Mathematics. — *"On a line complex determined by two twisted cubics."* By Prof. JAN DE VRIES.

(Communicated in the meeting of November 30, 1912).

§ 1. We will indicate the chords of the given twisted cubics ϱ^3, σ^3 by r, s.

Any plane π contains three chords r and three chords s, therefore nine points $P = rs$. In the *focal system* (P,π) each point has in general *one* focal plane, each plane *nine* foci ($\alpha = 1$, $\beta = 9$).

If π rotates about the line l, the points determined on l by two complanar chords r, s are conjugated to each other in a correspondence (3, 3). As each point of coincidence furnishes a point $P = rs$, l contains *six* points P, the focal planes of which pass through l; so the third characteristic number of (P, π) is *six* ($\gamma = 6$).

Let b represent one of the ten common chords of ϱ^3 and σ^3. Any point B of b admits ∞^1 focal planes, i.e. all the planes β through b. Any plane β admits four foci not lying on b, whilst at the same time any point B of b is focus. So the lines b are loci of *singular foci* and *singular focal planes*.

If P is assumed on ϱ^3, s is a definite chord of σ^3, whilst r may be any line connecting P with an other point of ϱ^3; then any plane through s can figure as focal plane π in which P counts for two of the nine foci. So the curves σ^3 and ϱ^3 are *singular curves* for the focal system (P, π).

§ 2. The polar planes of P with respect to the ∞^2 quadratic surfaces through ϱ^3 have a point R on r in common; P and R can be said to be separated harmonically by ϱ^3. If P describes any line l, the polar planes of P with respect to three quadratic surfaces of the net not belonging to the same pencil rotate about three definite lines and describe therefore three projective pencils. So the locus of R is a twisted cubic λ^3, intersecting ϱ^3 in four points; for on the four tangents r_0 of ϱ^3, resting on l, the point conjugated to P is every time the point of contact R_0 [1]).

We indicate by S the point on s harmonically separated from P by σ^3 and consider the relationship between R and S.

To any plane Σ as locus of S corresponds a cubic surface Π^3 of

[1]) This generally known involutory cubic transformation has been investigated thoroughly by Dr. P. H. SCHOUTE (Nieuw Archief voor Wiskunde, 2nd series, vol. IV. 1900, p. 90).

points P; as Π^3 intersects the twisted cubic λ^3 described by P when R moves along l in nine points, the correspondence (R,S) is of order *nine.*

A point of coincidence of R and S can only present itself when r and s coincide, i. e. on a common chord b. On any of the ten b the pairs (P,R) and (P,S) generate two involutions, of which H_1, H_2 may represent the common pair. By assuming R in H_k, we find H_l for P and H_k for S; so H_1 and H_2 are points of coincidence of (R,S). So this correspondence admits *twenty coincidencies* lying in pairs on the *ten* common chords b.

As a point R_0 of ϱ^3 corresponds to each point P of the tangent r_0 of R_0. R_0 corresponds to each point S of the twisted cubic $\sigma_0{}^3$ into which r_0 passes by the transformation (P,S); evidently $\sigma_0{}^3$ has four points in common with σ^3.

Consequently the curves ϱ^3 and σ^3 are *singular curves* of the correspondence (R,S).

If R describes the tangent r_0 of ϱ^3, P remains in the point of contact of r_0; so the point S^* conjugated to P is singular and corresponds to all the points of r_0. Evidently the locus of S^* is the rational twisted σ^9 into which ϱ^3 passes by the transformation (P,S).

So the correspondence (R,S) admits *two singular twisted curves of order nine*, σ^9 and ϱ^9.

As the developable with ϱ^3 as cuspidal curve cuts σ^3 in 12 points σ^3 and σ^9 have twelve points in common; likewise ϱ^9 rests in 12 points on ϱ^3.

§ 3. We now consider the lines $p \equiv RS$. If P describes the line l, p generates a scroll of order six; for we found above that the plane $\pi \equiv P_l$, passes through l in six positions (§ 1).

The line p generates a *complex*. We determine the number of lines p belonging to a pencil with vertex L and plane λ.

If R describes a ray l of pencil (L, λ), S generates a curve intersecting λ in nine points (§ 2); we conjugate to l the nine lines l' connecting these points with L. In this manner we get in the pencil a correspondence (9,9) each coincidence of which furnishes a line p connecting two points R and S corresponding to each other. So:

The complex (p) *is of order eighteen.*

Evidently the 20 points H are *principal points* of the complex; each complex cone passes through these 20 points.

§ 4. Any point R_0 of ϱ^3 is singular, for it bears the lines p connecting it with the points S of the corresponding curve $\sigma_0{}^3$ (§ 2) and so its complex cone degenerates. Consequently the curves ϱ^3 and σ^3 lie on the *singular surface* of the complex.

