Mathematics. — *"On a class of surfaces with algebraic asymptotic curves."* By Prof. W. A. VERSLUYS. (Communicated by Prof. J. CARDINAAL).

§ 1. Let a twisted curve $C\begin{pmatrix} p, q, s \\ a, b, c \end{pmatrix}$ be given by the equations:

$$x = at^p \quad , \quad y = bt^q \quad , \quad z = ct^s , \quad . \quad . \quad . \quad . \quad . \quad (1)$$

t being the arbitrary parameter, a, b, c constants and p, q, s positive integers not admitting a common divisor. In general we suppose

$$p < q < s.$$

By assigning to a, b, c all possible values we get a system of ∞^2 curves, which will be denoted as *the system* $C(p, q, s)$. All the curves of this system contain the origin O and the point at infinity on the axis OZ; through any other point of space only one curve of the system $C(p, q, s)$ passes. The curve determined by the point A shall be indicated by $C_A(p, q, s)$ or by C_A.

Let $P_1(x_1, y_1, z_1)$ be the point of the curve $C\begin{pmatrix} p, q, s \\ a, b, c \end{pmatrix}$ corresponding to the value t_1 of the parameter t; then

$$x_1 = at_1^p \quad , \quad y_1 = bt_1^q \quad , \quad z_1 = ct_1^s$$

The equation of the osculating plane in P_1 to C_{P_1} is:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ p\, a\, t_1^{p-1} & q\, b\, t_1^{q-1} & s\, c\, t_1^{s-1} \\ p(p-1)\, at_1^{p-2} & q(q-1)\, bt_1^{q-2} & s(s-1)\, ct_1^{s-2} \end{vmatrix} = 0,$$

or reduced

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ px_1 & py_1 & sz_1 \\ p^2x_1 & q^2y_1 & s^2z_1 \end{vmatrix} = 0,$$

or worked out

$$\frac{s-q}{p}\left(\frac{x}{x_1} - 1\right) + \frac{p-s}{q}\left(\frac{y}{y_1} - 1\right) + \frac{q-p}{s}\left(\frac{z}{z_1} - 1\right) = 0 \quad . \quad . \quad (2)$$

By putting

$$\frac{s-q}{p} = P \quad , \quad \frac{p-s}{q} = Q \quad , \quad \frac{q-p}{s} = S , \quad . \quad . \quad . \quad . \quad (3)$$

and replacing $P + Q + S$ by the value $-PQS$, equal to it, the equation of the osculating plane becomes

$$P\frac{x}{x_1} + Q\frac{y}{y_1} + S\frac{z}{z_1} + PQS = 0. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

§ 2. We now propose the question how to determine the three functions $\varphi_1, \varphi_2, \varphi_3$ in such a way that the twisted curve

$$x = \varphi_1(u) \quad , \quad y = \varphi_2(u) \quad , \quad z = \varphi_3(u)$$

admits in the point $P_1(x_1, y_1, z_1)$ corresponding to the value u_1 of u the plane (2) as osculating plane.

The twisted curve φ under discussion has to cut the plane (2) thrice in the point $u = u_1$, i. e. the equation

$$P\left(\frac{\varphi_1(u)}{x_1} - 1\right) + Q\left(\frac{\varphi_2(u)}{y_1} - 1\right) + S\left(\frac{\varphi_3(u)}{z_1} - 1\right) = 0$$

must admit three roots $u = u_1$

This gives the conditions

$$P\frac{\varphi'_1(u_1)}{\varphi_1(u_1)} + Q\frac{\varphi'_2(u_1)}{\varphi_2(u_1)} + S\frac{\varphi'_3(u_1)}{\varphi_3(u_1)} = 0, \quad . \quad . \quad . \quad . \quad (5)$$

$$P\frac{\varphi''_1(u_1)}{\varphi_1(u_1)} + Q\frac{\varphi''_2(u_1)}{\varphi_2(u_1)} + S\frac{\varphi''_3(u_1)}{\varphi_3(u_1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

As the equation (5) must hold for any value of u_1 the first differential coefficient of the first member must disappear. This gives by taking (6) into account:

$$P\left\{\frac{\varphi'_1(u_1)}{\varphi_1(u_1)}\right\}^2 + Q\left\{\frac{\varphi'_2(u_1)}{\varphi_2(u_1)}\right\}^2 + S\left\{\frac{\varphi'_3(u_1)}{\varphi_3(u_1)}\right\}^2 = 0 \quad . \quad . \quad . \quad (7)$$

As the equations (5) and (7) must hold for any value of u_1 they lead to the two sets of solutions:

$$\frac{\varphi'_1(u) : \varphi_1(u)}{p} = \frac{\varphi'_2(u) : \varphi_2(u)}{q} = \frac{\varphi'_3(u) : \varphi_3(u)}{s}, \quad . \quad . \quad . \quad (8)$$

and

$$\frac{\varphi'_1 : \varphi_1}{p(-p+q+s)} = \frac{\varphi'_2 : \varphi_2}{q(p-q+s)} = \frac{\varphi'_3 : \varphi_3}{s(p+q-s)} \quad . \quad . \quad . \quad (9)$$

By representing the three equal ratios (8) by $\psi(u)$ we find

$$\varphi_1 = ae^{p\int \psi(u)\,du}$$

which passes, by replacing $\int \psi(u)\,du$ by t, into

$$x = \varphi(u) = at^p \quad , \quad y = bt^q \quad , \quad z = ct^s,$$

i. e. into a curve of the system $C(p, q, s)$.

