_+ = TS$. So the *regulus* s_+ is a *cone* and consists of the *edges* of the cone projecting the singular conic σ^2 out of T .

Likewise the axes s^* form the *system of tangents* of a *conic* lying in the principal plane τ .

§ 5. The conics $(l)^2$ form a system ∞^4 admitting a representation on the lines of space. For through any two points F_1, F_2 one $(l)^2$ passes, which is completely determined by the line l common to the null-planes φ_1, φ_2 .

The cones $[l]_2$ each of which is the envelope of the null-planes of the points of a line l also form a system ∞^4 ; any of these cones can be determined by means of two planes φ_1, φ_2 the null-points of which indicate then the line l .

If l lies in a singular null-plane σ , the conic $(l)^2$ breaks up into the line s_+ bearing the null-points of σ and a second line l' which is bound to cut s_+ ; so the principal point T which also can figure as null-point of σ cannot lie outside s_+ . So we find once more that the *regulus* (s_+) is a *cone*.

If l passes through the vertex of Σ_s and bears therefore two singular planes, $(l)^2$ degenerates into two intersecting lines, the point of intersection coinciding evidently with the vertex of Σ_s ; for the null-point of any other plane through l must coincide with that vertex.

§ 6. A special trilinear null-system is determined by the tangential planes of a pencil of quadratic surfaces φ^2 touching each other along a conic σ^2 , where the point of contact forms the null-point.¹⁾

¹⁾ In the case of a general pencil with a twisted quartic as base we get a null-system $(1, 3, 2)$, treated at some length by Dr J. WOLFF ("Ueber ein Null-system quadratischer Flächen", Nieuw Archief voor Wiskunde 1911, vol. IX, page 85).

If the pencil is represented by

$$x_1^2 + x_2^2 + x_3^2 + \lambda x_4^2 = 0^1)$$

the tangential plane in (y) has the equation

$$y_1 x_1 + y_2 x_2 + y_3 x_3 + \lambda y_4 x_4 = 0,$$

λ being determined by

$$y_1^2 + y_2^2 + y_3^2 + \lambda y_4^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

So for its coordinates (η) we find

$$\eta_1 : y_1 = \eta_2 : y_2 = \eta_3 : y_3 = \eta_4 : \lambda y_4, \quad . \quad . \quad . \quad (2)$$

or

$$\eta_1 : y_1 y_4 = \eta_2 : y_2 y_4 = \eta_3 : y_3 y_4 = \eta_4 : -(y_1^2 + y_2^2 + y_3^2) \quad . \quad (3)$$

From (1) and (2) we deduce

$$\eta_1^2 + \eta_2^2 + \eta_3^2 + \frac{\eta_4^2}{\lambda} = 0,$$

i. e.

$$y_1 : \eta_1 \eta_4 = y_2 : \eta_2 \eta_4 = y_3 : \eta_3 \eta_4 = y_4 : -(\eta_1^2 + \eta_2^2 + \eta_3^2) \quad . \quad (4)$$

So (4) shows that any plane has only one null-point.

If the null-plane (η) passes through the fixed point $P(z_k)$ we have $\sum z_k \eta_k = 0$, so the equation of $(P)^2$ is

$$z_1 y_1 y_4 + z_2 y_2 y_4 + z_3 y_3 y_4 = z_4 (y_1^2 + y_2^2 + y_3^2) \quad . \quad . \quad (5)$$

The intersection of this surface with the surface belonging in the same way to the point $Q(w_k)$ breaks up into the singular conic

$$y_4^2 = 0, \quad y_1^2 + y_2^2 + y_3^2 = 0$$

and a second conic lying in the plane

$$(z_1 w_4 - z_4 w_1) y_1 + (z_2 w_4 - z_4 w_2) y_2 + (z_3 w_4 - z_4 w_3) y_3 = 0.$$

The latter contains the null-points of the planes passing through PQ . From this ensues that γ is equal to one.²⁾

All the surfaces $(P)^2$ pass through the principal point $y_1 = y_2 = y_3 = 0$. As could be expected, this point is the vertex of the quadratic cone touching all the surfaces of the pencil (Φ^2) along σ^2 .

The null-planes (η) of the points (y) of the plane ξ envelope the quadratic surface

$$\xi_1 \eta_1 \eta_4 + \xi_2 \eta_2 \eta_4 + \xi_3 \eta_3 \eta_4 = \xi_4 (\eta_1^2 + \eta_2^2 + \eta_3^2).$$

All these surfaces forming a system ∞^3 touching the plane $\eta_1 = \eta_2 = \eta_3 = 0$ ($x_4 = 0$) and the quadratic cone with the equation $\eta_4 = 0$,

¹⁾ Coefficients which might present themselves have been comprised into the definition of the coordinates.