The edges of the ∞^1 cones projecting the curves $\sigma_0{}^3$ from their corresponding point R_0 as vertices form a *congruence* of which we will determine order and class.

The locus of the curves $\sigma_0{}^3$ is the surface Σ^{12} into which the developable with ϱ^3 as cuspidal curve is transformed by (P,S).

The cubic cones with an arbitrary point M as vertex and ϱ^3 and σ^3 as director curves, intersect in 9 edges, each of which connects a point S of $\sigma_0{}^3$ with a point R' of ϱ^3; if R' coincides with the point R_0 to which $\sigma_0{}^3$ corresponds we have to deal with a ray of the congruence passing through M. We will conjugate these 9 points R' to R_0. The line MR' cuts the surface Σ^{12} mentioned above in 12 points S lying in general on different curves $\sigma_0{}^3$; so to R' correspond 12 points R. The correspondence (R_0, R') has therefore 21 coincidencies, i.e. the *order* of the congruence is 21.

Any plane μ contains 3 points R_0 and each of the corresponding curves $\sigma_0{}^3$ has 3 points S with μ in common; so the class is 9.

So the lines S_0R form a congruence (21,9) and an other congruence of the same type is formed by the lines S_0R. The two congruences admit successively ϱ^3 and σ^3 as *singular curve.*

§ 5. Any point S^* of the rational σ^9 (§ 2) is the vertex of a pencil of complex rays p the plane of which contains the corresponding tangent r_0. So the curves ϱ^9 and σ^9 lie also on the *singular surface.*

The ∞^1 pencils with vertices S^* form a *congruence* which we will study more closely.

In any plane μ lie 9 points S^*; the tangents r_0 corresponding to these points determine 9 rays p lying in μ; so the congruence is of *class nine.*

To any point S^* we make to correspond the 9 points S' of σ^9 which can be projected out of the arbitrary point M in a point of the corresponding tangent r_0. The line MS' cuts 4 tangents r_0, so S' is conjugated to 4 points S^*. As any coincidency $S' \equiv S^*$ is due to a ray of the pencil with vertex S^*, M bears 13 lines RS^*, i.e. the congruence is of *order thirteen.*

So the complex contains *two congruences* (13,9) each of which is built up of ∞^1 pencils. They admit successively σ^9 and ϱ^9 as *singular curve.*

§ 6. To the complex (p) belongs the system of generatrices of the developable determined by ϱ^3 and σ^3. Any tangent r_0 cuts four tangents s_0 and reversely; so the points of contact R_0 and S_0 of the tangents conjugated to each other in this way are in correspondence (4,4). By projecting the pairs of this correspondence out of a line a, the pencil of planes (a) is arranged in a correspondence (12,12). As each coincidency furnishes a line $p = R_0 S_0$ resting on a the *developable* under discussion is of *order* 24; it has ϱ^3 and σ^3 for *fourfold curves.*

Any chord r^* of ϱ^3 meeting σ^3 belongs to the complex, for in the common point of r^* and σ^3 the points P and S coincide. The chords r of ϱ^3 cutting a line l generate a scroll of order four with ϱ^3 as nodal curve; so the locus of the chords r is a scroll of order 12. On the latter surface ϱ^3 is a *sixfold curve,* for through any of its points pass the common edges of the two cones projecting ϱ^3 and σ^3.

So the complex (p) contains *two scrolls of order twelve*, the generatrices of which are chords of one of the curves ϱ^3, σ^3 and secants of the other.

Let p^* be a chord of ϱ^3 not meeting σ^3; then the tangent r_0 in one of the points R_0 common to ϱ^3 and that chord must contain the point P. If P moves along that tangent, S describes a curve $\sigma_0{}^3$; the cone projecting the latter curve out of R_0 has 6 edges in common with that of which ϱ^3 is director curve. So any point of ϱ^3 bears 6 rays p^*. As an arbitrary chord r can be cut by chords p^* in its points common to ϱ^3 only, so all in all by 12, the locus of the chords under discussion is of order 12.

So the complex contains *two scrolls of order twelve,* built up out of chords of one of the curves ϱ^3, σ^3.

Physics. — "*Determinations of the refractive indices of gases under high pressures.*" Second communication. "*On the dispersion of air and of carbon dioxide.*" By Prof. L. H. SIERTSEMA. (Communicated by Prof. H. KAMERLINGH ONNES).

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4. *The dispersion of air.*

This has already been repeatedly determined both for the visible spectrum and for the ultra-red and ultra-violet rays. The results, however, diverge considerably, and, moreover, the dispersion has never been measured under high pressure.

Through the kindness of Prof. KAMERLINGH ONNES compressed air was placed at my disposal with which dispersion determinations were made in exactly the same way as those for hydrogen described in a former paper.