We will conclude this communication with the schematic PT-figure of the phosphorus; the connection between the unary and the pseudo-binary system will be treated in a following publication.

When the calculated critical temperature 422° for liquid white phosphorus is correct about 18 atmospheres follows from the vapour pressure line for the critical pressure. The critical point is indicated by k_1, in the drawing. The vapour pressure line of molten red phosphorus exhibits probably a peculiarity that has never been met with as yet, viz. two critical points k_2 and k_1, the former of which is metastable.

It is of course also possible even probable that unmixing takes place in the pseudo-system between p and q, so in the metastable region. The point k_2 might, therefore, lie at even lower temperature and pressure than the point k_1. Possibly the continued investigation may give an indication with regard to this too.

It may finally be pointed out that when we apply VAN DER WAALS's equation,

$$log \frac{p_k}{p} = f\left(\frac{T_k}{T} - 1\right)$$

and write

$$log\, p = -\frac{fT_k}{T} + C$$

3,94 is found for the value of f.

This equation does not represent the observed vapour pressure line as well as the former, the cause of this may be that f is not constant as has been found indeed with several substances.

Anorg. Chem. Laboratory of the University.

Amsterdam, Nov. 29, 1912.

Mathematics. — *"On loci, congruences and focal systems deduced from a twisted cubic and a twisted biquadratic curve".* III. By Prof. HENDRIK DE VRIES.

(Communicated in the meeting of November 30, 1912).

17. If we assume that the line l itself is a ray of the complex without however belonging to the congruence deduced from Ω^6, then the two surfaces Ω^{20} and Ω^1 undergo considerable modifications. The surface Ω^{20} has no lowering of order; instead of the regulus, namely, which is the locus of the rays s conjugated to the points of l we now have a quadratic cone (passing likewise through the cone

vertices) whose vertex P_l is the focus of l, because the two conjugated lines of l, which cross each other in general and exactly therefore generate a regulus, now both pass through P; but P_l does not lie on Ω^6, because l is a ray of the complex, but not of the congruence. A generatrix of the cone therefore intersects Ω^6, as formerly a line of the regulus, in six points, from which ensues that l now again is a sixfold line of the surface. And to a plane λ through l corresponds as formerly a twisted cubic through the cone vertices and which now passes moreover through P_l, because l is a tangent of the complex conic lying in λ, but which now again intersects Ω^6, except in the cone vertices, in fourteen points; thus in λ lie 14 generatrices of the surface, so that this is indeed of order $6 + 14 = 20$. The curve k^{12}, the section of the cone with Ω^6, has also 6 nodal points lying on k^3, so that Ω^{20} contains 6 nodal generatrices.

The nodal curve of Ω^{20} undergoes a very considerable modification as regards the points it has in common with l. Through such a point namely must go 2 generatrices of the surface lying with l in one plane; but now l is itself a ray of the complex and three rays of the complex can then only pass through one point when the complex cone of that point breaks up into two pencils; so the only points which the nodal curve can have in common with l are the points of intersection of l with the four tetrahedron faces.

These points which in § 15 we have called S_i coincide with the points which were called $T_i{}^*$ in the same §. Let us assume the plane lT_1. As now again and for the same reason as before nine of the fourteen generatrices of Ω^{20} lying in this plane pass through T_1 (§ 13) the five remaining ones must pass through another point $T_1{}^*$ lying in τ_1 and whose complex conic breaks up into τ_1 and the plane $T_1{}^*l$; now however this point coincides with S_1. For the complex cone of S_1 likewise breaks up into two pencils, of which one lies in τ_1, the second in a plane through $T_1{}^*$ and T_1; now however, to this second pencil evidently belongs our ray l and so indeed the complex cone of S_1 degenerates in this way into τ_1 and a plane through l; so S_1 and $T_1{}^*$ are identical. To S_1, regarded as a focus, a ray s through T_1 is conjugated which lies at the same time on the quadratic cone, thus in other words the ray $P_l T_1$; the latter intersects Ω^6 besides in T_1 in 5 more points and the rays s conjugated to these are the 5 generatrices of Ω^{20} through $S_1 = T_1{}^*$ lying in the plane lT_1; the sixth generatrix through this point conjugated to T_1 lies in τ_1, but not in the plane lT_1.