Likewise the ratios (9) furnish

$$x = \alpha\tau^{p_1} \quad , \quad y = \beta\tau^{q_1} \quad , \quad z = \gamma\tau^{s_1},$$

i. e. a second curve belonging to a system $C(p_1, q_1, s_1)$ determined by the relations

$$\left.\begin{aligned} p_1 &= p\,(-p+q+s) \\ q_1 &= q\,(p-q+s) \\ s_1 &= s\,(p+q-s) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

So we have the theorem:

The equation (2) *represents the osculating plane in the arbitrarily chosen point* $P_1(x_1, y_1, z_1)$ *to both the curves* $C_{P_1}(p, q, s)$ *and* $C_{P_1}(p_1, q_1, s_1)$.

We also find easily for the equation of the osculating plane in P_1 to the curve $C_{P_1}(p_1, q_1, s_1)$

$$\frac{s_1-q_1}{p_1}\left(\frac{x}{x_1}-1\right)+\frac{p_1-s_1}{q_1}\left(\frac{y}{y_1}-1\right)+\frac{q_1-p_1}{s_1}\left(\frac{z}{z_1}-1\right)=0$$

so that

$$P_1=\frac{s_1-q_1}{p_1}=\frac{q-s}{p}=-P$$

and likewise

$$Q_1=-Q \quad , \quad S_1=-S \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

§ 3. *Definition.* We call $C(p_1, q_1, s_1)$ *the complementary system* of $C(p, q, s)$.

By determining the complementary system of $C(p_1, q_1, s_1)$ we find again the original $C(p, q, s)$, as we have

$$\begin{aligned} p_1(-p_1+q_1+s_1) &= p(-p+q+s)(p-q+s)(p+q-s), \\ q_1(p_1-q_1+s_1) &= q(-p+q+s)(p-q+s)(p+q-s), \\ s_1(p_1+q_1-s_1) &= s(-p+q+s)(p-q+s)(p+q-s). \end{aligned}$$

Therefore an exception presents itself if and only if we have

$$(-p+q+s)(p-q+s)(p-q+s)=0.$$

For $p<q<s$ this reduces to the possibility $p+q-s=0$; on this supposition we find $s_1=0$ and $p_1=q_1$, i. e. the system $C(p_1, q_1, s_1)$ is the system of the right lines intersecting the axis OZ and the line at infinity of the plane $z=0$.

We find $s_1>0$ for $s<p+q$ and $s_1<0$ for $s>p+q$; p_1 and q_1 are always positive.

For $p=0$ we also find $p_1=0$; then the two complementary systems $C(p, q, s)$ and $C(p_1, q_1, s_1)$ are both systems of plane curves situated in planes $x=$ constant.

If two of the three numbers p, q, s, e. g. p and q are equal, we find $p_1=q_1=ps$ and both systems $C(p, q, s)$ and $C(p_1, q_1, s_1)$ consist in plane curves complanar with the axis OZ.

The identities

$$\left.\begin{aligned} pP+qQ+sS &= 0, \\ p^2P+q^2Q+s^2S &= 0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

can immediately be verified. Likewise one finds

$$p_1P_1 + q_1Q_1 + s_1S_1 = 0$$

and as according to (11) $P_1 = -P$, etc. we find

$$\left.\begin{aligned} p_1P + q_1Q + s_1S = 0, \\ p_1^2P + q_1^2Q + s_1^2S = 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and also

§ 4. Let O_{cc_1} be a surface determined by the equations

$$\left.\begin{aligned} x &= a u^p v^{p_1}, \\ y &= b u^q v^{q_1}, \\ z &= c u^s v^{s_1}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where the coordinate lines $v =$ constant are curves of the system $C(p, q, s)$ and the coordinate lines $u =$ constant curves of the complementary system $C(p_1, q_1, s_1)$. The two coordinate lines passing through any point $P_1(x_1, y_1, z_1)$ of O_{cc_1} admit in this point the same osculating plane. This common osculating plane contains the tangents in P_1 to both the coordinate lines and as the director cosines of these tangents are proportional to

$$\left.\begin{aligned} px_1,\ qy_1,\ sz_1, \\ p_1x_1, q_1y_1, s_1z_1, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and

these tangents do not coincide and the common osculating plane is at the same time the tangent plane of O_{cc_1} in P_1.

This proves the theorem:

The two systems of coordinate lines are the systems of asymptotic curves of the surface O_{cc_1} given by (14).

In any point P_1 of O_{cc_1} the tangents to $C_{P_1}(p,q,s)$ and $C_{P_1}(p_1,q_1,s_1)$ are the principal tangents as these curves are the asymptotic curves. So in any real point of O_{cc_1} the principal tangents (see (15)) are real and different from one another; so we have the theorem:

All the points of O_{cc_1} are hyperbolic.

The equation of the surface O_{cc_1} is:

$$\left(\frac{x}{a}\right)^{Pk} \left(\frac{y}{b}\right)^{Qk} \left(\frac{z}{c}\right)^{Sk} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

k being the lowest common multiple of the numerators P, Q, S of (3) after reduction of these fractions to their simplest values. Indeed the values (14) of the coordinates of any point of O_{cc_1} satisfy the equation (16) for arbitrary values of u and v, as according to (12) and (13) we have the identities

$$pP + qQ + sS = 0,$$

$$p_1P + q_1Q + s_1S = 0.$$

On account of $p < q < s$ we have $Q < 0$, so we prefer to trasnform (16) into

$$x^{Pk} z^{Sk} = By^{-Qk} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Corollary I. The degree of the surface O_{cc_1} is $(P + S)k$.

