the system $L+G$. The appearance of such a point has no influence on fig. 3 unless this accidentally coincides with the point $F$ of one of the previously examined figures. Such a singular point, that at each $T$ occurs only at a definite $P$, proceeds in the component triangle along a curve which may happen to pass through $F$. If this should take place, and if this point is a statonary point, then, in the case of the correlated $P$ and $T$, the vapour and liquidum line of the heterogeneous region $L+G$ and the theoretical liquidum vapour line pass through $F$; if this point is a maximum or minimum one these three lines coincide in $F$. From this it follows that in fig. 3 the singular point must always lie simultaneously on the lines $g' S g$, $e' E e$ and $f' K f$. The coincidence of a singular point with the point $F$ therefore causes the above three curves of fig. 2 to have one point in common; from other considerations it follows that they get into contact with each other.

This point of contact may lie in the solid as well as in the liquidum-gas region; in the first case, the system liquid $F$ + vapour $F$ is metastable, in the second case it is stable.

This point of contact may — but this is not very likely — also coincide with point $S$ of fig. 3. The system solid $F$ + liquid $F$ + vapour $F$ would then occur in the stable condition and the sublimation and melting point curves would then continue up to the point $S$. (To be continued).

**Mathematics.** — *"On complexes which can be built up of linear congruences"*. By Prof. JAN DE VRIES.

(Communicated in the Meeting of December 28, 1912).

§ 1. We will suppose that the generatrices $a$ of a scroll of order $m$ are in (1,1)-correspondence with the generatrices $b$ of a scroll of order $n$, and consider the complex containing all the linear congruences admitting any pair of corresponding generatrices $a, b$ as director lines. The two scrolls admit the same genus $p$; as the edges of a complex cone are in (1,1)-correspondence with the generatrices $a, b$ on which they rest, $p$ is also the genus of all the complex cones [1]). The rays of a pencil are arranged in a correspondence $(m, n)$ by the generatrices of the scrolls $(a), (b)$; so in general the complex is of order $m+n$.

[1]) For $m = n = 1$ (two pencils) we get the *tetrahedral complex*. In a paper "*On a group of complexes with rational cones of the complex*" (Proceedings of Amsterdam, Vol. VII, p. 577) we already considered the case of a pencil in (1,1) correspondence with the tangents of a rational plane curve.

The double edges of a complex cone are rays resting on two pairs $a, b$; they belong to a congruence contained in the complex, of which congruence both *order* and *class* are equal to the number of double edges of the cone.

Evidently any point common to two corresponding generatrices $a, b$ is a *principal point*, the plane containing these lines a *principal plane* of the complex. If one of the scrolls is plane, the bearing plane is a *principal plane* too; if one of them is a cone, the vertex is a *principal point* [1]).

Any point $P$ of a principal plane is *singular*, the pencil with vertex $P$ lying in that plane forming a part of the complex cone of $P$. The same degeneration presents itself for any point of each of the given scrolls; so these surfaces are loci of *singular points*. Likewise any plane through a generatrix $a$ or $b$ and any plane through a principal point is *singular*.

By means of one scroll only can also be obtaine complexes consisting of linear congruences. So we can arrange the generatrices of a scroll in groups of an involution $I$ and consider any pair of any group as director lines of a linear congruence [2]).

In the following lines we treat the *biquadratic complex* which can be derived in the manner described above from *two projective reguli*. After that we will investigate the particular cases of plane scrolls or cones.

§ **2.** We use the general line coordinates $x_k$, introduced by KLEIN, which are linear functions of the coordinates $p$ of PLÜCKER and satisfy the identity $(x^2) = \sum\limits_6 x_k^2 = 0$, while $\sum\limits_6 x_k y_k = 0$ or $(xy) = 0$ indicates that $x$ and $y$ intersect each other.

Then a regulus is characterized by the six relations

$$a_k = p_k \lambda^2 + 2q_k \lambda + r_k,$$

satisfying the conditions:

$$(p^2) = 0, (r^2) = 0, (pq) = 0, (qr) = 0, 2(q^2) + (pr) = 0.$$

Likewise we represent the second regulus by

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1) In our paper "*Sur quelques complexes rectilignes du troisième degré*" (Archives Teyler, 2nd series, vol. IX, p. 553—573) we have considered among others the case that one of the scrolls is a pencil whilst the other is formed by the tangents of a conic.