So we see that through S_1 pass five generatrices of Ω^{20} lying in the same plane; *so the four points S_i are* $\frac{1}{2} \cdot 5 \cdot 4 = 10$*-fold points* *for the nodal curve; this curve cannot have other points in common with l. So it cuts l in four tenfold points* (i. e. the 40 points of before have changed into four tenfold ones) *and so it is again of order* $40 + 91 = 131$.

Also the surface Ω^4 undergoes considerable modifications as the conic lying in a plane λ must now always touch the line l. The complex cone for a point P of l contains the ray l; the two tangential planes through l to the cone coincide therefore; from which ensues that for each point P of l the two conics passing through it, coincide. The most intuitive representation of this fact is obtained by imagining instead of the point of contact of a k^2 with l two points of intersection lying at infinitesimal distance, if then on l we assume three of such like points, then through 1 and 2 passes a conic and through 2 and 3 an other differing but slightly from it, so that really through point 2 pass two conics. *The loci of the conics is thus now again a* Ω^4 *with nodal line l, but this line has become a cuspidal edge,* i. e. whereas formerly an arbitrary plane intersected Ω^4 along a plane curve with a nodal point on l and only the planes through the four points S_i (§ 15) furnished curves with cusps, now every arbitrary plane of intersection contains a curve with a cusp on l (and with a cuspidal tangent in the plane of the conic through that cusp). Furthermore we must notice that as the points $T_i{}^*$ coincide with S_i, the four nodal points T_i will be found on the nodal line itself, thus forming in reality no more a tetrahedron proper, nevertheless the property of the simultaneous circumscription round about and in each other remains if one likes.

18. The curve of intersection of order eighty of Ω^4 and Ω^{20} is again easy to indicate; it consists of the line l counted twelve times (for a cuspidal edge remains a nodal edge), and of a curve of contact of order 34 to be counted double (§ 15) which has with a plane λ through l fourteen points lying outside l in common and therefore twenty lying on l; these last however can be no others than the four points S_i, for otherwise a generatrix of Ω^{20} would have to touch a k^2 of Ω^4 on l, which could only be possible (as l itself touches k^2) if a generatrix of Ω^{20} could coincide with l which is as we know not possible. The curve of contact of Ω^4 and Ω^{20} passes thus five times through each of the four points S_i which corresponds to the fact that five generatrices of Ω^{20} touch in S_i, the degenerated conic (viz. the pair of points S_i, T_i) lying in the plane lT_i.

The method indicated in § 14 to determine the number of torsal lines of the first kind undergoes no modification whatever; we can however control this method here because we have to deal here with a cone instead of a regulus. The first polar surface of P_l namely with respect to Ω^6 is a Ω^5 containing k^3 one time, and therefore cutting Ω^6 along k^3 counted twice and a residual curve of order 24, so that the circumscribed cone at the vertex P_l is of order 24. Now this cone cuts the quadratic cone $[P_l]$ in 48 edges, so 48 edges of $[P_l]$ touch Ω^6 and therefore k^{12}. The number of torsal lines of the first kind is thus indeed 48, and that this same number must now be found in general follows from the law of the permanency of the number.

These numbers 6 and 48, as well as the number of points (namely 40) which the nodal curve of Ω^{20} has in common with l can be controlled with the aid of the symmetrical correspondence of order 70 existing between the planes λ through l (§ 16). To the 140 double planes δ belong, as we saw before. the planes through l and the nodal lines and those through l and the torsal lines of the first kind, together appearing there at a number of 54, but representing 60 double planes. The nodal curve of Ω^{20} has with l only the 4 points S_i in common which however count for 10 each and which have the property that five of the six generatrices through each of those points lie in one plane; such a plane is thus undoubtedly a manyfold plane of the correspondence, the question is only how many single double planes it contains. Now there lie in the plane lT_1, e.g. 9 generatrices through T_1 cutting l in different points; through each of the last pass five other generatrices, and so we find so far 45 planes conjugated to the plane lT_1.