²⁾ This can also be found by considering the involution determined on PQ by the pencil (Φ^2) ; one of the coincidences lies in the plane of the conic τ^2 , the other is point of contact with one of the quadratic surfaces.

$\eta_1^2 + \eta_2^2 + \eta_3^2 = 0$, also represented by $x_1^2 + x_2^2 + x_3^2 = 0$. So we find once more that the plane of σ^2 is the principal plane and that the common enveloping cone of the surfaces Φ^2 touches all the singular null-planes.

By replacing σ^2 by the imaginary circle common to all the spheres, we find the metric null-system in which any plane has for null-point the foot of the normal out of the fixed point T .

We also find a trilinear null-system in the following way. Let σ^2 be any conic and T any point. We then consider as null-plane of any variable point Y the polar plane of TY with respect to the cone with Y as vertex and σ^2 as directrix.

By assuming O_4 in T and representing σ^2 by

$$x_1^2 + x_2^2 + x_3^2 = 0, \quad x_4 = 0,$$

we find for the null-plane of X the equation

$$y_4(y_1x_1 + y_2x_2 + y_3x_3) = (y_1^2 + y_2^2 + y_3^2)x_4.$$

So the coordinates η of this plane satisfy

$$\eta_1 : y_1y_4 = \eta_2 : y_2y_4 = \eta_3 : y_3y_4 = \eta_4 : -(y_1^2 + y_2^2 + y_3^2).$$

As these relations are identical to those of (3) this null-system is equal to the former.

§ 7. We now pass to *bilinear null-systems* where $\gamma = 2$.

Then the locus of the null-points of the planes of a pencil with axis l is a *twisted cubic curve* $(l)^3$ cutting l twice.

Analogously the null-planes of the points of a line l envelope a developable with index 3 (torse of the third class), i.e. they osculate a twisted cubic.

The locus of the null-points of the planes passing through a point P is a *cubic surface* $(P)^3$.

Two surfaces $(P)^3$ and $(Q)^3$ have the curve $(l)^3$ determined by the line $l = PQ$ in common. In general they admit as completing intersection a *twisted sextic* σ^6 , cutting $(l)^3$ in *eight* points and forming the locus of the *singular null-points*, each of which bears a pencil of null-planes. (If these planes were to envelope a cone σ^6 has to be manifold curve on $(P)^3$ and this is impossible if we surmise that the intersection of $(P)^3$ and $(Q)^3$ breaks up into *two* parts only).

The axes s^* of the pencils of null-planes through the points S of σ^6 form a *scroll of order eight*; for the points of intersection of σ^6 and $(l)^3$ determine eight null-planes through l , each of which has a point S as null-point and contains therefore an axis s^* .

The surfaces $(P)^3$, $(Q)^3$ and $(R)^3$ have the singular curve σ^6 in common and moreover one point only, the null-point of the plane

PQR . For $(R)^3$ meets the curve $(l)^3$ belonging to $l = PQ$ in eight points on σ^6 and therefore in one point-outside σ^6 .

Evidently σ^6 is base curve of the linear complex of surfaces $(P)^3$.

§ 8. A special null-system $(1, 1, 2)$ can be obtained in the following manner. We start from two pairs of non intersecting lines a, a' and b, b' . We assign to any point F the plane φ of the two transversals t and u through F over a, a' and b, b' .

The hyperboloids (laa') and (lbb') admit a curve $(l)^3$ of which l is a chord as completing intersection. So we have indeed $\gamma = 2$. Also a, a', b, b' are chords of $(l)^3$.

Here the *singular curve* σ^6 is represented by the lines a, a', b, b' and their *quadriseccants* q, q' . So the figure of singularity has eight points in common with $(l)^3$.

For any point S of a the transversal u is determined while we can assume for t any ray of the pencil (Sa') . So the null-planes of S form a pencil with axis u . So the scroll (s^*) breaks up here into the four *reguli* with the director lines $(a, b, b'), (a', b, b'), (b, a, a'), (b', a, a')$.

For any point of q the transversals t and u coincide and the same happens for any plane through q' . So the lines q, q' are not only loci of *singular points* but also envelopes of *singular planes*. As this is also the case with the lines a, a', b, b' the two dually related figures of singularity are united.

§ 9. For a line l intersecting a in A the locus $(l)^3$ breaks up into a conic $(l)^2$ and a line u containing the null-points of the singular plane (la) ; the conic lies in the plane (Aa') and passes through A , this point being the null-point of the plane connecting l with the transversal u , through A .

If l meets q , the curve $(l)^3$ degenerates in q and an $(l)^2$. The lines l determining conics $(l)^2$ form therefore *six special linear complexes*; so there are ∞^3 conics $(l)^2$.

If l meets both lines q and q' the hyperboloids (laa') and (lbb') intersect in l, q, q' and a fourth line l' meeting q, q' as l does. So the relation between l and l' is involutory; each of them contains the null-points of the planes passing through the other, the planes containing either q or q' discarded.

