For, on account of the uniform continuity of the correspondence between M and R, to a sequence of points of M possessing only one limiting point, a sequence of points of R likewise possessing only one limiting point, must correspond, and reciprocally. On this ground the given correspondence already admits of an extension to a one-one transformation of the cube with its boundary in itself of which we have still to prove the continuity in the property that a sequence $\{g_{m\nu}\}$ of limiting points of M converging to a single limiting point $g_{m\omega}$, the sequence $\{g_{r\nu}\}$ of the corresponding limiting points of R converges likewise to a single limiting point. For this purpose we adjoin to each point $g_{m\nu}$ a point m_ν of M possessing a distance $< \varepsilon_\nu$ from $g_{m\nu}$, the distance between $g_{r\nu}$ and the point r_ν corresponding to m_ν likewise being $< \varepsilon_\nu$, and for ν indefinitely increasing we make ε_ν to converge to zero. Thus $\{m_\nu\}$ converging exclusively to $g_{m\omega}$, $\{r_\nu\}$ likewise possesses a single limiting point $g_{r\omega}$, and also $\{g_{r\nu}\}$ must converge exclusively to $g_{r\omega}$.

On account of the invariance of the number of dimensions [1]) we can enunciate as a corollary of theorem 10:

THEOREM 11. *For $m < n$ the geometric types η^m and η^n are different.*

As, however, for normally connected sets in general the notion of uniform continuity is senseless, the *indeterminateness of the number of dimensions of everywhere dense, countable, multiply ordered sets*, as expressed in theorem 9, must be considered as irreparable.

Mathematics. — *"An involution of associated points."* By Prof. JAN DE VRIES.

(Communicated in the meeting of February 22, 1913).

§ 1. We consider three pencils of quadric surfaces (a^2), (b^2), (c^2), the base curves of which may be indicated by α^4, β^4, γ^4. By the intersection of any surface a^2 with any surface b^2 and any surface c^2 an *involution of associated points*, I^8, consisting of ∞^3 groups, is generated. Any point outside α^4, β^4, γ^4 determines one group.

Through any point A of α^4 passes one surface b^2 and one surface c^2; these quadrics have a twisted quartic $(A)^4$ in common, intersected by the surfaces of pencil (a^2) in ∞^1 groups of seven points A' completed by A to groups of the I^8. The points of the three base curves are *singular*.

1) Comp. Math. Annalen 70, p. 161.

The locus of the quartic $(A)^4$ corresponding to the different points A of α^4 is a surface which may be indicated by **A**. The curves $\varrho^4 = (b^2, c^2)$ passing through a given point B of β^4 lie on a c^2 meeting α^4 in eight points A; so B lies on eight curves $(A)^4$, i. e. β^4 is an eightfold curve of **A** and the same result holds for γ^4. A quadric b^2 meets α^4 in eight points A and contains therefore eight curves $(A)^4$; moreover it has with **A** the eightfold curve β^4 in common. We conclude from this that **A** is a *surface of order* 32.

§ **2.** The lines joining two points P, P' belonging to the same group of I^8 form a complex Γ; we are going to determine its order.

The curves $\varrho^4 = (b^2, c^2)$ generate a bilinear congruence [1]. Any line is chord of one ϱ^4; the points Q, Q' determined on the lines m through M by the ϱ^4 with m as chord lie on a surface $(Q)^5$ with M as threefold point; the tangential cone in M projects the ϱ^4 passing though M.

The two surfaces a^2 passing through Q and Q' cut m in two other points R, R'. The locus (R) of the points R, R' has in M a sevenfold point, any plane μ through M cutting $(Q)^5$ in a curve μ^5 with threefold point M and the surface a^2 through M in a conic μ^2; so the seven points Q common to μ^2 and μ^5 and differing from M bring seven points R in M. So (R) is a surface of order nine with sevenfold point M.