Now we have moreover the plane through l and the 6th generatrix through S_1 (lying in τ_1); however by regarding, just as we have done at the beginning of § 16, a plane λ in the immediate vicinity of lT_1 and in which thus five generatrices cut each other n e a r l y in one point of l we can easily convince ourselves that this plane counts for 5 coinciding planes conjugated to lT_1. To lT_1 are conjugated $45 + 5 = 50$ planes *not* coinciding with lT_1 and thus 20 planes coinciding with lT_1; i. e. just as in the general case a plane λ through two generatrices cutting each other on l counts for two double planes, so here each plane lT_i containing five such generatrices counts for 5×4 double planes; so the four planes lT_i represent 80 double planes, and they furnish with the 60 already found the 140 double planes as they ought to.

As by the transition to a ray of the complex all numbers have remained unchanged, the surface Ω^{20} contains now again 58 torsal lines of the 2nd kind; the $4 \times 131 = 524$ points of intersection of Ω^4 with the nodal curve of Ω^{20} lie now however a little differently. The points T_i remain 36-fold for the nodal curve and they therefore furnish $4 \times 72 =$ **288** points of intersection, the 58 torsal lines of the 2nd kind give **58**, the 6 nodal edges give $3 \times 6 =$ **18** other ones; the 4 points $S_i = T_i^*$ however absorb each of them 40 points of intersection. Let us namely imagine our figure variable and in particular l continuously passing into a complex ray, we then see how the 4 points T_i^* tend more and more to S_i, but at the same time how the 40 points of intersection of l with the nodal curve group themselves more and more into 4 groups of 10 in such a way that each group is as it were attracted by one of the points S_i; now each of those 40 points counts for 2, each point T_i^* for 20 points of those we looked for; so on the moment that T_i^* as well as the 10 points of the corresponding group coincide with S_i this point counts for 40, so the four together for **160** and the sum of the four numbers printed in heavy type is again 524.

19. More considerable are the modifications if finally we now assume that l becomes a ray of the congruence; nothing is to be noticed at Ω^4, as l remains a ray of the complex, but the other locus becomes a surface Ω^{18}, for which l is only a fivefold line. The regulus of before is namely now again replaced by a cone $[P_l]$, but the vertex itself P now lies on Ω^6, because l is a ray of the congruence, thus itself a generatrix. It even appears twice as a generatrix, for the cone cuts Ω^6 according to a k^{12} which has now a. o. also a nodal point in P_l and to this nodal point the line l corresponds twice. A generatrix of the cone $[P_l]$ cuts Ω^6 in P_l and in five other points; so through the corresponding focus on l pass five generatrices not coinciding with l, i. e. *l is a fivefold line.*

To a plane λ through l a twisted cubic is conjugated containing the four vertices of the cones and P_l and cutting Ω^6 in 13 points more; so in a plane λ lie besides l 13 generatrices, i. e. *our surface is a Ω^{18} of order* 18 *with a fivefold line l.*

Among the generatrices of the cone $[P_l]$ there are two touching k^{12} in P_l and likewise among the twisted cubics; the foci of the former are the points of intersection proper of l with two generatrices coinciding with l, the planes conjugated to the latter being the connecting planes; thus two particular torsal planes and pinch points (see § 20).

The line $P_l T_i$ is a generatrix of the cone $[P_l]$ and it cuts Ω^6 besides in these two points in four more; the corresponding four rays s pass through $S_i = T_i^*$ and lie in the plane lT_i whilst the ray s conjugated to T_i lies in τ_i, but not in $l\,T_i$, so the points S_i are $\frac{1}{2}\,.\,4.3 = 6$-fold points for the nodal curve and others this curve can evidently not have in common with l. So it has 24 points united in 4 sixfold points in common with l, and as there are in a plane ε through l $\frac{1}{2}\,.\,13.12 = 78$ points not lying on *l the order of the nodal curve now amounts to* $24 + 78 = 102$. The number of nodal points of a plane section of Ω^{18} amounts thus now to $102 + 6 + 10 = 118$, and from this ensues for the class $18.17 - 2.118 = 70 = \varepsilon\beta$; the formula $\varepsilon\sigma = 2.\,\varepsilon\beta - 2.\,\varepsilon g$ furnishes therefore $\varepsilon\sigma = 2\,.\,70 - 2\,.\,18 = 104$ torsal lines of both kinds.