If l meets a and b , the curve $(l)^3$ breaks up into a line u in the plane (al) , a line t in the plane (bl) and a line l' cutting t and u containing the null-points of the other planes through l .

If we assume for l a transversal t , the curve $(l)^3$ is represented by the lines u and u' of the planes $(al), (a'l)$ and by t itself. This

line evidently contains the null-points of the remaining planes through t ; therefore it is *singular*.

We derive from this that the surface $(P)^3$ contains the transversals t and u passing through P ; so the null-plane of P is a threefold tangential plane. The third line of $(P)^3$ lying in that plane admits the property that the null-planes of its points envelop a cubic cone with P as vertex.

If a, a', b, b' form a skew quadrilateral each null-plane touches one of the quadratic surfaces of the pencil with those four lines as base. Then the surfaces $(P)^3$ have four nodes in common, the vertices of the tetrahedron with a, a', b, b', q, q' as edges.

§ 10 We still examine an *other null-system* (1,1,2) the singular curve of which degenerates.

Let us assume the conic σ^2 in the plane τ and a pair of non intersecting lines. Through F we draw the transversal t over a, a' ; then the polar plane of t with respect to the cone $F(\sigma^2)$ may figure as null-plane of F .

Reversely, if the plane τ is cut by φ according to the line d and D is the pole of d with respect to σ^2 , the transversal through D determines in φ the null-point F .

If φ rotates around l , the line d describes a pencil around the trace R of l as vertex and D describes a line of r . But then t describes a regulus with a, a', r as director lines, in projective correspondence with the pencil of planes (φ) . Consequently the null-point F then describes a twisted cubic $(l)^3$ with l as chord. The two points common to (l) and $(l)^3$ lie on the regulus.

Each point A of the line a is *singular*. The transversal t describes a pencil in the plane (Aa') , its trace D with the plane τ describes a line e bearing the trace A'_0 of a' . So the polar line d rotates round a point E (pole of e); the null-plane of A describes therefore a pencil with axis AE .

If A describes the line a , the line e keeps passing through A'_0 and therefore E describes the polar line of A'_0 . So the axes of the pencils of null-planes corresponding to the singular points A form a *regulus*. A second regulus contains the axes of the pencils corresponding to the singular points A' of a' .

The conic σ^2 too is *singular*. Any point S of it admits as null-planes all the planes touching σ^2 in S .

All the surfaces $(P)^3$ have in common the singular curve σ^2 , the singular lines a, a' and also the line s through the traces A_0 and A'_0 of a and a' with τ , containing two points S_1, S_2 of σ^2 .

For any point of s the cone projecting σ^2 degenerates into the plane τ counted twice, so its null-plane is indefinite and this explains why s must lie on each surface $(P)^3$.

Indeed the plane τ is *principal plane*; for the null-plane of any point of τ lying neither on σ^2 nor on s coincides with τ as polar plane of a line t not situated in τ .

In connection with this result the cubic torse of the null-planes of the points lying on l always contains the plane τ , i.e. τ is common tangential plane of all the surfaces of class three enveloped by the null-planes of the points of a plane.

The trace d of a *singular* plane must be incident with the pole D , i.e. it must touch σ^2 . In this case t is transversal of a, a', σ^2 and each of its points may figure as null-point. The locus of these transversals is a *quartic scroll* $[t]^4$ with a and a' as double director lines and the line s mentioned above as double generatrix.

The polar surface of any point P with respect to $[t]^4$ intersects σ^2 in six points; the planes touching $[t]^4$ in these points are *singular null-planes*. So these planes envelope a *torse of class six*.

§ 11. In the null-system considered in the preceding article the transversals t form a bilinear congruence. If we replace it by a congruence $(1, n)$ we get a null-system $(1, 1, n+1)^1$. If the plane φ rotates once more around the line l , in which case its trace d describes a pencil in τ and the pole D a line r , then the ray t resting on r describes a scroll of order $n+1$. So the null-point of φ lies $(n+1)$ times on l ($\gamma = n+1$) and describes a twisted curve $(l)^{n+2}$.

Let the congruence $(1, n)$ be determined by the director curve a^n and the director line a , which is to have $(n-1)$ points in common with a^n .

Each point of a^n is *singular* and bears a pencil of null-planes (see § 10). From a point of a the curve a^n is projected by a cone of order n with an $(n-1)$ -fold edge a . To the trace of this cone, considered as locus of D corresponds a curve of class n , the envelope of the trace d of the null-plane φ . So each point A of a bears ∞^1 null-planes enveloping a cone of class n . So a is an n -fold line on the surface $(P)^{n+2}$.

Here also any point of the *singular conic* σ^2 bears a pencil of null-planes, the axis of which touches σ^2 .

