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Mathematics. — *"On Steinerian points in connexion with systems of nine ϱ-fold points of plane curves of order* 3ϱ." By Dr. W. VAN DER WOUDE. (Communicated by Prof. P. H. SCHOUTE).

(Communicated in the meeting of December 28 1912).

§ 1. In a former communication [1]) has been indicated what is the locus of the point forming with eight given points a system of nine nodes of a non degenerated plane sextic curve; here will be treated a more general problem including the preceding one as a particular case.

To that end we remark that by nine arbitrarily chosen points $D_1, D_2, \ldots, D_9$ a curve of order 3ϱ passing ϱ times through these points is determined: in general however this $C_{3\varrho}$ is a cubic curve counted ϱ times. So the problem we propose now is: "Eight points $D_1, D_2, \ldots, D_8$ being given, to determine the locus of the point D_9 under the condition that the nine points D_i can be ϱ-fold points of a curve $C_{3\varrho}$ not degenerating in the manner mentioned.

§ 2. As we shall find by and by this problem is very closely related to the following one: "Let $B_1, B_2, \ldots, B_9$ be the base points of a pencil (β') of cubic curves, and u_3 any curve of this pencil. On u_3 lie (ϱ^2-1) points S each of which forms with B_9 a Steinerian pair [2]) of order ϱ. To determine the locus of these points S, if u_3 describes the pencil (β')".

§ 3. We start by treating the first of the two problems.

So the eight points $D_1, D_2 \ldots, D_8$ are given and we have to determine the locus of the ninth point D_9 satisfying the condition stated. In the quoted memoir the case $\varrho = 2$ has been treated; for convenience sake we repeat here the principal results.

Then we occupy ourselves with the case $\varrho = 3$ before passing to

[1]) W. v. D. WOUDE. *"Double points of a c_6 of genus 0 or 1* (Proceedings of Amsterdam, vol. XIII, p. 629)

Compare also Dr. V. SNIJDER, *"The involutorial birational transformation of the plane of order 17"* (American Journal of Mathematics, vol XXXIII, p. 328).

[2]) Two points P and Q of u_3 form a Steinerian pair of order n, if it be possible to inscribe in u_3 one and therefore an infinity of closed polygons with $2n$ vertices, the sides of which pass alternately through P and Q. *Literature*: STEINER (Journal of Crelle, vol. XXXII, p. 182); KÜPPER (Math. Ann., vol XXIV, p. 1); SCHRÖTER (Theorie der ebenen Kurven dritter Ordnung. § 31). For the treatment by means of elliptic functions see CLEBSCH: Vorlesungen über Geometrie.

the general case of an arbitrary ρ. But we wish to give just now one theorem where ρ has already any arbitrary value:

"If $D_1, D_2, \ldots, D_9$ are ρ-fold points of a non degenerated curve $C_{3\rho}$ of order 3ρ, these points are at the same time the base points of a pencil of curves of order 3ρ, each of which passes ρ times through $D_1, D_2, \ldots, D_9$".

For the proof it will suffice to remark, that the nine points lie on a cubic curve u_3; so the pencil mentioned is represented by

$$c_{3\rho} + \lambda u_3{}^\rho = 0.$$

§ 4. By $D_1, D_2, \ldots, D_8$ we will henceforth denote arbitrarily chosen points; we represent by (β') the pencil of curves c_3 passing through them, by B_9 the ninth base point of this pencil. So the principal results, obtained for $\rho = 2$, are the following:

I. "The locus of the point forming with $D_1, D_2, \ldots, D_8$ a set of nine nodes of a non degenerated [1]) c_6 is a curve j_9 of order nine passing three times through $D_1, D_2, \ldots, D_8$".

II. "This curve j_9 is also the locus of the points corresponding with B_9 in tangential point on the curves of pencil (β')".

III. "Let u_3 be any cubic of (β') and c_6 any sextic passing three times through $D_1, D_2, \ldots, D_8$. Then the line joining the last two points common to u_3 and c_6 will meet u_3 for the third time in the tangential point T of B_9 on u_3".