The formula

$$\varepsilon = p + q - g$$

now again applied to determine the number of generatrices of the cone $[P]$ touching k^{12} and thus of the number of torsal lines of the first kind gives the following results. The plane of the condition p cuts k^{12} in 12 points; through each of these passes a generatrix of the cone cutting Ω^6 besides in P_l in four points more; so the number p is equal to 48, a and likewise q. The line of the condition g cuts the cone in two points and through each of these passes a generatrix of that cone, on which lie besides P_l five points of k^{12}; so g is $= 2.20$, and thus $\varepsilon = 2.48 - 2.20 = 56$. Among these however are included the six nodal lines counted twice; *the number of torsal lines of the first kind amounts thus to* $56 - 2 \times 6 = 44$.

To control this we again consider the first polar surface of P_l with respect to Ω^6, a Ω^5 touching Ω^6 in P_l and passing through k^3. The intersection with Ω^6 consists therefore of k^3 counted twice and a residual curve of order $30 - 2.3 = 24$ which however is projected out of P_l by a cone of order 22 only, because P_l itself is a nodal point of that curve (for Ω^6 and Ω^5 touch each other in P_l); this cone has with the cone $[P_l]$ 44 generatrices in common, and these touch k^{12}.

The number of torsal lines of the 2nd *kind of* Ω^{18} *amounts to* $104 - 6 - 44 = 54$.

The correspondence of the planes λ through l is now of order 52 with 104 double planes. For, in a plane λ lie besides l thirteen generatrices of Ω^{18} and through each of the 13 points in which these cut l four others pass; so to each plane λ $4 \times 13 = 52$ others are conjugated. The double planes are 1. the planes through the 44 torsal lines of the first kind; 2. the planes through the 6 nodal edges, each counted twice; 3. the 4 planes lT_i each counted twelve times, because in each such like plane 4 generatrices pass through the point S_i (comp. § 18); so we find $44 + 2.6 + 4.12 = 104$ double planes.

And as regards finally the number of $4 \times 102 = 408$ points of intersection of the nodal curve with Ω^4, in the four points T_i lie again 288 (comp. § 18), in the pinch points of the torsal lines of the second kind 54, in those of the six nodal edges 18 and in the four points S_i, which are sixfold for the nodal curve, 48, together $288 + 54 + 18 + 48 = 408$.

20. The two particular pinch points on l which we have found in the preceding § were the two foci of the ray of the congruence l and the two torsal planes the two focal planes; for, in these points l was cut by a ray of the congruence at infinitesimal distance. If henceforth with a slight modification in the notation the line l is called s_0, the focus P_0, then P_0 lies on Ω^6 and it is in general an ordinary point of this surface. Let us assume the tangential plane in this point and in it an arbitrary line t through P_0; then this has two conjugated lines crossing each other, and if therefore a point P describes the line t, the ray s of the complex conjugated to P will generate a regulus to which also belongs our ray s_0, a ray of the congruence. As however t is a tangent of Ω^6, a second generatrix of the regulus lying at infinitesimal distance from s_0 will belong to the congruence, however without cutting s_0. If however, we now imagine the complex cone at point P_0 and if we intersect it by the tangential plane, we find two lines t which are at the same time lines s, viz. rays of the complex, and whose two conjugated lines cut each other. Now the lines s conjugated to the points P of t will describe two cones containing also s_0, and having their vertices on s_0 whilst we know out of our former considerations that these vertices are nothing but the foci of the two rays t; and now s_0 will be cut in each of these foci by a ray of the congruence at infinitesimal distance; the two cone vertices are thus the foci of s_0. So: *we find the foci of a ray s_0 of the congruence by determining the focus P_0 (lying on Ω) of s_0, by intersecting the complex cone of this point by the tangential plane in P_0 to Ω^6, and by taking the foci of the two lines of intersection t. And the two focal planes are the tangential planes through s_0 to the complex cones of the foci.*

If P_0 is a point of the nodal curve k^3 of Ω^6 then s_0 is a double ray of the congruence (§ 12); the complex cone of P_0 intersects the two tangential planes of P_0 in twice two rays t, so that we now have on s_0 two pairs of foci and through s_0 two pairs of focal planes; and as the focal surface of the congruence is touched by each ray of the congruence in the two foci, so each double ray will touch the focal surface four times. The four tangential planes are the focal planes, however in such a way that if one pair of foci is called F_1, F_2 the focal plane of F_1 is tangential plane in F_2 and reversely.