The curve ϱ^9 common to (R) and μ cuts μ^5 in $9 \times 5 - 7 \times 3 = 24$ points S differing from M, which can be arranged into two groups. In any point of the first group MS is touched by an a^2. So these points lie on the polar surface M^3 of M with respect to the pencil (a^2) [2]. Consequently the first group counts $3 \times 5 - 3 = 12$ points.

In any point S of the second group a point R coincides with a point Q'; then the point Q coincides with R' in a second point S and both points S lie on the same a^2; so these points are associated and belong to the same group of I^8. So the plane μ contains six pairs P, P' collinear with M; in other words: *the pairs of points of the involution I^8 lie on the rays of a complex of order six.*

§ **3.** The complex cone of M contains the seven rays joining M to the points M' belonging with M to the same group of I^8. So

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[1] We have treated this congruence in a paper *"A bilinear congruence of twisted quartics of the first species"*, These Proceedings, vol. XIV, p. 255.

[2] The polar surface of (y) with respect to $a^2_x + \lambda a'^2_x = 0$ is generated by means of this pencil and the pencil of planes $a_y a_x + \lambda a'_y a'_x = 0$; so it is represented by $a'_y a'_x a^2_x = a_y a_x a'^2_x$.

M is sevenfold on the locus of the pairs P, P' collinear with M, and this locus is a twisted curve $(P)^{19}$ passing seven times through M.

The curve $(P)^{19}$ is common to the surfaces $(Q)^5$ and $(R)^9$, intersecting each other moreover in the curve of order 15 common to $(Q)^5$ and the polar surface M^3; so the residual intersection consists of 11 lines. The lines are singular chords of the bilinear congruence[1]) of the curves $\varrho^4 = (b^2, c^2)$, i.e. any of these lines contains ∞^1 pairs (Q, Q'); these lines are not singular for I^8, as these quadratic involutions have only one pair in common.

Amongst these 11 lines we find two chords of β^4 and two chords of γ^4. So the complex Γ^6 contains three congruences (2, 6) and three congruences (7, 3) the rays of which are singular chords of a bilinear congruence (ϱ^4).

There are 120 lines g each of which contains ∞^1 pairs of the I^8, i.e. the common bisecants of the base curves $\alpha^4, \beta^4, \gamma^4$ taken two by two. A common bisecant of α^4 and β^4 forms, in combination with a twisted cubic, the intersection of an a^2 and a b^2; evidently any pair of the involution determined on it by the pencil (c^2) is a pair of I^8. So this involution admits 120 *singular chords*.

The curve $(P)^{19}$ cuts each of the base curves in 20 points, as the surface $(Q)^5$ corresponding to M has 20 points Q in common with α^4; the surface a^2 containing the corresponding point Q' also contains Q, i.e. Q, Q' is a pair of the I^8.

The three polar surfaces of M with respect to the pencils (a^2), (b^2), (c^2) intersect each other in M and 26 points more; in any of these points R the line MR is touched by three surfaces a^2, b^2, c^2. So R is a coincidence $P \equiv P'$ of the I^8, the bearing line passing through M. So the twisted curve $(P)^{19}$ admits the particularity that 26 of its tangents concur in the sevenfold point M.

§ 4. If M describes a plane λ, the three polar surfaces generate three projective nets. The locus of the points of intersection consists of the plane λ and a surface Δ containing all the coincidencies of the I^8.

We deduce from

$$\begin{vmatrix} A^3_x & A'^3_x & A''^3_x \\ B^3_x & B'^3_x & B''^3_x \\ C^3_x & C'^3_x & C''^3_x \end{vmatrix} = 0$$

that this surface is of order eight.[2])

[1]) loc cit.

[2]) This result is in accordance with a theorem of Mr. G. AGUGLIA (Sulla super-

The coincidencies of the involutions Γ^8 lie on a surface Δ^8 passing through the base curves $\alpha^4, \beta^4, \gamma^4$.

The surface Δ^8 also contains the three curves of order 14 containing the points of contact of surfaces of two of the pencils.