Before continuing our considerations we wish to correct the preceding communication. We have indicated there that B_9 does not lie on j_9; indeed this is so, but one of the proofs — the geometrical one — may give rise to difficulties. Therefore we once more prove here: B_9 does not lie on j_9. To that end we consider j_9 as the locus of the points on any curve of (β') corresponding with B_9 in tangential point. Now B_9 will be a point of j_9, if and only if one of these points coincides with B_9, which only can happen if B_9 is a node for one of the curves of (β'). Of these nodes — the 12 so called "critical points" of the pencil — none however coincides with one of the base points, if — as it is the case here — eight of the base points have been chosen arbitrarily. So B_9 does not lie on j_9.

§ 5. We now pass to the case $\rho = 3$.

We still denote by $D_1, D_2, \ldots, D_8$ arbitrarily chosen points,

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[1]) Here by *non degenerated* is meant a curve not breaking up into a c_3 to be counted twice. In this manner is to be interpreted henceforth the expression *non degenerated* c_6 used now and then.

whilst u_3, (β'), B_9 and T keep the signification assigned to them in art. 4. Now the question is to determine the locus of the point forming with $D_1, D_2, \ldots, D_8$ a set of threefold points of a non degenerated c_9.

In order to determine a curve c_9 passing three times through $D_1, D_2, \ldots, D_8$ we can imply to it the condition of containing six arbitrary points. Of these six points however no more than two [1]) may lie on u_3; then the last point common to u_3 and c_9 is determined inequivocally. We will show immediately how the latter point can be found; provisionally we start from any c_9 with the eight given threefold points, cutting c_3 in an arbitrarily chosen fixed point X. This c_9 cuts u_3 in two points more; the line connecting these two points has still a third point E with u_3 in common; according to the *Residual Theorem* of SYLVESTER the latter point is a fixed point, i. e. independent from the chosen curve c_9 passing through X. Now we first determine the point E; to that end we choose a c_9 breaking up into a curve v_3 of pencil (β') and a curve c_6 passing twice through $D_1, D_2, \ldots, D_8$ and passing moreover through X. We have seen that this c_6 cuts u_3 in one point Y more, being collinear with X and T (§ 4, III); moreover u_3 and v_3 have B_9 in common. So the point E is the third point of intersection of the line YB_9 and u_3.

If now we fix on u_3 two points X, X' and consider a curve c_9 with threefold points in $D_1, D_2, \ldots, D_8$ and cutting u_3 in X and X', then the last point of intersection of this c_9 and u_3 can be found as follows: we first determine in the manner indicated the point E; then the third point of intersection of the line EX' and u_3 is the point looked out for.

Remark. We have stated, that any c_9 with $D_1, D_2, \ldots, D_8$ as threefold points meets u_3 in three points more; evidently this does not hold if this c_9 breaks up into two curves one of which coincides with u_3. In this case the residual curve of order six must be determined in such a manner that it admits on u_3 nine nodes, eight of which lie in $D_1, D_2, \ldots, D_8$. So we fall back on the case $\varrho = 2$, but we can discard this by requiring that D_9 has been determined in such a way that the c_9 under discussion does not break up, neither into a c_3 to be counted thrice nor in two curves c_3 and c_6, the latter of which admits a node in any of its points of intersection with the former.

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1) See e.g. SALMON FIEDLER: Höhere ebene Kurven, p. 23.

§ 6. It is now immediately clear that there are four points in which any c_9 with the threefold points $D_1, D_2, \ldots, D_8$ and passing through X can touch u_3, i. e. in any of the four points admitting E as tangential point; likewise that any c_9 with the threefold points $D_1, D_2, \ldots, D_8$ touching u_3 will cut this curve in X.

We now will try to determine X in such a way that it coincides with one of the four points of which E is the tangential point; in that case any c_9 with threefold points in $D_1, D_2, \ldots, D_8$ and touching u_3 in X, will have in X a third point in common with u_3.