For we have:

$$P + Q + S = -PQS > 0,$$
$$P + S > -Q.$$

Corollary II. The surfaces O_{cc_1} on which the lines of the systems $C(p, q, s)$ and $C(p_1, q_1, s_1)$ are the asymptotic curves form a pencil.

Corollary III. The base curve of the pencil of surfaces O_{cc_1} is formed by the sides of the skew quadrilateral $OX_\infty Y_\infty Z_\infty O$, each of these sides counted a certain number of times.

Corollary IV. The complex of the principal tangents of the pencil of surfaces O_{cc_1} is formed by the tangents to the curves of both the systems $C(p, q, s)$ and $C(p_1, q_1, s_1)$.

§ 5. Reversely we start from the equation

$$x^L y^M z^N = B \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad , \quad (18)$$

where L, M, N are integers admitting no factor common to all three, in order to investigate under which restrictions with respect to these numbers the surface represented by (18) admits as asymptotic curves the lines of a system $C(p, q, s)$ and therefore also those of the complementary system $C(p_1, q_1, s_1)$. This will be the case if the surface (18) contains curves of both systems; to that end we must have

$$\left.\begin{aligned} pL + qM + sN &= 0 \\ p_1L + q_1M + s_1N &= 0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and

or

$$(p+q+s)(pL+qM+sN) - 2(p^2L+q^2M+s^2N) = 0,$$

what can be replaced, on account of (19), by

$$p^2L + q^2M + s^2N = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where p, q and s are integers.

From (19) and (20) we deduce:

$$\frac{p}{q} = \frac{-LM \pm \sqrt{\{-LMN(L+M+N)\}}}{L(L+N)}.$$

As p and q have to be integers the expression $-LMN(L+M+N)$ under the root sign must be positive and a square; so L, M, N cannot have the same sign. Let a^2 be the highest integer square by which LMN and b^2 the highest integer square by which $L+M+N$ can be divided, so that $LMN : a^2$ and $(L + M + N) : b^2$ contain

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prime factors only occurring only once in each expression; then we must have

$$LMN : a^2 = -(L+M+N) : b^2 \quad . \quad . \quad . \quad . \quad . \quad (21)$$

By substituting the value of $-(L+M+N)$ following from it into the expression for $p:q$ given above we easily find:

$$\frac{p}{M(a \pm bN)} = \frac{q}{-a(L+N)} = \frac{s}{M(a \mp bL)} \quad . \quad . \quad . \quad (22)$$

So, as soon as L, M, N satisfy the condition (21) we find sets of numbers (p, q, s), (p', q', s') and therefore also two sets of curves $C(p, q, s)$, $C(p', q', s')$ lying on the surface (18). After some reductions we find $p' : q' : s' = p_1 : q_1 : s_1$ as the deduction of (p, q, s) and (p', q', s') requires. So we have proved the theorem:

A surface

$$x^L y^M z^N = B$$

admits as asymptotic curves the curves of the systems

$$C(M(a+bN), -a(L+N), M(a-bL)),$$

and

$$C(M(a-bN), -a(L+N), M(a+bL)),$$

as soon as L, M, N satisfy the condition

$$L + M + N = -\frac{b^2}{a^2} LMN,$$

a and b being integers.

The simplest example of a surface $x^L y^M z^N = B$, where the condition $L+M+N = -\frac{b^2}{a^2} LMN$, holds, is the hyperbolic paraboloid

$$xz = By.$$

In this case the equations (22) become

$$p = 0, \; q = s \;\text{ and }\; s_1 = 0, \; p_1 = q_1 .$$

The systems $C(0, 1, 1)$ and $C(1, 1, 0)$ are systems of right lines forming on the paraboloid the asymptotic lines.

§ 6. Any surface O_{cc_1} contains besides the two systems of asymptotic lines $C(p, q, s)$ and $C(p_1, q_1, s_1)$ other systems of curves belonging to the systems

$$C(p + \lambda p_1, q + \lambda q_1, s + \lambda s_1)$$

and this holds for any rational value of λ either positive or negative.

Let, in order to show this, $P_1(x_1, y_1, z_1)$ be any point of O_{cc_1}, so that we have

$$x_1^{Pk} z_1^{Sk} = B y_1^{-Qk},$$

then O_{cc_1} contains any point of the curve

$$x = x_1 t^{p+\lambda p_1}, \quad y = y_1 t^{q+\lambda q_1}, \quad z = z_1 t^{s+\lambda s_1}$$

as from the identities (12) and (13) we can deduce

$$P(p+\lambda p_1) + Q(q+\lambda q_1) + S(s+\lambda s_1) = 0.$$

If λ_1 and λ_2 represent any two definite values of λ, the cross ratio of the four tangents in a point P_1 of O_{cc_1} to the curves through P_1 of the systems

$$C(p, q, s), C(p_1, q_1, s_1), \quad C(p+\lambda_1 p_1, q+\lambda_1 q_1, s+\lambda_1 s_1),$$
$$C(p+\lambda_2 p_1, q+\lambda_2 q_1, s+\lambda_2 s_1),$$

is always equal to $\lambda_1 : \lambda_2$ and therefore independent of x_1, y_1, z_1. So this cross ratio is constant all over the surface.

If we put e.g.

$$\lambda_1 = 1 : (p+q+s)$$
$$\lambda_2 = (p_1+q_1+s_1) : (-p+q+s)(p-q+s)(p+q-s),$$

the two systems of curves corresponding to these two values of λ are the systems $C(p^2, q^2, s^2)$ and $C(p_1^2, q_1^2, s_1^2)$.