2) This has been applied to a developable in our paper '*On complexes of rays in relation to a rational skew curve*" (Proceedings of Amsterdam, vol. VI, p. 12) and on a rational scroll in "*A group of complexes of rays whose singular surfaces consist of a scroll and a number of planes*". (Proceedings of Amsterdam, vol. VIII, p. 662).

$$b_k = p'_k \lambda^2 + 2q'_k \lambda + r'_k.$$

Then we find for the rays $x$ of the congruence with director lines $a$, $b$

$$(px)\lambda^2 + 2(qx)\lambda + (rx) = 0,$$
$$(p'x)\lambda^2 + 2(q'x)\lambda + (r'x) = 0,$$

which we abridge into

$$P\lambda^2 + 2Q\lambda + R = 0 \quad , \quad P'\lambda^2 + 2Q'\lambda + R' = 0.$$

By elimination of $\lambda$ we get the equation of the *biquadratic complex* under discussion. It is

$$(PR' - P'R)^2 = 4(PQ' - P'Q)(QR' - Q'R),$$

or, what comes to the same,

$$(PR' - 2QQ' + P'R)^2 = 4(PR - Q^2)(P'R' - Q'^2).$$

From this ensues that the complex can be generated in two different ways by *two projective pencils of quadratic complexes.* This is shown by the equations

$$PR' - P'R = 2\mu(PQ' - P'Q),$$
$$\mu(PR' - P'R) = 2(QR' - Q'R)$$

and

$$PR' - 2QQ' + P'R = 2\mu(PR - Q^2),$$
$$\mu(PR' - 2QQ' + P'R) = 2(P'R' - Q'^2).$$

The equation $(ab) = 0$ expressing the condition that two corresponding generatrices $a$, $b$ have a point in common, gives rise to a biquadratic equation in $\lambda$. So there are *four principal points* and *four principal planes.*

§ **3.** We now occupy ourselves with the congruence of the rays $x$ each of which rests on *two pairs* of homologous generatrices $(\lambda)$. For such a ray $x$ the two equations

$$P\lambda^2 + 2Q\lambda + R = 0 \quad , \quad P'\lambda^2 + 2Q'\lambda + R' = 0$$

must be satisfied for the same values of $\lambda$; so we have the condition

$$\begin{Vmatrix} P & Q & R \\ P' & Q' & R' \end{Vmatrix} = 0.$$

This equation leads to a *congruence* (3,3). For the quadratic complexes $PQ' = P'Q$ and $PR' = P'R$ have the congruence $P = 0$, $P' = 0$ in common and the latter congruence does not belong to the complex $QR' = Q'R$.

This result is in accordance with the fact that the complex cones (and curves) must be rational and have to admit therefore *three double edges* (and *three double tangents*).

Both the characteristic numbers of the congruence can also be found as follows. A plane through any point $A_0$ of the generatrix $a_0$ and the corresponding generatrix $b_0$ cuts both reguli respectively in a conic $\alpha_0^2$ and a line $\beta_0$. On these sections the other pairs of corresponding lines $a,b$ determine two projective ranges of points $(A)$, $(B)$. As these arrange the rays of a pencil in the plane $(A_0 b_0)$ in a correspondence (1,2), the lines $AB$ envelop a *rational curve* of *class three* with $\beta_0$ as *double tangent.* Each of the three lines $AB$ passing through $A_0$ rests on two pairs $a,b$ and belongs therefore to the congruence.

The curve of class three just found and the pencil with $A_0$ as vertex form together the *complex curve* of plane $(A_0 b_0)$. Likewise the *complex cone* of $A_0$ breaks up into this pencil and a rational cubic cone.

Any point and any tangential plane of the quadratic scrolls $(a)$, $(b)$ is *singular.* Moreover the points of the principal planes and the planes through the principal points are *singular.*

§ 4. If we add the relation $(p'r') = 0$ to the conditions enumerated in § 2, it follows from $2(q'^2) + (p'r') = 0$ that the coordinates $q'_{i}$ also determine a line, which is to cut $p'$ and $r'$ on account of $(p'q') = 0$, $(q'r') = 0$ without belonging to the regulus. So it lies either in the plane $\tau$ through $p'$ and $r'$ or on a quadratic cone with the point of intersection $T$ of $p'$ and $r'$ as vertex.