Let P_0 be a point of k^4, lying as a single curve on Ω^6; then s_0 is the tangent to k^4 in P_0 and it belongs to the congruence. The complex cone of P_0 intersects the tangential plane in this point to Ω^6 according to s_0 itself and an other generatrix; so of the two foci of s_0 point P_0 is one whilst the other is the focus of the second generatrix of the complex cone of P_0 lying in the tangential plane; and of the two focal planes the osculation plane of k^4 in P_0 is one, because this really contains two rays of the congruence intersecting each other in P_0 and lying at infinitesimal distance (viz. two tangents of k^4); so it touches the focal surface in the other focus, i. e. the surface of tangents of k^4 which is of order 8 envelops the focal surface, and the curve k^4 itself lies on the focal surface.

The question how the cone vertices T_i bear themselves with respect to the congruence, is already answered in § 11; Ω^6 intersects the plane τ_i according to a plane k^6 and the rays s conjugated to these form a cone of order 9 with the vertex T_i and with three nodal edges and three fourfold edges, the latter of which coincide with the three tetrahedron edges through T_i.

Let us assume an arbitrary point P of k^6, then to this a ray s through T_i is conjugated; now the complex cone of P degenerates into a pair of planes, of which τ_i is one component, whilst the other passes through T_i, and this degenerated cone cuts the tangential plane in P to Ω^6 along the tangent t in P to k^6 and according to an other line t^* through P. To that tangent the point T_i is conjugated as focus, so that for each ray of the congruence through T_i this point itself is one of the foci, the other being the focus of the line t^*.

In order to find the focal plane of the considered ray s in the point T_i we should have to know according to the preceding the complex cone of T_i which is in first instance entirely indefinite; let us however bear in mind that in the general case that complex cone is at the same time the locus of the ray s conjugated to the points of the tangent t; then in this case also we can have a definite cone, viz. the cone which replaces the regulus if the line l passes into a complex ray s, and which contains in general the four cone vertices and which will contain here, where T_i itself is the cone vertex, the three tetrahedron edges through this point. On this cone lie the two rays s conjugated to the two points of k^6 lying at infinitesimal distance from each other on t, and the plane through these is the focal plane of our ray s in T_i; but those edges of the quadratic complex cone lying at infinitesimal distance lie of course also on the cone of order 9 (see above); so we can say more briefly that for each ray of this cone T_i is one of the foci and the tangential plane to the cone is one of the focal planes.

Each ray of the congruence through T_i, so each generatrix of the cone of order nine with this point as vertex, must have in P_i two coinciding points in common with the focal surface; so T_i is for the focal surface a manifold point, however without the cone of order 9 being the cone of contact; for the tangential planes of this cone touch the focal surface in the foci of its generatrices *not* coinciding with T_i; the cone of contact in T_i is enveloped by the focal planes of this last category of foci.

21. Over against the question which complex rays through T_i belong to the congruence, is the other one which complex rays out of τ_i belong to the congruence. In the preceding we have repeatedly come across these rays. Indeed, any surface Ω^{20} formed by the congruence rays which cut a line l or a complex ray s, and any surface Ω^{18} formed by the congruence rays which cut a congruence ray s contained such a ray as we proved above; we shall now show that all these rays form a pencil. To that end we imagine the tangential plane ϱ in T_i to Ω^{6} and we cut it according to the line r by τ_i. We now saw in the preceding that the rays s conjugated to the points of τ_i form a quadratic cone with T_i as vertex and containing the three tetrahedron edges through T_i; if the base curve of this cone lying in τ_i is k^2, then reversely the points of k^2 are the foci of the rays s lying in ϱ and passing through T_i, for the rays s conjugated to the points of a line pass through the focus of that line and the ray s conjugated to a point of τ_i passes moreover through T_i.