The intersection of two surfaces $(P)^{n+2}$ breaks up into a curve $(l)^{n+2}$, the curves a^n and σ^2 , the line a (to be counted n^2 -times) and

¹⁾ For $n=0$ we get the null-system of § 6, for $n=1$ that of § 10.

the n rays of the congruence lying in τ . As in § 10 the lines these n rays partake of the property that the null-plane of any of their points is indefinite.

The *singular null-planes* touch in the points of σ^2 the scroll $[t]^{2n+2}$ with σ^2, α^n, a as director lines and n double generatrices in τ . The polar plane of P cuts σ^2 in $2(2n+1)$ points each of which bears a singular null-plane; so the *singular null-planes* envelope a *torse* of class $(4n+2)$.

Evidently τ is once more *principal plane*.

The bisecants of a twisted cubic α^3 determine in an analogous way a *null-system* $(1, 1, 4)$. Here each point S of the singular curve α^3 is vertex of a quadratic cone enveloped by the null-planes of S .

Now two surfaces $(P)^6$ have in common the singular curve α^3 , to be counted four times, the singular conic σ^2 , a curve $(l)^6$ and finally the three chords of α^3 lying in τ .

§ 12. By the considerations of § 11 we have shown that bilinear null-systems with $\gamma > 2$ do exist.

Now we will prove that the locus of the singular points of a null-system $(1, 1, \gamma)$, with the condition $\gamma > 2$, *cannot be a single curve*.

Evidently the curve $(l)^{\gamma+1}$ containing the null-points of the planes through l is rational, l being a γ -fold secant. The null-points of the planes through P lie on a surface $(P)^{\gamma+1}$ touched in P by the null-plane of P .

The surfaces $(P)^{\gamma+1}$ and $(Q)^{\gamma+1}$ have a curve $(l)^{\gamma+1}$ in common. Now let us suppose that the completing intersection is a curve σ of order $\gamma(\gamma+1)$.

In order to determine the number of points common to (l) and σ we first determine the number H of transversals passing through any given point O and resting on (l) and σ .

For this number the known relation

$$m(\mu-1)(\nu-1) = 2h + H$$

holds, where μ, ν are the orders of both the surfaces, whilst m is the order of the first curve and h the number of its apparent double points.

Here we have $\mu = \nu = m = \gamma + 1$, $2h = \gamma(\gamma-1)$, as (l) is rational. So we get $H = \gamma(\gamma^2 + 1)$.

The transversals under consideration are common edges of the cones projecting (l) and σ out of O ; the remaining common edges pass through the points of intersection of both the curves.

For the number of these points we find therefore $\gamma(\gamma+1)^2 - \gamma(\gamma^2+1) = 2\gamma^2$.

Now the surface $(R)^{\gamma+1}$ has in common with (l) besides the $2\gamma^2$ points lying on σ and the null-points of the plane PQR still $\gamma(2-\gamma)$ more points and this is only possible for either $\gamma=1$ or $\gamma=2$.

So we may conclude that for $\gamma > 2$ the singular points must be arranged at least on *two* curves.

Mathematics. — “On plane linear null-systems”. By Prof. JAN DE VRIES.

(Communicated in the meeting of January 25, 1913).

§ 1. By a plane *null-system* (α, β) we understand a correlation between the points and lines of the plane in which to any point F correspond α null-rays f passing through it and to any ray f correspond β null-points situated on it.

We restrict ourselves to the case $\alpha=1$ in which any point F bears only one null-ray (*linear null-system*) and represent by k the second characteristic number.

If the ray f rotates around a point P , its k null-points describe a curve of order $k+1$ passing through P and touching in P the null-ray of P ; we denote that curve by $(P)^{k+1}$.

The curves $(P)^{k+1}$ and $(Q)^{k+1}$ have the k null-points of PQ in common; any of the remaining $(k+1)^2 - k$ points of intersection bears a ray through P and another ray through Q , therefore a pencil of null-rays; so these points are *singular*.

Therefore a *null-system* $(1, k)$ admits $k^2 + k + 1$ *singular points*.

The curves P^{k+1} form together a net with $k^2 + k + 1$ base points; through any pair of arbitrarily chosen points X, Y passes one curve determined by the point common to the two null-rays x, y .

A pencil of curves φ^n with n^2 base points determines a linear null-system, in which to any point F corresponds the tangent f in F to the curve passing through F . This pencil intersects an arbitrary line f in the groups of an involution of order n , admitting $2(n-1)$ double points, therefore $k=2(n-1)$. This null-system admits $(4n^2-6n+3)$ singular points. To these belong the n^2 base points, lying on ∞^1 tangents; the remaining ones must be nodes of curves φ^n . So we fall back on the known property of the pencil (φ_n) to contain $3(n-1)^2$ curves possessing a node.