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For the number of these points we find therefore $\gamma(\gamma+1)^2 - \gamma(\gamma^2+1) = 2\gamma^2$.

Now the surface $(R)^{\gamma+1}$ has in common with (l) besides the $2\gamma^2$ points lying on σ and the null-points of the plane PQR still $\gamma(2-\gamma)$ more points and this is only possible for either $\gamma=1$ or $\gamma=2$.

So we may conclude that for $\gamma>2$ the singular points must be arranged at least on *two* curves.

Mathematics. — "*On plane linear null-systems*". By Prof. JAN DE VRIES.

(Communicated in the meeting of January 25, 1913).

§ 1. By a plane *null-system* (α, β) we understand a correlation between the points and lines of the plane in which to any point F correspond α null-rays f passing through it and to any ray f correspond β null-points situated on it.

We restrict ourselves to the case $\alpha=1$ in which any point F bears only one null-ray (*linear null-system*) and represent by k the second characteristic number.

If the ray f rotates around a point P, its k null-points describe a curve of order $k+1$ passing through P and touching in P the null-ray of P; we denote that curve by $(P)^{k+1}$.

The curves $(P)^{k+1}$ and $(Q)^{k+1}$ have the k null-points of PQ in common; any of the remaining $(k+1)^2-k$ points of intersection bears a ray through P and another ray through Q, therefore a pencil of null-rays; so these points are *singular*.

Therefore a *null-system* $(1, k)$ admits k^2+k+1 *singular points*.

The curves P^{k+1} form together a net with k^2+k+1 base points; through any pair of arbitrarily chosen points X, Y passes one curve determined by the point common to the two null-rays x, y.

A pencil of curves φ^n with n^2 base points determines a linear null-system, in which to any point F corresponds the tangent f in F to the curve passing through F. This pencil intersects an arbitrary line f in the groups of an involution of order n, admitting $2(n-1)$ double points, therefore $k=2(n-1)$. This null-system admits $(4n^2-6n+3)$ singular points. To these belong the n^2 base points, lying on ∞^1 tangents; the remaining ones must be nodes of curves φ^n. So we fall back on the known property of the pencil (φ_n) to contain $3(n-1)^2$ curves possessing a node.

§ 2. The *bilinear null-system* (1,1) has three singular points A, B, C. The line AB admits A and B as null-points and bears therefore ∞^1 null-points. So the sides a, b, c of triangle $A\,B\,C$ are *singular lines*.

If F describes any line l, the null-ray f envelops a conic touching a, b, c and l (the latter in its null-point).

The conic $(P)^2$ degenerates if P lies on a singular line. If we assume P on a the null-points of the other lines through P lie on the line PA.

Let f be a line cutting a, b, c in A', B', C' and F its null-point. If f rotates around A' the point F describes a line through A, and the cross ratio $(A'\,B'\,C'\,F)$ remains constant $= \delta$. If f rotates around B', the point F describes a line through B and $(A'B'C'F)$ is once more $= \delta$. So this cross ratio has the same value for all the rays and is characteristic of the null-system. Now, according to a known theorem, we have also $F\,(A\,B\,C\,f) = \delta$.

So any null-system (1,1) *consists of the pairs* (F, f) *connected with each other with respect to the singular triangle* $A\,B\,C$ *by the relation* $F\,(A\,B\,C\,f) = const.$

In his "*Lehre von den geometrischen Verwandtschaften*" (vol IV, p. 461) M. R. Sturm proves that this construction furnishes a (1,1) but probably it has escaped him that we can get *any* (1,1) in this way.

A pencil of conics touching each other in two points A, B determine a (1,1) by its tangents. Then the singular points are A, B and the point C common to the common tangents in A and B.

If in any collineation with the coincidencies A, B, C the point F' corresponds to F, the line $f = FF'$ admits F as null-point in a bilineair null-system [1]).