If a point P describes one of the rays of the pencil $[T_i]$ lying in ϱ, say s_0, then the rays s conjugated to the points P form the complex cone of the focus P_0 of s_0, which point lies on k^2; this complex cone breaks up however into a pair of planes, viz τ_i and a plane through P_0 and T_i, and the line of intersection t_i of these two planes is the ray of the congruence conjugated to T_i, in as far as this point is regarded as a point of the ray s_0; so the question is how the rays t_i bear themselves when s_0 describes the pencil $[T_i]$ or, what comes to the same, how the planes $T_i t_i$ bear themselves in those circumstances. We shall try to find how many of those planes through an arbitrary ray s_1 pass through T_i. In each arbitrary plane through s_1 the complex conic breaks up into two pencils; one has the vertex T_i, the other a point T_i^* lying in τ_i.

In each plane through s_1 lies however one such point T_i^*; but if S_1 is the point of intersection of s_1 with τ_i, then also the complex cone of S_1 breaks up into a pair of planes of which one component is of course again τ_i, the other being a plane through $S_i T_i$; so S_1 is itself a point T_i^*, and the consequence of this is that T_i^* describes a conic k^{*2} which passes in the first place through S_1 and in the second place, as is easy to see, through the three cone vertices lying in τ_i; for if a plane through s_1 passes also through a second vertex, then the complex conic breaks up into the two pencils at T_i and at that second cone vertex.

All rays through a point T_i^* of k^{*2} cutting s_1 are according to the preceding rays of the complex; from this ensucs reversely that the complex cones of all points of s_1 in τ_i have the same base curve, namely k^{*2}. If now the degenerated complex cone of a point of k^2 is to pass through s_1, then that point must evidently lie also on k^{*2} and of such points there exists apart from the three cone vertices lying in τ_i, only one; in the pencil $[T_i]$ there is thus only one ray for which the (degenerated) complex cone of its focus passes through an indicated ray s_1, i. e. the second components of the complex cones of the foci of the rays of the pencil $[T_i]$ form a pencil of planes, or *the rays of* τ_i *belonging to the congruence form a pencil.*

The axis a of the pencil of planes must of necessity cut the curve k^2; for, if this were not so, then an arbitrary plane through a would cut k^2 in two points, and then the complex curve in that plane would break up into three pencils (among which one at T_i is always included) instead of into two. This objection does not exist when a cuts the curve k^2 in a point A; for then each plane through a cuts k^2 besides in A in only one point T_i^* more, and A itself is a point T_i^* for the plane through a which touches k^2. *The axis a is simply that line which has the property that the complex cones of its points have as common base curve the conic k^2 itself;* for, for each plane through a the point T_i lying on k^2 must lie at the same time on k^{*2}, so k^2 and k^{*2} coincide.

For each ray of the pencil $[A]$ lying in τ_i point A is evidently one focus and τ_i the corresponding focal plane, for each ray is cut in A by an adjacent one of the pencil; the other focus is the second point of intersection T_i^* with k^2 and here the second focal plane passes through T_i. *The focal surface must therefore touch τ_i along the conic k^2; the point A itself is however a singular point, for here any plane through a is a tangential plane.*

For the tangent in A to k^2 the two foci coincide evidently with

A; the focal planes, however, do not coincide, for one is τ_i and the other connects the tangent to T_i.

22. Order and class of the focal surface can be immediately determined by means of two dualistically opposite equations of SCHUBERT, viz.

$$\varepsilon\sigma p^2 = \sigma p g_e + \sigma p h_e - \sigma \widehat{pe},$$

and

$$\varepsilon\sigma e^2 = \sigma e g_p + \sigma e h_p - \sigma \widehat{pe}\ ^{1)}.$$

We conjugate to each ray g of the congruence all other rays as rays h, we then obtain a set of ∞^4 pairs of rays and we can apply to these the two equations just quoted. The symbol σ indicates that the two rays of a pair must intersect each other, ε that they lie at infinitesimal distance and p^2 that the point of intersection p must lie in two planes at a time, thus on an indicated line; so $\varepsilon\sigma p^2$ is evidently the order of the focal surface. The condition $\sigma p g_e$ indicates the number of pairs which cut each other, whilst the point of intersection p lies in a given plane and the ray g likewise in a given plane, now there lie in a given plane 14 rays of our congruence, thus 14 rays g; each of these intersects the plane of the condition p in one point and through each of these pass 5 more rays of the congruence, $\sigma p g_e$ is therefore $14 \times 5 = 70$, and $\sigma p h_e$ means the same and is thus likewise $= 70$.