In the first case each line of $\tau$ belongs to the complex and even *twice* as it cuts two generatrices of the regulus $(a)$. In other words: $\tau$ is a *double principal plane.*

In the second case an analogous reasoning shows that $T$ is a *double principal point.*

§ 5. In the two latter particular cases the complex has lost the property of corresponding *dually with itself.* On the contrary this property is still preserved by the complex generated by two projective reguli the first of which consists of the tangents of a conic $\alpha^2$ (in plane $\alpha$) and the second is formed by the edges of a *quadratic cone* $\beta^2$ (with vertex $B$).

The range of points $B_0$ on the section $\beta_0^2$ of $\beta^2$ and $\alpha$ is in (1,1)-correspondence with the system $(a)$. So the points $B_0$ are in (2,2)-correspondence with the points of intersection $A_0$ of the generatrices $a$ and the conic $\beta_0^2$. So the complex admits *four principal points,* each of which bears a *principal plane.*

Furthermore $\alpha$ is a *double principal plane,* $B$ a *double principal point.*

The complex cone of point $P$ has $PB$ for double edge; for $PB$ cuts two generatrices $a$ and at the same time the corresponding lines $b$. So the congruence (3,3) of the general case must break up here into a (1,0), a (0,1) and a (2,2).

In order to check this we consider the correspondence between the points $A = a_1 a_2$ and the corresponding planes $\beta = b_1 b_2$. If $A$ describes a line, $a_1$ and $a_2$ generate an involution; as $b_1$ and $b_2$ do then likewise, $\beta$ will rotate about a fixed axis. So the correspondence $(A, \beta)$ is a *correlation.* Therefore plane $\alpha$ contains a conic $\alpha_0^2$, each point $A_0$ of which is incident with the trace $b_0$ of the homologous plane $\beta_0$. So each point $A_0$ is the vertex of a pencil belonging to the complex and lying in plane $\beta_0$. These pencils generate a *congruence* (2, 2). For their planes envelop a quadratic cone with vertex $B$, two tangential planes $\beta_0$ of which pass through the arbitrarily chosen point $P$; so the lines connecting $P$ with the homologous points $A_0$ belong to the congruence in question, which evidently is *dual in itself.*

§ **6.** We will now suppose that the tangents $a$ of the conic $\alpha^2$ in plane $\alpha$ and the tangents $b$ of the conic $\beta^2$ in plane $\beta$ are in (1, 1)-correspondence. Then the congruence with any pair of corresponding tangents $a$, $b$ as director lines generates once more a complex of order four, evidently *not* dual in itself.

By the correspondence $(a, b)$ the points of the line $c$ common to $\alpha$ and $\beta$ are arranged in a (2,2)-correspondence. The *four* coincidencies are *principal points* of the complex and the lines $a$, $b$ concurring in any of these points determine a *principal plane* So we have indicated four sheaves of rays and four fields of rays belonging to the complex.

The planes $\alpha$ and $\beta$ are also fields of rays of the complex; for any line $s$ of $\alpha$ is cut on $c$ by two lines $b$ but also by the corresponding lines $a$; so $s$ belongs twice to the complex.

We account for this by saying that $\alpha$ and $\beta$ are *double principal planes.*

The complex cone of any point $P$ meets $c$ in four points, i. e. in the four principal points; so we deal with a *biquadratic complex.*

The *complex cone* is rational, its edges corresponding one to one to the tangents of $\alpha^2$; therefore it has to admit *three double edges.* Likewise the *complex curve* of any plane has to admit *three double tangents.*

§ **7.** In order to investigate this more closely we consider the relationship between any point $A$ of $\alpha$, as point common to two tangents $a_1, a_2$, and the point $B$ common to the corresponding tangents $b_1, b_2$.

If $A$ describes a line $l_A$ its polar line with respect to $\alpha^2$ will rotate about a fixed point, whilst the pair $a_1, a_2$ generates an involution. But then $b_1, b_2$ must also generate an involution, so that $B$ describes a line $l_B$. So the point fields $(A)$, $(B)$ are in projective correspondence (collinear. homographic).