So the cross ratio of the tangents in any point of O_{cc_1} to the four curves through this point belonging to the systems

$$C(p, q, s), \quad C(p_1, q_1, s_1), \quad C(p^2, q^2, s^2), \quad C(p_1^2, q_1^2, s_1^2),$$

is therefore

$$\lambda_1 : \lambda_2 = \frac{(-p+q+s)(p-q+s)(p+q-s)}{(p+q+s)(p_1+q_1+s_1)} = p_1 q_1 s_1 : \{pqs(p+q+s)(p+q_1+s_1)\}.$$

This cross ratio becomes zero or infinite if two of the four tangents coincide with each other, then the curves touching these coinciding lines also coincide. For p, q, and s positive and $p < q < s$ the cross ratio becomes zero under the condition $p + q = s$ only and infinite for $p_1 + q_1 + s_1$ only. In the first case the systems $C(p_1, q_1, s_1)$ and $C(p_1^2, q_1^2, s_1^2)$ coincide, in the second case the systems $C(p, q, s)$ and $C(p_1^2, q_1^2, s_1^2)$.

Reversely, if a curve of a system $C(p', q', s')$ lies on the surface O_{cc_1} it will always be possible to find a value of λ for which the system $C(p + \lambda p_1, q + \lambda q_1, s + \lambda s_1)$ coincides with the system $C(p', q', s')$.

So we have to prove that it is possible to find values λ and μ satisfying the three equations:

$$p + \lambda p_1 + \mu p' = 0,$$
$$p + \lambda q_1 + \mu q' = 0,$$
$$s + \lambda s_1 + \mu s' = 0.$$

These three equations are not mutually independent; multiplying the first members by P, Q and S and adding the results we get $0 = 0$, for the condition under which the curve $C(p', q', s')$ lies on O_{cc_1} is

$$Pp' + Qq' + Ss' = 0.$$

For this proof we have not made use of the fact that the two systems $C(p, q, s)$ and $C(p_1, q_1, s_1)$ are complementary; so this theorem also holds for any surface generated by curves of the system $C(p, q, s)$ meeting a curve of a system $C(p_2, q_2, s_2)$.

§ 7. Let $P_1(x_1, y_1, z_1)$ be once more an arbitrary point of O_{cc_1}. Then the two ruled surfaces osculating O_{cc_1} along the curve through P_1 of the system $C(p+\lambda p_1,\ q+\lambda q_1,\ s+\lambda s_1)$ are the surfaces the generatrices of which are the principal tangents of O_{cc_1} in the points of this curve. So these two osculating ruled surfaces are represented by the following two sets of equations·

$$\left.\begin{aligned} x &= x_1 t^{p+\lambda p_1}(1+pv), \\ y &= y_1 t^{q+\lambda q_1}(1+qv), \\ z &= z_1 t^{s+\lambda s_1}(1+sv). \end{aligned}\right\} (I) \qquad \left.\begin{aligned} x &= x_1 t^{p+\lambda p_1}(1+p_1 v), \\ y &= y_1 t^{q+\lambda q_1}(1+q_1 v), \\ z &= z_1 t^{s+\lambda s_1}(1+s_1 v). \end{aligned}\right\} (II)$$

As soon as two surfaces are generated by curves of a same system $C(p, q, s)$ the intersection of these surfaces consists exclusively of curves of this system, whether single or degenerated ones; for the curve $C(p, q, s)$ passing through any common point of the two surfaces must lie on both. So, as the two surfaces I and II osculating O_{cc_1} partake of the property of O_{cc_1}, of being generated by curves of the system $C(p+\lambda p_1, q+\lambda q_1, s+\lambda s_1)$, or $C(\lambda)$ for short, the intersection of each of these two surfaces with O_{cc_1} and their mutual intersection must break up into curves of the system $C(\lambda)$.

In the case $\lambda = 0$, i.e. if the system $C(\lambda)$ coincides with the system $C(p, q, s)$, the ruled surface I is developable. If $C(\lambda)$ coincides with the system $C(p_1, q_1, s_1)$ the ruled surface II is developable.

For $\lambda_2 = -\lambda_1$ the cross ratio $\lambda_1 : \lambda_2 = -1$ (see § 6) and the four tangents form a harmonic quadruple. Then the tangents to the curves of the systems $C(\lambda_1)$ and $C(-\lambda_1)$ are two conjugate diameters of the indicatrix, the tangents to the curves of the systems $C(p, q, s)$ and $C(p_1, q_1, s_1)$ being the asymptotes of the indicatrix. So the two systems of curves corresponding to λ_1 and $-\lambda_1$ are *conjugate* on O_{cc_1}.

So the developable enveloping O_{cc_1} according to a curve of the system $C(\lambda_1)$ is represented by the equations

$$\left.\begin{aligned} x &= x_1\,(1 + v\,(p - \lambda_1\, p_1))\, t^{p+\lambda_1 p_1}, \\ y &= y_1\,(1 + v\,(q - \lambda_1\, q_1))\, t^{q+\lambda_1 q_1}, \\ z &= z_1\,(1 + v\,(s - \lambda_1\, s_1))\, t^{s+\lambda_1 s_1}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (III)$$

Indeed this ruled surface proves to be developable, as it is possibe to determine v in such a way that the director cosines of the tangent to the curve of the system $C(\lambda_1)$ situated on this surface III and corresponding to this value of v become proportional to the director cosines of the generatrices.

Indeed it is possible to find values v and w satisfying the equations

$$p + \lambda_1 p_1 + v(p^2 - \lambda_1^2 p_1^2) + w(p - \lambda_1 p_1) = 0,$$
$$q + \lambda_1 q_1 + v(q^2 - \lambda_1^2 q_1^2) + w(q - \lambda_1 q_1) = 0,$$
$$s + \lambda_1 s_1 + v(s^2 - \lambda_1^2 s_1^2) + w(s - \lambda_1 s_1) = 0,$$

as the sum of the three first members, multiplied respectively by P, Q, and S disappears.

