Mathematics. — *"On the correspondence of the pairs of points separated harmonically by a twisted quartic curve."* By Prof. JAN DE VRIES.

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§ 1. We indicate by P and Q two points, lying on a chord of a *twisted quartic curve of the first kind*, separated harmonically by this curve ϱ^4. As any point P lies generally on two chords, in the correspondence (P, Q) to any point P two points Q are conjugated.

If P moves along a line l, Q describes a curve λ^6 of *order six*. For any plane Δ through l cuts ϱ^4 in four points S_k and contains therefore six points Q_{kl}, where Q_{kl} lies on a chord S_kS_l and is harmonically conjugated to the points P_{kl} common to that chord and l. If l is an arbitrary line, Q never lies on l when Δ rotates about l.

The line $Q_{13}Q_{23}$ is separated harmonically from l by $P_{12}S_3$ and S_1S_2. By assuming a position for Δ in which S_1 and S_2 coincide with Q_{12} we find for $Q_{13}Q_{23}$ a tangent of λ^6 separated harmonically from l by $P_{12}S_1$ and $P_{12}S_3$, whilst an other tangent of λ^6 takes the place of $Q_{14}Q_{24}$. So each of the eight tangential planes of ϱ^4 contains two tangents of λ^6; so the *rank* of this curve is sixteen.

Moreover we find that λ^6 has *eight* points in common with ϱ^4.

§ 2. The line p connecting the two points Q, Q' conjugated to P describes a regulus Δ^2 if P moves along l. For p is the polar line of P with respect to ϱ^4, i. e. the intersection of the polar planes of P with respect to any two quadratic surfaces through ϱ^4, and these polar planes describe two projective pencils.

Let us now consider one of the two lines p cutting l. The corresponding point P bears two chords S_1S_2 and S_3S_4 lying in the plane $\Delta = lp$. The points Q_{12} and Q_{34} lie on p, the points $Q_{13}, Q_{23}, Q_{14}, Q_{24}$ lie on a line m through P harmonically separated from l by the chords S_1S_2 and S_3S_4. As λ^6 lies on the regulus Δ^2, m is a line of Δ^2. Any tangential plane of Δ^2 contains therefore a quadrisecant of λ^6 and both the reguli of Δ^2 are arranged by λ^6 in a correspondence $(2, 4)$. Evidently the quadrisecants q are the polar lines of l with respect to the quadratic surfaces through ϱ^4.

§ 3. If we assume for l a *chord* of ϱ^4, the locus of Q breaks up into four parts, i. e. the chord l itself, the tangents r and r' in the points R, R' common to l and ϱ^4, and a *twisted cubic* λ^3. The polar line p now connects a point Q of l with the point Q' of the second chord k passing through P. This line describes a regulus having with A^2 the line l in common. So the locus of $Q' = kl$ is a curve λ^3 through R and R', as l is to have two points in common with it and R and R' correspond amongst other points with themselves; the curves λ^3 and ϱ^4 have four more points in common.

§ 4. If l is a unisecant of ϱ^4 in R, the locus (Q) degenerates into the tangent r and a λ^5. Any plane through l contains besides R three points Q; of these two must be combined with R, if the plane contains the tangent r. The quadrisecants q of l become here trisecants; for r rests on each of the polar lines q of l (§ 2). The plane qr touches ϱ^4 in R and contains therefore two points Q united in R. In relation with the results obtained we conclude from this that by the correspondence (P,Q) to a unisecant of ϱ^4 a twisted curve of *order five* is conjugated having a node in the point common to the unisecant and ϱ^4, the nodal tangents lying in the plane lr.
So the curve is of *rank ten*. Through l pass six common tangential planes of ϱ^4 and λ^5.

§ 5. The vertices T_k of the four quadratic cones containing ϱ^4 are *singular points* of the correspondence (P,Q). For T_1 bears ∞^1 chords and the corresponding points Q lie on the conic τ^2_1 common to the polar plane $\tau_1 = T_2T_3T_4$ of T_1 and the quadratic cone with T_1 as vertex.
To the line T_1T_2 as locus of points P correspond in the first place the two *conics* τ_1^2 and τ_2^2 and moreover the *line* T_3T_4 *counted twice*. For the points S_k in any plane through S_1S_2 form a complete quadrangle of which T_1 and T_2 are diagonal points; in the third diagonal point Q_{13} and Q_{24} coincide, whilst of the remaining four points Q two lie in τ_1 and two in τ_2. So to any point of T_1T_2 correspond two points of T_3T_4 and inversely.
If l contains the point T_1 only, the six points Q lying in a plane λ through l consist of two points in τ_1 and on τ_1^2 and of four points lying on the line common to λ and the polar plane of l with respect to the cone projecting ϱ^4 out of T_1. Then the curve (Q) breaks up into the *conic* τ_1^2 and a *plane curve* λ^4. In the two tangential planes of the cone passing through l the two points Q lying on τ_1^2 coincide with two of the remaining four in a point of intersection of τ_1^2 and λ^4 where the latter is touched by the edge of contact.