The three polar surfaces generate three projective pencils if M describes a line l. These surfaces generate the line l and moreover a twisted curve σ forming the locus of the coincidencies $P \equiv P'$, the bearing lines of which rest on l. If the three pencils are indicated by

$$A^3_x + \lambda A'^3_x = 0 \quad , \quad B^3_x + \lambda B'^3_x = 0 \quad , \quad C^3_x + \lambda C'^3_x = 0,$$

the twisted curve under consideration can be deduced from

$$\left\| \begin{array}{ccc} A^3_x & B^3_x & C^3_x \\ A'^3_x & B'^3_x & C'^3_x \end{array} \right\| = 0.$$

So the degree of this curve is $6^2 - 3^2 - 1 = 26$.[1])

The line l bears 8 coincidencies, so it is an eightfold secant of σ^{26}.

§ 5. We now consider the locus of the points P' associated to the points P of the line l. The curve α^4 contains 32 points P', as l intersects $\mathbf{A}^{32}$ in 32 points. So any surface a^2 contains these 32 points and moreover the two sets of seven points P' associated to the two points common to a^2 and l. So the groups associated to the points of a line lie on a *twisted curve of order* 23, intersecting each of the three base curves in 32 points. In its points on Δ^8 the line l meets its curve λ^{23}; so l *eightfold secant* of λ^{23}.

A plane φ through l meets λ^{23} in 15 points not lying on l; as these points are associated to 15 points P of l, *the locus of the associated pairs lying in a plane is a curve of order* 15.

This curve, φ^{15}, has threefold points in the 12 traces of the curves $\alpha^4, \beta^4, \gamma^4$ on φ. The curve (A^4) corresponding to any of these traces meets φ in three other points, each of which forms with A a pair of the Γ^8.

§ 6. The sets of seven points P' associated to the points P of a plane φ lie on a surface Φ^{23} intersecting φ according to the curve φ^{15} containing the pairs P, P' lying in φ and to the curve σ^8 of the coincidencies lying in φ.

The curve $(A)^4$ corresponding to the point A of α^4 (§ 1) meets φ

ficie luogo di un punto in cui le superficie di tre fasci toccano una medesima retta, Rend. del Circolo Mat. di Palermo, t. XX, p. 305).

[1]) AGUGLIA, l. c. p. 321.

in four points associated to A; so Φ^{23} passes four times through the base curves $\alpha^4, \beta^4, \gamma^4$. This is in accordance with the fact, that each trace of a base curve is threefold on φ^{15} and onefold on δ^8.

The curve δ^8 contains 18 coincidencies the bearing lines of which lie in the plane, for the curve δ^{26} (§ 4) corresponding to a line l of φ meets l eight times. These 18 coincidencies lie on φ^{15}; so φ^{15} and δ^8 touch one another in 18 points. Moreover they have 36 points in common in the 12 traces of the base curves; each of the remaining 48 common points belongs as coincidence to a group of the I^8 containing still one more point of φ^{15}.

§ **7.** The plane φ contains a finite number of associated triplets. As these triplets have to lie on φ^{15} we determine the order of the locus of the sextuples of points P'' associated to the pairs P, P' of φ^{15}.

The surface $\mathbf{A}^{32}$ passes eight times through β^4, γ^4 and one time through α^4. As φ^{15} has threefold points in the 12 traces of the base curves it meets $\mathbf{A}^{32}$ elsewhere in $15 \times 32 - 4 \times 3 - 2 \times 4 \times 3 \times 8 = 276$ points forming 138 pairs P, P' corresponding to 138 points P'' of α^4. A surface a^2 cuts φ^{15} in the four threefold points A and in 9 pairs P, P' more, each pair of which determines six points P'' on a^2. So the locus under discussion has $138 + 6 \times 9 = 192$ points with a^2 in common and is therefore a curve φ^{96}. Of its points of intersection with φ a number of 48 lie in the points common to φ^{15} and δ^8 indicated above. Evidently the remaining 48 traces of φ^{96} are formed by 16 triplets of the I^8. So *any plane contains sixteen triplets of associated points.*

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§ **8.** If the bases of the pencils (a^2), (b^2), (c^2) have the line g in common, three surfaces a^2, b^2, c^2 intersect each other in *four* associated points; so we then get an involution I^4 of associated points.