With $\sigma\widehat{pe}$ we must pay more attention to the point of intersection of the two rays and to the connecting plane than to the rays themselves; $\sigma\widehat{pe}$ indicates namely the number of pairs of rays which cut each other and where the point of intersection lies on a given line and at the same time the connecting plane passes through that line; this number is evidently the third of the three characteristics of the congruence, thus the rank, however multiplied by 2 because each pair of rays of the congruence represents 2 pairs gh; so $\sigma\widehat{pe}$ is $= 80$, *so that the order of the focal surface is equal to* $70 + 70 - 80 = 60$.

$\varepsilon\sigma e^2$ indicates the number of pairs of rays at infinitesimal distance whose connecting plane passes through 2 given points, so through a given line, i. e. the class of the focal surface. Now $\sigma e g_p$ indicates the number of pairs of rays whose connecting plane passes through a given point, whilst also the ray g passes through a given point. So there are 6 rays g and in the plane through one of those rays and the point of the condition e lie besides g still 13 others; $\sigma e g_p$

1) SCHUBERT l. c. page 62.

and $\sigma e h_p$ are thus each $= 6 \times 13 = 78$, and $\sigma p e$ was 80, so the class of the focal surface $= 78 + 78 - 80 = 76$.

I may be permitted to point out in passing a slight inaccuracy committed by SCHUBERT on page 64 of his "Kalkül" where he gives formulae for order and class of the focal surface of a congruence taking the number $\widehat{\sigma p e}$, called by him c, only once into account; in PASCAL-SCHEPP's well known "Repertorium" vol. II, page 407 we find indicated the exact formulae, with the rank number r counted twice.

In a congruence of rays appear in general ∞^1 rays whose two foci coincide; these too are easy to trace in our congruence. For, according to § 20 in order to find the foci of an arbitrary ray s_0 we must apply in the focus P_0 the complex cone and the tangential plane to Ω^6 and intersect these by each other; the foci of the lines of intersection are the foci of s_0 and the tangential planes through s_0 to the complex cones of the foci the focal planes. So as soon as the complex cone of P_0 touches the tangential plane Ω^6 along a line t, the two foci of s_0 will coincide in the focus of t and the focal planes will coincide in the tangential plane through s_0 to the complex cone of the only focus.

The points P_0 whose complex cones touch Ω^6 are to be found again with the aid of SCHUBERT's "Kalkul". We conjugate the two rays s, along which the complex cone of a point P_0 of Ω^6 cuts the tangential plane in that point, to each other; so we obtain in that manner a set of ∞^2 pairs of rays and we apply to it the formula:

$$\varepsilon\sigma p = \sigma g_e + \sigma h_e + \sigma p^2 - \sigma p e \text{ }^{1)};$$

The left member namely indicates the number of coincidences whose points of intersection lie in a given plane, that is thus evidently the order of the curve which is the locus of the points P_0 to be found. σg_e indicates the number of pairs of rays whose component g lies in a given plane; this plane cuts out of Ω^6 a plane curve k^6 which possesses no other singularities than three nodes and which is so of class $6 . 5 - 2 . 3 = 24$. and all the complex rays in this plane envelop a conic; so there lie 48 complex rays g in this plane touching Ω^6. If we apply in one of the points of contact the tangential plane to Ω^6, then there lies in it one ray h; so σg_e is 48 and likewise of course σh_e.