§ 6. Let us now consider the surface of the points Q corresponding to the points P of a plane Π. If S_k are the points common to Π and ϱ^4, the six lines S_kS_l form the intersection of Π with the

60*

locus under discussion. So it is of *order six.* As it contains at the same time the lines touching ϱ^4 in S_k, these points are nodal points. To the two points of τ_1^2 lying in Π correspond two points Q coinciding with T_k, whilst to the point of Π lying on $T_k T_l$ two points on $T_m T_n$ correspond. From this ensues that the four points T have to be also nodes of Π^6.

So to a plane corresponds a *surface of order six with eight nodes and ten lines.*

§ 7. Let us now consider the correspondence between two points P, Q separated harmonically by a *twisted quartic curve of the second kind* σ^4. As P bears three chords of σ^4, it is conjugated to three points Q. To the points P of a line l correspond the points Q of a twisted curve λ^6; for each plane through l contains six points Q.

The three points Q corresponding to P lie in the polar plane of P with respect to the quadratic surface H^2 through σ^4. The plane Π rotates about the polar line l' of l, if P moves along l. So l' is a trisecant of λ^6.

The scroll of the chords of σ^4 cutting l is of order nine; so nine of these chords also intersect l'. To these nine belong the two trisecants of σ^4 cutting l, each of which represents three chords; they have to meet l', as they lie on the hyperboloid H^2 and are at the same time trisecants of λ^6. The remaining three chords cutting l and l' determine the three points Q on l'.

§ 8. Each of the six tangential planes of σ^4 passing through l contains a point and two tangents of λ^6; so this curve is of *rank twelve* and rests in *six* points on σ^4. By S_k we represent the points of σ^4 lying in a plane drawn through l; the chord $b = S_1 S_2$ is paired to the chord $b' = S_3 S_4$ and now we consider the correspondence between the points P and P' in which b and b' intersect l. As P bears three chords we find a (3,3). If b and b' intersect l in the same point P, only the third chord through P furnishes a point P' not coinciding with P; from this ensues that the coincidencies of the (3,3) coincide by two in a double coincidency. So through l three planes pass for which b and b' intersect in l; the line h separating l harmonically from b and b' then contains four out of the six points Q, the remaining two lying on b and b'.

So the curve λ^6 admits three quadrisecants.

§ 9. Let l be a chord of σ^4 and S_1 and S_2 the points it has in common with σ^4. Through any point P of l pass two more chords

b,b' of σ^4. So the locus of the points Q lies on a cubic scroll Λ^3 with double line l.

In the plane bl two points Q coincide in S_1, two other ones in S_2, whilst Q_{12} lies in l and Q_{34} in b. If P moves along l, $q = Q_{12}Q_{34}$ describes a cubic scroll Φ^3 with double line l; for through Q_{12} pass two lines q,q' to the points Q_{34} of the chords b,b' concurring in the point P corresponding to Q_{12}.

The scrolls Λ^3, Φ^3 have the trisecants t_1, t_2 of σ^4 passing through S_1 and S_2 in common. For if P coincides with S_1, t_1 becomes a chord b and, as Q_{12} coincides then with S_1, at the same time a line q.

As l is nodal line for both scrolls, these surfaces have still a twisted cubic λ^3 containing the points Q_{34} in common. In the planes touching σ^4 in S_1 and S_2 the point Q_{34} coincides with the point of contact; so S_1S_2 is a chord of λ^3. This curve intersects σ^4 in the two points the tangents of which intersect S_1S_2; it has for chords the single director lines of the scrolls Λ^3, Φ^3.

So by the transformation (P,Q) the *chord* l passes into the system consisting of l itself, the tangents s_1,s_2 and a *twisted cubic.*

Evidently a *trisecant* t is transformed into that line to be counted thrice and the tangents in the three points it has in common with σ^4.

If l touches σ^4 in S_{12}, the scroll Φ^3 becomes a cone with nodal edge l. In the osculating plane of σ^4 in S_{12} q lies along l; so this plane is common tangential plane of Λ^3 and Φ^3, having still in common the trisecant through S_{12}. The residual intersection λ^3 touches in S_{12} the tangent of σ^4.

§ 10. If l is unisecant of σ^4 in S the curve λ^6 breaks up into the tangent s of σ^4 in S and a curve λ^5. The polar line l' of l becomes chord of λ^5, s being one of the three chords cutting l and l'. The plane $l'S$ touches H^2 in S and is therefore polar plane of $P \equiv S$; it contains the tangent s and the trisecant of σ^4 on which S lies. Of the three variable points Q common to λ^5 and a plane through l', two coincide with S and only one lies outside S.

Any plane through l contains besides S three points Q and has therefore in S two points with λ^5 in common. Also the plane $l'S$ not passing through l has in S two points in common with λ^5; so S is a node of λ^5. The plane ls contains beside S only one point Q; so it passes through the nodal tangents of the node. So to a unisecant corresponds a *twisted quintic with a node.*

The curve is of *rank eight,* through l passing four common tangential planes of σ^4 and λ^5.