This developable also cuts O_{cc_1} according to curves of the system $C(\lambda_1)$ to which also belongs the curve of contact.

§ 8. By assuming for λ_1 the value $-\frac{s}{s_1}$ we find $s - \lambda_1 s_1 = 0$ and the system $C(\lambda)$ becomes the system $C(p(s-p), q(s-q), 0)$. The conjugated system, i.e. the system corresponding to the value $\lambda_2 = -\lambda_1 = s:s_1$, is then the system $C(pq, pq, s_1)$. Then the first system consists of curves lying in the planes $z =$ constant and the second of curves lying in planes through the axis OZ.

The developable D circumscribed to O_{cc_1} along a curve of the system $C(p(s-p), q(s-q), 0)$ is generated by the tangents to the curves $C(pq, pq, s_1)$ and admits therefore the equations:

$$x = x_1 (1 + pqv)\, t^{p(s-p)},$$
$$y = y_1 (1 + pqv)\, t^{q(s-q)},$$
$$z = z_1 (1 + s_1 v),$$

x_1, y_1, z_1 satisfying the relation

$$x_1^{Pk} z_1^{Sk} = B y^{-Qk}.$$

As the system of curves conjugated to the curves of contact consists of curves situated in planes through the axis OZ, the developable must be a cone (according to the theorem of KOENIGS [1]), the vertex of which lies on OZ. It is easily verified that all the generatrices of D pass through the point $\left\{0, 0, z_1\left(1 - \frac{s_1}{pq}\right)\right\}$.

The developable D' circumscribed to O_{cc_1} along a curve of the system $C(pq, pq, s_1)$ is represented by the equations:

[1]) G. DARBOUX, Théorie gén. des surfaces T. I § 91.

$$x = x_1 \{1 + p(s-p) v\} t^{pq},$$
$$y = y_1 \{1 + q(s-q) v\} t^{pq},$$
$$z = z_1 t^{s_1}.$$

The direction of the generatrices of this developable being constant, D' is an enveloping cylinder.

For $\lambda_1 = -p : p_1$ and $\lambda_1 = -q : q_1$ we obtain analogous results. So the theorems hold:

I. *The plane sections of O_{cc_1} by planes through any edge of the tetrahedron of coordinates are conjugated to those by planes containing the opposite edge.*

II. *The developable circumscribed to O_{cc_1} along a plane curve, the plane of which contains an edge of the tetrahedron of coordinates is a cone the vertex of which lies on the opposite edge.*

III. *Any of these enveloping cones cuts O_{cc_1} according to curves of the system $C(\lambda)$ to which belongs the curve of contact.*

§ 9. Let $A(a, b, c)$ be an arbitrary point. Then the curve of contact of the enveloping cone of O_{cc_1} with A as vertex lies on O_{cc_1} itself, the equation of which surface is

$$x^{Pk} y^{Qk} z^{Sk} = B,$$

and on the first polar surface of A with respect to O_{cc_1} with the equation

$$Pax^{Pk-1} y^{Qk} z^{Sk} + Qbx^{Pk} y^{Qk-1} z^{Sk} + Scx^{Pk} y^{Qk} z^{Sk-1} - (P+Q+S) B = 0.$$

By eliminating B between these two equations we find:

$$Payz + Qbxz + Scxy - (P+Q+S)\, xyz = 0 \quad . \quad . \quad . \quad (23)$$

So the curve of contact always lies on a cubic surface O^3_A represented by (23). The equation (23) of O^3_A being independent of B, this surface O^3_A is the same for all the surfaces O_{cc_1}; so we have theorem:

The locus of the curves of contact of all the surfaces O_{cc_1} with the enveloping cones with common vertex A is a cubic surface O^3_A.

The tangential planes of O_{cc_1} being at the same time the osculating planes of the systems $C(p, q, s)$ and $C(p_1, q_1, s_1)$, the surface O^3_A is also the locus of the points P for which the osculating planes to $C_P(p, q, s)$ and to $C_P(p_1, q_1, s_1)$ pass through A; this can easily be proved directly by making use of the equations (4).

The surface O^3_A containing the six edges of the tetrahedron of coordinates, four of which also lie on O_{cc_1}, the intersection of O_{cc_1}

and O_A^3 breaks up into the curve of contact and these four edges. The tangential plane of O_A^3 in any point of one of these edges is the same for all the points of this edge and different from the faces of the tetrahedron of coordinates. As we always have $S < -Q$ and we suppose provisionally that $P > -Q$, the tangential planes of O_{cc_1} along the four edges coincide with faces of the tetrahedron of coordinates. So each of the four edges belongs to the intersection a number of times indicated by its multiplicity on O_{cc_1}.

Now the edge OX_∞ is always Sk-fold on O_{cc_1} and $Y_\infty Z_\infty$ is always $(S+Q+P)k$-fold, while for $P > -Q$ the edge $X_\infty Y_\infty$ is Sk-fold and the edge OZ_∞ is $-Qk$-fold. So the four edges represent together $(3S+P)k$ common right lines. The total intersection of O_A^3 and O_{cc_1} being of the order $3(P+S)k$, there remains a curve of contact of order $2Pk$.

In the case $P < -Q$ the edge $X_\infty Y_\infty$ counts $(P+Q+S)k$ times on O_{cc_1} and the edge OZ_∞ counts Pk times. Then the four edges represent $(3S+3P+2Q)k$ common right lines belonging to the intersection and therefore the curve of contact is of order $-2Qk$.