§ 3. From a given linear null-system (F, f) we derive a new one (F, f^*), if we replace f by the line f^* normal to it in F. In this construction f and f^* are harmonically related with respect to the absolute pair of points. By a *harmonic* transformation we will understand the transformation of a null-system in which f and f^* are harmonically separated by the tangents from F to a given curve φ^2 of class two.

For any point F of φ^2, the null-ray f passes into the tangent f^* of φ^2 in F; if f touches φ^2 in F_0 we may assume for f^* any line through F_0 and F_0 is a *singular point* of the new null-system

[1]) From $y_k = c_k x_k$, $\Sigma_3 \xi_k x_k = 0$, $\Sigma_3 \xi_k y_k = 0$ we deduce $\rho \xi_1 = (c_2 - c_3)\, x_2 x_3$, etc. i.e $\sigma \xi_1 x_1 = c_2 - c_3$, etc.

$(1, k^*)$. As any singular point of $(1, k)$ remains singular, k^* must surpass k.

In order to determine k^* we bear in mind that all the rays f, which pass into a definite ray f^* by means of the transformation considered, must pass through the pole P^* of f^* with respect to φ^2. So the null-points of f^* lie on the curve $(P^*)^{k+1}$ corresponding to P^* in the null-system $(1, k)$.

So a $(1, k)$ *passes into a* $(1, k+1)$ *by the harmonic transformation.*

From these facts we can derive that $2(k+1)$ singular points of $(1, k+1)$ must lie on φ^2. We can confirm this result as follows. Let G be the second point of intersection of φ^2 with a ray f admitting a null-point F on φ^2. Then the curve $(G)^{k+1}$ cuts φ^2 in G and in $2k+1$ points F more. In any of the $2(k+1)$ coincidencies of the correspondence (F, G), the ray f touches φ^2 and f^* can be taken arbitrarily through F; then F is singular.

By repeating the transformation (F, f^*) must pass reversely into the original null-system $(1, k)$. The null-points of f lie on the curve $(P)^{k+2}$ corresponding to the pole P of f in the null-system $(1, k+1)$. On this curve we also find the points of contact of φ^2 with the tangents passing through P; these points are null-points of f in the special null-system $(0, 2)$ of the pencils the centres of which lie on φ^2. So the null-system $(1, k+1)$ is transformed into the combination of $(1, k)$ and a $(0, 2)$ admitting exclusively singular points (the points of φ^2).

If a is a singular ray of a null-system $(1, k)$, harmonic transformation with respect to a pair of points lying on a generates once more a $(1, k)$. For in this case [1]) the pole P^* of a ray f^* lies on a, which implies that the locus $(P^*)^{k+1}$ breaks up into a and a curve cutting f^* in k null-points F^*.

§ 4. In the case of the null-system $(1, 2)$ the curves $(P)^3$ form a net with 7 base points. *Any net of cubic curves with 7 base points determines a null-system* $(1, 2)$, in which any line f admits as null-points two base points of a pencil belonging to the net. For the curves of the net generate on f a cubic involution of the second rank, the neutral pair of which belongs to ∞^1 triples, i. e. consists of two base points of a pencil.

The figure of singularity has no special characteristic, as we can choose the base points of the net arbitrarily. As soon as three singular

[1]) So the null-system $(1, 1)$ of the tangents of a pencil of conics in double contact passes by transformation with respect to the absolute pair of points into the null-system of the normals.

points are collinear, the line bearing them is singular, as it contains three and therefore ∞^1 null-points.

Though we can determine any (1, 2) by a net of cubic curves we do not judge it superfluous to point out some null-systems (1, 2) which can be obtained otherwise.

If the points F and F' correspond to each other in an involutory quadratic transformation (quadratic involution) they may be considered as null-points of the connecting line f. Then any line is cut by the conic into which it is transformed in its null-points. Then the figure of singularity contains the four points of coincidence and the three fundamental points and consists therefore in the vertices and the co-vertices of a complete quadrangle, the *six* sides of which are singular lines.