Any point A of the curve α^3 completing g to the base of (a^2) belongs to ∞^1 quadruples. These quadruples lie on the twisted cubic $(A)^3$ common to the surfaces b^2, c^2 passing through A and they are determined on $(A)^3$ by the pencil (a^2).

In the same way any point B of the base curve β^3 and any point C of the base curve γ^3 belongs to ∞^1 quadruples.

We determine the order of the locus $\mathbf{A}$ of the curves $(A)^3$. By means of the points A the surfaces of (b^2) and (c^2) are arranged in a correspondence (4, 4), any surface b^2 or c^2 containing four points A; so the surface $\mathbf{A}$ is of order 16.

In any plane through g the pencils (b^2), (c^2) determine two pencils in (4, 4)-correspondence with the traces B and C of β^3 and γ^3 lying outside g as vertices. So $\mathbf{A}^{16}$ is cut according to g and to a curve of order eight with fourfold points in B and C.

So, *the triplets of points associated to the points of one of the base curves lie on a surface of order sixteen, passing eight times through g and four times through each of the other two base curves.*

§ **9.** Any point G of g also belongs to ∞^1 quadruples. If G is to be a point common to three cubic curves (a^2b^2), (b^2c^2), (a^2c^2) the surfaces a^2, b^2, c^2 must admit in G the same tangential plane.

We now consider in the first place the locus Φ^4 of the curve (a^2b^2), intersection of surfaces a^2, b^2 touching one another in G. Any plane φ through g cuts these projective pencils (a^2), (b^2) according to two projective pencils, the vertices of which are the traces A and B of α^3 and β^3 outside g. These pencils of lines generate a conic passing through G, the lines AG and BG determining with g two surfaces a^2, b^2 touching φ in G. So g is double line and G is threefold point of Φ^4.

In the same way the pencils (a^2) and (c^2) determine a second monoid ψ^4. The monoids Φ^4 and ψ^4 have the base curve α^3 and the line g to be counted four times in common; the residual intersection, locus of the three points associated to G, is of order nine. The cubic cones touching the monoids in G intersect in g and in five other edges; so G is *fivefold point* of the curve $(G)^9$. Any plane through g cuts Φ^4 and ψ^4 according to two conics passing through G and a point A; in each of the two other points of intersection three homologous rays of three projective pencils with vertices A, B, C concur. So g is cut, besides in G, in two more points G^*, each of which forms with G a pair of associated points. So the pairs of the I^4 lying on g are arranged in an involutory correspondence (2, 2), i. e. g bears four coincidencies. This proves moreover that g is a *sevenfold line* of the locus $\mathbf{G}$ of the curves $(G)^9$; for in the first place any point G is fivefold on the corresponding $(G)^9$ and it lies furthermore on two suchlike curves corresponding to other points of g.

The curve (a^2b^2) meeting γ^3 in a point C rests in two points G on g; so C lies on two curves $(G)^9$, i.e. γ^3 is double curve of $\mathbf{G}$. The curve (a^2b^2) contains the two triplets of points associated to the points of intersection G with g. Moreover it has in common with the surface $\mathbf{G}$ in each of these two points G seven points and two points in each of the eight points in which it rests on α^3 and β^3. So we find that $\mathbf{G}$ is of order 12. So, *the points associated to the* *points of g lie on a surface of order twelve, passing seven times through g and twice through each of the base curves.*

If the point G of g lies on α^3, the surfaces a^2 admit in G a common tangential plane, the plane through g and the tangent t in G to α^3; so these surfaces determine on the curve (b^2c^2) touching t in G an I^3 of associated points. The cone k^2 projecting α^3 out of G cuts any curve (b^2c^2) through G in a triplet of associated points; therefore these points lie on the intersection of k^2 with the monoid χ^4 containing all these curves. So, for any of the six points common to g and a base curve, $(G)^9$ breaks up into a twisted cubic and a twisted sextic.