For $P = -Q$ which implies $S = S + P + Q$ the tangential plane of O_{cc_1} along OZ_∞ is no more constant and therefore this plane does not coincide with the tangential plane of O_A^3 along this edge which is constant, likewise for the edge $Y_\infty Z_\infty$. So the multiplicity of these edges as parts of the intersection still remains equal to their multiplicity on O_{cc_1}.

Now the edge $X_\infty Y_\infty$ is Sk-fold on O_{cc_1} and the edge OZ_∞ is Pk-fold. The order of the curve of contact is $2Pk = -2Qk$.

From $P = -Q$ we deduce

$$(s-p-q)(q-p) = 0,$$

i.e. either $s = q + p$, or $p = q$. In the first case O_{cc_1} is a ruled surface (see § 3, § 14), in the second a plane (see § 3).

As in general the point A does not lie on the surface O_{cc_1} it neither lies on the curve of contact and the order of the enveloping cone to O_{cc_1} with vertex A is equal to the order of the curve of contact. So we find the theorem:

The order of the enveloping cone to O_{cc_1} with an arbitrary vertex A is the larger of the two numbers $2Pk$ and $-2Qk$.

If A lies in one of the faces of the tetrahedron of coordinates, O_A^3 breaks up into the plane of that face and into a quadratic cone the vertex of which coincides with the opposite vertex of the tetrahedron.

If A lies on one of the edges of the tetrahedron of coordinates,

O_A^3 breaks up into the two faces through A and into a third plane. Then the curve of contact is plane (see § 8).

§ 10. The class of the enveloping cone is equal to the class of O_{cc_1}; the class of O_{cc_1} being $(P+S)\,k$, as we shall see immediately, the class of the enveloping cone also is $(P+S)\,k$.

The class of O_{cc_1} is equal to its order, the reciprocal polar figure of O_{cc_1} being also a surface O_{cc_1}. The homogeneous plane coordinates $(\alpha, \beta, \gamma, \delta)$ of a tangential plane to O_{cc_1}, i. e. of an osculating plane to a curve $C(p, q, s)$ satisfy the conditions (see § 1, equation 4):

$$\frac{\alpha}{P : x_1 u_1^{p} v_1^{p_1}} = \frac{\beta}{Q : b u_1^{q} v_1^{q_1}} = \frac{\gamma}{S : c u_1^{s} v_1^{s_1}} = \frac{\delta}{PQS};$$

where (x_1, y_1, z_1) are the coordinates of any point of O_{cc_1} and u_1, v_1 the parameter values corresponding to the point of contact. By replacing $1 : u_1$ and $1 : v_1$ by u' and v' we find:

$$\alpha : \delta = \frac{u'^{p} v'^{p_1}}{QSx_1},$$

$$\beta : \delta = \frac{u'^{q} v'^{q_1}}{PSy_1},$$

$$\gamma : \delta = \frac{u'^{s} v'^{s_1}}{PQz_1}.$$

So, but for constant factors, the coordinates of the pole of the tangential plane to O_{cc_1} with respect to the quadric

$$x^2 + y^2 + z^2 + 1 = 0 \quad \ldots \ldots \quad (24)$$

are equal to the coordinates of a point of O_{cc_1} (see § 4, equation 14).

So, if the equation of O_{cc_1} is

$$x^{Pk} y^{Qk} z^{Sk} = B,$$

the equation of the reciprocal polar figure with respect to (24) is

$$x^{Pk} y^{Qk} z^{Sk} = \frac{1}{B\,\{P^{S+Q}\,Q^{P+S}\,S^{P+Q}\}^k}.$$

So the product of the parameters corresponding to two reciprocal polar surfaces of the pencil O_{cc_1} is constant, viz.

$$\{P^{Q+S}\,Q^{S+P}\,S^{P+Q}\}^{-k}.$$

§ 11. In the case $s_1 = 0$ the asymptotic lines of the system $C(p_1, q_1, s_1)$ are right; so according to a known theorem four arbitrary asymptotic curves of the system $C(p, q, s)$ must intersect all the generatrices in four points with a constant cross ratio. This theorem not only holds for the ruled surfaces on which the curves

$C(p, q, s)$ are asymptotic curves, but also for any ruled surface generated by these curves.

Proof: Let the ruled surface be represented by the equations

$$\left.\begin{aligned} x &= (a + \alpha v)\, t^p \\ y &= (b + \beta v)\, t^q \\ z &= (c + \gamma v)\, t^s \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (25)$$

Let P_1, P_2, P_3, P_4 be the four points of intersection of the four curves $C(p, q, s)$ corresponding to the four parameter values v_1, v_2, v_3, v_4, with the generatrix corresponding to the parameter t_1. The cross ratio of these four points is equal to that of the four projections of these points on the axis OX and in its turn this cross ratio is equal to that of the four points of OX for which the x coordinate has the values

$$a + \alpha v_1 \ , \quad a + \alpha v_2 \ , \quad a + \alpha v_3 \ , \quad a + \alpha v_4 .$$

These four coordinates being independent of t, the cross ratio of the last group of four points does not vary with t_1. So the cross ratio of the four points P_1, P_2, P_3, P_4 is independent of t_1, i. e. this cross ratio is the same for any group of four points determined by the four curves $C(p, q, s)$ corresponding to the parameter values v_1, v_2, v_3, v_4 on any generatrix.

Example. The curves of the system $C(1, 2, 3)$ intersecting a given right line lie on a ruled surface of order four, for which one of the twisted cubics $C(1, 2, 3)$ is double curve (nodal curve, isolated curve or cuspidal curve). According to the theorem just proved any definite group of four curves of the system $C(1, 2, 3)$ cuts all the generatrices in four points with a constant cross ratio.