By projecting the field $(A)$ out of any point $P$ unto $\beta$ we obtain in $\beta$ two projective collocal fields, admitting three coincidencies. So the congruence of the lines $AB$ is of *sheaf degree* (order) *three*. Its *field degree* (class) however is *one*; for if $A$ describes the section of $\alpha$ with any plane $\Pi$, $B$ will arrive once in $\Pi$, i. e. $\Pi$ contains only *one* line $AB$.

The congruence $(3, 1)$ found here is generated, as we know, by the *axes* (= biplanar lines) of a twisted cubic $\gamma^3$, i. e. any line $AB$ lies in two osculating planes of $\gamma^3$.

Evidently any line $AB$ is double edge of the complex cone of any of its points $P$. However the complex rays through $A$ form the pencil $A(\alpha)$ counted twice and the pencils determined by the lines $b_1, b_2$; for $B$ the analogous property holds.

§ **8**. Evidently the *three double edges* of the *complex cone* of $P$ are the mutual intersections of the *three osculating planes* of $\gamma^3$ passing through $P$.

Likewise the *complex curve* in $\Pi$ has for *double tangent* the axis of $\gamma^3$ lying in that plane, the other *two double tangents* coinciding with the intersections of $\Pi$ with $\alpha$ and $\beta$. For, each of the lines $b', b''$ which concur in the point $c\Pi$ determines a complex ray lying in $\Pi$, which lines coincide both with $\alpha\,\Pi$.

An *osculating plane* $\Omega$ of $\gamma^3$ contains $\infty^1$ axes, enveloping a conic $\omega^2$. Any plane $\Omega$ is *singular* for the congruence $(AB)$. So the complex curve in $\Omega$ is the conic $\omega^2$ counted twice.

As the congruence (3,1) cannot admit singular points, no point bearing more than three planes $\Omega$, no complex cone can degenerate but those corresponding to the principal points and the points of the principal plane. We already remarked this for $\alpha$ and $\beta$; for any point of a single principal plane the complex cone breaks up into a pencil and a rational cubic cone.

The complex cone of any point of the *developable* with $\gamma^3$ as cuspidal edge admits an edge along which two sheets touch each other (the plane section has two branches touching each other). For any point of $\gamma^3$ the cone possesses an edge along which two sheets osculate each other (the section has two branches with a common point of inflexion touching each other).

A *cuspidal edge* connects any point $A_0$ of $\alpha^2$ with the corresponding point $\beta_0$ of $\beta^2$. The locus of the line $A_0B_0$ is a *biquadratic scroll*, of which $\alpha$ and $\beta$ contain two generatrices. Any point of this scroll admits a complex cone with a *cuspidal edge*.

Evidently the biquadratic scroll is *rational*, so it has a *twisted cubic* as *nodal curve*: For any point $P$ of this curve the complex cone has *two cuspical edges*.

By replacing the two conics $\alpha^2$, $\beta^2$ (as bearers of flattened reguli) by two quadratic cones we obtain a complex evidently dually related to that treated above.

If $\alpha^2$ and $\beta^2$ touch the line $c = \alpha\beta$ whilst $c$ corresponds to itself in the relationship between $a$ and $b$, the complex degenerates into the special linear complex with axis $c$ and a cubic complex. Evidently the same holds for the general biquadratic complex (§ 2) if the reguli admit a common generatrix corresponding to itself.

**Chemistry.** — *"On the system phosphorus"*. By Prof. A. SMITS, J. W. TERWEN, and Dr. H. L. DE LEEUW. (Communicated by Prof. A. F. HOLLEMAN).

(Communicated in the meeting of November 30, 1912).

In a previous communication on the application of the theory of allotropy to the system phosphorus [1]) it was pointed out that the possibility existed that the line for the internal equilibrium of molten white phosphorus is not the prolongation of the line for the internal equilibrium of molten red phosphorus, in consequence of the appearance of critical phenomena below the melting-point of the red modification. The latter could namely be the case if the system $\alpha P$—$\beta P$ belonged to the type ether-anthraquinone, which did not seem improbable to us.

This supposition was founded on the following consideration. In the first place it follows from the determinations of the surface-tension carried out by ASTON and RAMSAY [2]), that the white phosphorus would possess a critical point at 422°. Hence the critical point of

1) Zeitsch f. phys. Chem. **77**, 367 (1911).
2) J. Chem. Soc. **65**, 173 (1894).
Cf. also SCHENCK, Handb. ABEGG $III_3$ 374.