The same figure of singularity is found in the case of the null-system, where any line has for null-points its points of contact with two conics of a pencil.

Another null-system (1, 2) is determined by a pencil of cubic curves admitting three collinear points of inflexion B_1, B_2, B_3 with common tangents b_1, b_2, b_3. The cubic involution determined by the curves of this pencil on any line f has a threefold point on the threefold line $b_0 \equiv B_1B_2B_3$; so f is touched by two cubic curves only. We generate a (1, 2) by considering their points of contact as the null-points of f. Three of the singular points coincide with the vertices of the triangle $b_1b_2b_3$, whilst B_1, B_2, B_3 are three others; the seventh is node of a non degenerating cubic curve. Evidently there are *four singular lines.*

By applying the harmonic transformation to a null-system (1, 1) with $ABC = abc$ as singular triangle in such a way that the conic φ^2 touches a, b, c respectively in A', B', C' we get a null-system (1, 2) of which A, B, C, A', B', C' are singular points whilst the seventh can be found by a linear construction. Here a, b, c are *singular lines.*

§ 5. For any null-system $(1, k)$ the curves P^{k+1} form a net with the singular points as base points. Here any line f bears an involution of order $k+1$ and the second rank admitting a neutral group formed by the k null-points F. But for $k > 2$ the net is not more a general one; for this would cut any line in an involution with $\frac{1}{2}k(k-1)$ neutral pairs. Indeed a general net of curves φ^{k+1} admits at most $\frac{1}{2}k(k+5)$ base points, whilst the curves $(P)^{k+1}$ pass through (k^2+k+1) fixed points and the latter number surpasses the former by $\frac{1}{2}(k-1)(k-2)$.

Evidently a null-system $(1, k)$ can be determined by the equations

$$\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 = 0,$$
$$\xi_1 a_x^k + \xi_2 b_x^k + \xi_3 c_x^k = 0.$$

The null-points of the line (ξ) are its points of intersection with the curve indicated by the second equation.

For the curve $(P)^{k+1}$ corresponding to the point $P(y)$ we find, by means of the relation

$$\xi_1 y_1 + \xi_2 y_2 + \xi_3 y_3 = 0,$$

the equation

$$\begin{vmatrix} y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \\ a_x^k & b_x^k & c_x^k \end{vmatrix} = 0.$$

So the singular points are determined by

$$\begin{Vmatrix} x_1 & x_2 & x_3 \\ a_x^k & b_x^k & c_x^k \end{Vmatrix} = 0.$$

By harmonic transformation with respect to the conic $\alpha_\xi^2 = 0$ we find a null-system $(1, k+1)$, in which the line (η) indicated by $\alpha_\xi \alpha_\eta = 0$ corresponds to the point (x).

If we put for short

$$x_2 c_x^k - x_3 b_x^k = A_x^{k+1}, \quad x_3 a_x^k - x_1 c_x^k = B_x^{k+1}, \quad x_1 b_x^k - x_2 a_x^k = C_x^{k+1},$$

then we find

$$\xi_1 : \xi_2 : \xi_3 = A_x^{k+1} : B_x^{k+1} : C_x^{k+1},$$

i. e.

$$(\alpha_{11} A + \alpha_{12} B + \alpha_{13} C)\eta_1 + (\alpha_{12} A + \alpha_{22} B + \alpha_{23} C)\eta_2 + (\alpha_{13} A + \alpha_{23} B + \alpha_{33} C)\eta_3 = 0,$$

and this equation determines with

$$x_1 \eta_1 + x_2 \eta_2 + x_3 \eta_3 = 0$$

the new null-system.

That it is impossible to deduce any arbitrary $(1, k+1)$ by harmonic transformation from null-systems $(1, k)$ can be shown already by remarking that the $2(k+1)$ new singular points furnished by this transformation lie on a conic, which does not happen generally for $k > 2$.