§ 12. In the case of a rectangular system of coordinates we easily find for the first differential coefficient of the length of arc σ in the point $P(x, y, z)$ of the curve $C_P(p, q, s)$ corresponding to the parameter value t the expression

$$\frac{d\sigma}{dt} = \frac{1}{t}\left\{ p^2x^2 + q^2y^2 + s^2z^2 \right\}^{\frac{1}{2}}.$$

Let $\Delta\theta$ be the angle between the binormals of the curve $C(p, q, s)$ in the points corresponding to the values t and $t + \Delta t$; then we easily find:

$$\frac{d\theta}{dt} = \frac{PQS\{p^2x^2 + q^2y^2 + s^2z^2\}^{\frac{1}{2}}}{t\,x\,y\,z\left\{\frac{P^2}{x^2} + \frac{Q^2}{y^2} + \frac{S^2}{z^2}\right\}}.$$

So the radius of torsion ϱ becomes:

$$\varrho = \frac{d\sigma}{d\theta} = \frac{xyz}{PQS}\left(\frac{P^2}{x^2} + \frac{Q^2}{y^2} + \frac{S^2}{z^2}\right).$$

For the radius of torsion ϱ_1 of the curve $C_P(p_1, q_1, s_1)$ in the same point we get

$$\varrho_1 = \frac{x\,y\,z}{P_1 Q_1 S_1}\left(\frac{P_1^2}{x^2} + \frac{Q_1^2}{y^2} + \frac{S_1^2}{z^2}\right),$$

and, as $P_1 = -P$, $Q_1 = -Q$, $S_1 = -S$ (see § 2, equation 11),

$$|\varrho| = |\varrho_1|\ [1].$$

Of the screws osculating the asymptotic lines of the surface $O_{\iota c_1}$ in any point the one is righthanded, the other lefthanded, as the determinant

$$\begin{vmatrix} x' & x'' & x''' \\ y' & y'' & y''' \\ z' & z'' & z''' \end{vmatrix} = \frac{xyz}{t^6} p^2 q^2 s^2 (P + Q + S)$$

assumes opposite signs for the two asymptotic lines.

Let X, Y, Z represent the director cosines of the binormal and d the distance of the origin to the osculating plane in the point (x, y, z); then we easily find:

$$1 : \varrho = XYZ \left\{\frac{P^2}{x^2} + \frac{Q^2}{y^2} + \frac{S^2}{z^2}\right\}^{\frac{1}{2}} = XY\frac{S}{z} = YZ\frac{P}{x} = ZX\frac{Q}{y};$$

$$\varrho = \frac{d}{XYZ\,PQS} = \frac{xyz\,PQS}{d^2}.$$

Let $\Delta\varphi$ be the angle between the tangents to a curve of the system $C(p, q, s)$ in the points corresponding to the values t and $t + \Delta t$; then we have:

$$\frac{d\varphi}{dt} = \frac{pqs\,xyz\left\{\frac{P^2}{x^2} + \frac{Q^2}{y^2} + \frac{S^2}{z^2}\right\}^{\frac{1}{2}}}{t\,(p^2x^2 + q^2y^2 + s^2z^2)}$$

by means of which we find for the radius of curvature R:

$$R = \frac{d\sigma}{d\varphi} = \frac{(p^2x^2 + q^2y^2 + s^2z^2)^{\frac{3}{2}}}{pqs\,xyz\left(\frac{P^2}{x^2} + \frac{Q^2}{y^2} + \frac{S^2}{z^2}\right)^{\frac{1}{2}}},$$

or, if α, β, and γ are the angles between the tangent and the axes of coordinates

$$\frac{1}{R} = \cos\alpha\cos\beta\cos\gamma \left\{\frac{P^2}{x^2} + \frac{Q^2}{y^2} + \frac{S^2}{z^2}\right\}^{\frac{1}{2}}$$

So we get:

[1] PASCAL, Rep. di Mat. Sup. Cap. 16; § 9.

$$R = \frac{d}{PQS \cos\alpha \cos\beta \cos\gamma},$$

and

$$\frac{R}{\varrho} = \frac{XYZ}{\cos\alpha \cos\beta \cos\gamma}.$$

Likewise, if $\alpha_1, \beta_1, \gamma_1$ are the angles between the tangent in the point P (x, y, z) to C^P (p_1, q_1, s_1) and the axes of coordinates, and R_1 is the radius of curvature of this curve in this point, we find:

$$R_1 = \frac{d}{P_1 Q_1 S_1 \cos\alpha_1 \cos\beta_1 \cos\gamma_1},$$

and therefore

$$\left|\frac{R}{R_1}\right| = \left|\frac{\cos\alpha_1 \cos\beta_1 \cos\gamma_1}{\cos\alpha \cos\beta \cos\gamma}\right|.$$

§ 13. The tangent in the point P_1 to the curve $C_P(p_1, q_1, s)$ admitting the director cosines

$$p'x_1 \ , \ q'y_1 \ , \ s'z_1 \ ,$$

this line is normal, in the case of rectangular axes, in P_1 to the quadric of the pencil

$$p'x^2 + q'y^2 + s'z^2 = \mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

passing through P_1 So the surfaces of this pencil (26) cut all the curves of the system $C(p, q, s)$ and consequently also all the surfaces generated by curves of the system $C(p, q, s)$ under right angles. Moreover the pencil (26) cuts any surface generated by curves $C(p, q, s)$ according to the orthogonal trajectories of these curves.