Any common transversal d of g, α^3, β^3 and γ^3 forms with g the partial intersection of three surfaces a^2, b^2, c^2 with two more points in common; these two points form a group of the I^4 with any pair of points of g.

The transversals of g, α^3, and β^3 generate a scroll of order six with g as fivefold line; for the cubic cones projecting α^3 and β^3 out of any point G of g admit g as double edge and intersect each other in five lines of this scroll. On g this scroll has 10 points in common with γ^3, so it cuts γ^3 outside g in 8 points. So, *the base lines g, α^3, β^3, γ^3 admit eight common transversals and therefore eight pairs of points belonging to ∞^2 groups of the I^4.*

Evidently the eight lines d lie in the surface Δ^8 of the coincidences; of this surface g is a *fivefold line.*

§ **10.** The pencils (a^2), (b^2) determine a bilinear congruence of twisted cubics ϱ^3. In general any ray m of a pencil (M, μ) is bisecant of one ϱ^3; the locus of the points Q, Q' common to m and this ϱ^3 is a curve $(Q)^4$ with a double point in M. In the manner of § 2 we introduce as auxiliary curve the locus of the points R, R' still common to m and the surfaces c^2 through Q and Q'. The surface c^2 through M cuts $(Q)^4$ in M and in six points Q; so M is a sixfold point of the curve (R) and this curve is of order eight.

The polar curve of M with respect to the pencil of intersection of (c^2) and μ intersects $(Q)^4$ in M and $4 \times 3 - 2 = 10$ other points, lying also on $(R)^8$. So $4 \times 8 - 2 \times 6 - 10 = 10$ points are arranged in associated pairs. So, *the pairs of points of the involution I^4 lie on the rays of a complex of order five.*

Any point G of g is associated to two points of g, the points common to g and to the curve $(G)^9$ corresponding to G. So g is a *singular line* of the I^4; the pairs of points lying on it generate an involutory (2,2).

Also the 27 common bisecants of α^3, β^3, γ^3 taken two by two are *singular* lines of the I^4. A common chord of α^3, β^3 bears ∞^1 pairs of points determined on it by the pencil (c^2).

§ **11.** We now consider the locus λ of the points P' associated to the points P of a line l. To the points common to l and each of the surfaces $\mathbf{A}^{16}$, $\mathbf{G}^{12}$ correspond respectively 16 points of α^3 and 12 points of g. Any surface a^2 contains these 28 points P and moreover the two triplets corresponding to the points common to a^2 and l. So the locus λ is a *curve of order* 17.

As l contains eight coincidencies $P \equiv P'$ it is an eightfold secant of the curve λ^{17}; so any plane φ through l contains 9 points P' associated to points of l. So, *the pairs of associated points lying in a plane generate a curve of order nine.*

The curve $(G)^9$ corresponding to the trace G of g meets φ in four points; so G is a *fourfold point* of the curve φ^9. In an analogous way the nine traces A_k, B_k, C_k of the base curves are *double points* of φ^9.

The intersection σ^8 of φ and the surface of coincidencies has a fivefold point in G. So φ^9 and σ^8 intersect each other in $9 \times 8 - 4 \times 5 - 9 \times 2 = 34$ points differing from the traces of the bases. To these points belong the points of contact of the curves, corresponding to coincidencies of the I^4 the bearing lines of which are contained in φ.

In order to determine their number we consider the three pencils of conics common to φ and (a^2), (b^2), (c^2). The polar curves of these pencils with respect to a point P describing a line l generate three projective pencils (a^3), (b^3), (c^3). The first and the second generate a curve c^5 with G as node and passing through the three base points A_k of α^3 and the double points of the three pairs of lines. The curve b^5 generated by the pencils (a^3) and (c^3) also contains these points. So b^5 and c^5 admit $25 - 4 - 3 - 3 = 15$ points of contact of three corresponding conics forming therefore coincidencies of the I^4 with a bearing line lying in φ.

So φ^9 and σ^8 have four coincidencies in common the bearing lines of which intersect the plane φ.