With σp^2 we must trace the number of pairs of rays whose points of intersection lie in two given planes at the same time, thus on a given line; this line intersects Ω^6 in six points and in the tangential

1) SCHUBERT l. c. page 62.

plane lie two rays of the complex cone and thus also two pairs gh, because each of the two rays can be either g or h; so $\sigma p^2 = 12$. For σpe finally the point of contact must lie in a given plane, the tangential plane must pass through a given point; so we can either apply the tangential planes in the points of a plane section of Ω^6 and determine the class of the developable enveloped by it, or we can construct the circumscribed cone and calculate the order of the curve of contact. The latter is the simplest; for the curve of contact is the intersection of Ω^6 with the first polar surface of the vertex of the cone and therefore of order $6 . 5 - 2 . 3 = 24$, because the first polar surface contains the nodal curve k^3 and the latter counted twice separates itself from it. But the two complex rays through the point of contact and in the tangential plane count again for two pairs and so $\sigma pe = 48$, from which ensues $\varepsilon \sigma p = 48 + 48 + 12 - 48 = 60$: *so there lies on Ω^6 a certain curve k^{60} of order 60 having the property that the rays s conjugated to its points have coinciding foci and focal planes.*

We can ask how the curve k^{60} will bear itself with respect to the four cone vertices T_i, where the complex cone becomes indefinite. We now know however out of § 21 that in the plane τ_i only one ray with coinciding foci lies, viz. the tangent in A to k^2, *so k^{60} will pass once through the four cone vertices.* That for that tangent in A to k^2 the two focal planes do n o t coincide, is an accidental circumstance, which is further of no more inportance; this result was based namely on the supposition that through an edge of the cone passes only one tangential plane of that cone; however, for the point A the complex cone breaks up into a pair of planes whose line of intersection is just the tangent in A to k^2, the tangential plane through that line to the cone is thus in first instance indefinite.

The rays of the congruence with coinciding foci determine a scroll of which we will finally determine the order. To that end the scroll must be intersected by an arbitrary line and we now know that all rays of the congruence meeting a line l form a regulus Ω^{20} and that the foci of those rays are situated on a curve k^{12} lying on Ω^6 and passing singly through the 4 cone vertices. It is clear that to a point of intersection of k^{12} and k^{60} a ray corresponds with coinciding foci and cutting l with the exception of the cone vertices; for, to T_i is conjugated as regards k^{60} the tangent in A to k^2, on the other hand as regards k^{12} the connecting line of the point of intersection of l and τ_i with A, as we now know. Now k^{12} is, as we know, the complete intersection of Ω^{20} with a regulus; *so* the complete number of points of intersection of k^{12} and k^{60} amounts to 120. If we set apart from these the four cone vertices, we then find as result that *the rays of the congruence with coinciding foci form a regulus of order* 116. The curve k^{60} intersects τ_i besides in the three cone vertices lying in this plane in 57 points more, lying of course on the section k^6 of Ω^6 and τ_i; to each of these points a ray through T_i is conjugated with coinciding foci; *the* 4 *cone vertices are thus for the surface* Ω^{116} 57-*fold points*.

Physics. — "*Some remarkable relations, either accurate or approximative, for different substances.*" By Prof. J. D. VAN DER WAALS.

(Communicated in the meeting of November 30, 1912).

In a previous communication (June 1910 These Proc. XIX p. 113) I pointed out the perfectly accurate or approximative equality of the ratio of the limiting liquid density to the critical density, and the ratio of the critical density to that which would be present for T_c, p_c and v_c, if $\frac{pv}{RT}$ should always be equal to 1. With the symbols used there

$$2(1+\gamma)=\varphi s$$

I have added the factor φ, which must then be equal to 1 or must differ little from 1.

The rule given there has attracted some attention. For first of all Dr. JEAN TIMMERMANS has informed me that he has found this rule entirely confirmed for six substances, for which the observations made were perfectly trustworthy For a seventh substance there was a great difference, but he thought that for this real association might perhaps occur, as is the case for acetic acid [1]). Besides this rule has also been adopted by KAMERLINGH ONNES and KEESOM in their recent work for the Encyklopädie: Die Zustandsgleichung. The rule is indeed apt to rouse some astonishment, because it pronounces the equality between two quantities, which, at least at the first glance, have nothing in common.

It is to be expected that this approximative equality will have to be explained by the way in which the quantity b varies with v; but it is seen at the same time that perfect equality cannot be put

[1]) The numerical values have been communicated in the "Scientific Proceedings of the Royal Dublin Society", October 1912.

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