The surface O_{cc_1} being generated by curves of any system C (γ), see § 7, we find the theorems:

I. *Any quadric of the net*

$$px^2 + qy^2 + sz^2 + \lambda (p_1x^2 + q_1y^2 + s_1z^2) = \mu \quad . \quad . \quad . \quad (27)$$

cuts any surface O_{cc_1} under right angles.

II. *The orthogonal trajectories of the curves C (λ_1) situated on O_{cc_1} are the intersections with surfaces of the pencil.*

$$px^2 + qy^2 + sz^2 + \lambda_1 (p_1x^2 + q_1y^2 + s_1z^2) = \mu.$$

III. *Any curve of order four forming the base of a pencil of quadrics belonging to the net* (27) *cuts any surface O_{cc_1} under right angles.*

IV. *In particular the orthogonal trajectories of the asymptotic curves of O_{cc_1} are determined by the intersection with the two pencils of quadrics*

$$p\,x^2 + q\,y^2 + s\,z^2 = \mu,$$
$$p_1x^2 + q_1y^2 + s_1z^2 = \mu_1.$$

§ 14. We now suppose $s = p + q$; then the numbers p, q, s are mutually prime two by two. We then find $p_1 = q_1 = 2pq$, $s_1 = 0$; so the complementary system $C(p_1, q_1, s_1)$ is a system of right lines resting on the axis OZ and on $X_\infty Y_\infty$. The surface O_{cc_1} is a ruled surface with two right director lines.

Furthermore we find:

$$P = 1, Q = -1, S = \frac{q-p}{q+p};$$

so the lowest common multiple of the denominators of P, Q, S is either $q + p$ or $(q + p) : 2$ according to the numbers q and p being either one even and the other odd, or both odd.

We suppose in the first place that one of the numbers p, q is even (see § 15, examples I and III).

Then the equation of the ruled surface O_{cc_1} is:

$$x^{q+p} z^{q-p} = B y^{q+p};$$

so the ruled surface is of order $2q$. The enveloping cone is of order $2Pk = 2(q+p)$, see § 9, and of class $2q$..

If p and q are both odd and therefore $p - q$ and $p + q$ both even (see § 15, examples II, IV and V), the equation of O_{cc_1} is

$$x^{(p+q):2} z^{(p-q):2} = B y^{(p+q):2},$$

so O_{cc_1} is a ruled surface of order q. The enveloping cone with arbitrary vertex A is of order $q+p$, see § 9, and of class q.

The ruled surface osculating O_{cc_1} along a generatrix l is generated by the principal tangents of O_{cc_1} in the points of l which do not coincide with l, i.e. by the tangents of the curves of the system $C(p, q, p+q)$. So this osculating ruled surface is represented by the equations:

$$x = x_1 (1+pv) t$$
$$y = y_1 (1+qv) t,$$
$$z = z_1 \{1 + (p+q) v\}.$$

or by the equation

$$\frac{xy_1 - yx_1}{px_1y - qy_1x} = \frac{z - z_1}{(p+q) z_1}$$

§ 15. *Example I.* Suppose $p = 1$, $q = 2$, $s = 3$; then we have $s = p + q$, $p_1 = q_1 = 4$, $s_1 = 0$, $P = -Q = 1$, $S = \frac{1}{3}$. So the equation of the ruled surface with the twisted cubics of the system $C(1, 2, 3)$ as asymptotic lines is

$$x^3 z = B y^3.$$

Example II. For $p = 1$, $q = 3$, $s = 4$; we find $s = p + q$, $q_1 = p_1 = 6$, $s_1 = 0$. $P = -Q = 1$, $S = \frac{1}{2}$.

So the surface admitting as asymptotic lines the twisted quartics of the system $C(1, 3, 4)$ with two stationary tangents, is the cubic surface

$$x^2 z = By^2.$$

The section of O_{cc_1} by a plane $x =$ constant breaks up into the line at infinity of this plane and a curve of the system $C(0, 1, 2)$. The ruled surface osculating O_{cc_1} along this section is represented by the equations:

$$x = x_1 (1 + v),$$
$$y = y_1 (1 + 3v) t,$$
$$z = z_1 (1 + 4v) t^2.$$

The equation of this osculating ruled surface is

$$y^2 z_1 (4x - 3x_1) = z y_1^2 (3x - 2x_1)^2.$$

The intersection of this cubic surface and O_{cc_1} consists of the conic of contact counted thrice and of the two right directors of O_{cc_1}.

Example III. Suppose $p = 2$, $q = 3$, $s = p + q = 5$. Then O_{cc_1} is a ruled surface of order $2q = 6$, the equation of which is

$$x^5 z = By^5.$$

So this ruled surface admits a system of asymptotic lines of order five.

Example IV. Suppose $p = 1$, $q = 5$, $s = p + q = 6$. Then O_{cc_1} is a ruled surface of order $q = 5$ with the equation

$$x^3 z^2 = By^3.$$

So this ruled surface of order five admits a system of asymptotic lines of order six.

Example V. Suppose $p = 3$, $q = 5$, $s = p + q = 8$. Here O_{cc_1} is a ruled surface with the equation

$$x^4 z = By^4.$$

Example VI. If the first system of asymptotic lines is formed by curves of the system $C(1, 3, 6)$, then the asymptotic lines of the second system belong to the system $C(2, 3, -3)$. So both systems are curves of order six.

The equation of O_{cc_1} is

$$x^9 z = By^5.$$

Example VII. If the first system of asymptotic lines belongs to the system $C(1, 2, 4)$ the second system belongs to the system $C(5, 6, -4)$.

Then the equation of O_{cc_1} is

$$x^8 z = By^6.$$