Let us suppose that the point X has been determined so as to satisfy the condition mentioned; then we can describe in u_3 closed hexagons the successive sides of which pass alternately through B_9 and X. If we choose B_9 as first vertex and P is the third point of intersection of B_9X and u_3, then there is a closed hexagon with the successive sides B_9B_9T, TXY, YB_9E, EXX, XB_9P, PXB_9 (fig. 1).

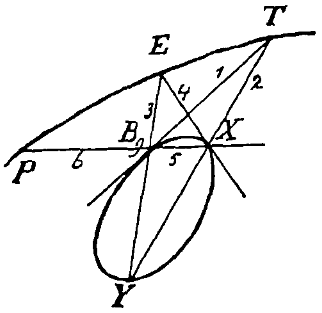

Fig. 1.

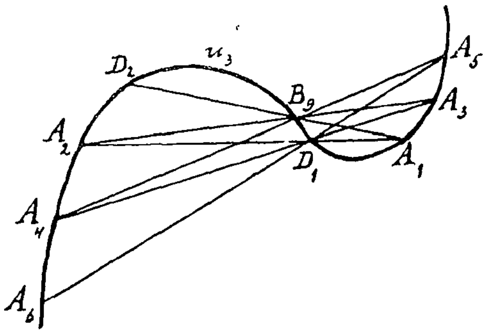

Fig. 2.

So the points X to be determined are the eight points each of which forms with B_9 on u_3 a Steinerian pair of order three. [1])

§ 7. We now choose one point out of these 8 and call it X_1. If we then require that C_9 has threefold points in $D_1, D_2 \ldots, D_8$ and touches u_3 in X_1, we can assume arbitrarily four more points K, L, M, N of this curve, which as we have seen above has in X_1 still a third point in common with u_3. Provisionally we suppose L, M, N to be fixed points but K to describe a right line k through

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1) That B_9 and the point X satisfying the imposed condition form on u_3 a Steinerian pair of order three can also easily be shown by representing the points of u_3 by means of an elliptic parameter. If β is the parameter value for B_9 and x that for the point X taken provisionally at random, we find for the values corresponding to T, Y and E respectively -2β, $2\beta - x$ and $x - 3\beta$. So the condition that E be the tangential point of X is $3\beta \equiv 3x$. Chiefly for the cases $\rho = 4, 5, \ldots$ presenting themselves later on the use of this parameter proves to be very convenient. Compare CLEBSCH: Vorlesungen über Geometrie (p. 615).

X_1 different from the tangent to u_3 in X_1; then the C_9 describes a pencil, one curve of which passes through any point of k. The coincidence of K with X_1 then furnishes a C_9 having in X_1 three points in common with u_3 and two points with k; so this C_9 has a node in X_1 and one of its branches touches u_3. If now we allow L to move along LX_1 to X_1 and afterwards M along MX_1 to X_1, we generate a C_9 having still threefold points in $D_1, D_2 \ldots, D_8$, now admitting a ninth threefold point in X_1 and passing moreover through an arbitrarily chosen point N (compare § 3). So the point X_1 is a point of the curve j_x under discussion. Therefore:

The curve j_x cuts any curve of (β') besides in the base points in 8 points more. It is at the same time the locus of the points forming with B_9 on the curves of (β') a Steinerian pair of the third order.

§ 8. In order to determine the curve j_x more closely it is necessary to know the order of multiplicity of the points $D_1, D_2, \ldots, D_8$ on it, i. e. how many times each of these points happens to form with B_9 a Steinerian pair of order three on a curve of (β'). Let u_3 (fig. 2) be once more an arbitrary curve of (β'); then we project B_9 out of D_2 on u_3 (i. e. we determine the third point A_1 common to D_2B_9 and u_3), from this point A_1 we project D_1 on u_3 into A_2, from A_2 we once more project B_9 on u_3 into A_3 and so on. alternately projecting B_9 and D_1. Then we allow u_3 to describe the pencil (β') and determine the loci of the points $A_1, A_2, \ldots, A_n$; then every coincidence of A_6 with D_2 points to a curve out of (β') on which B_9 and D_1 form a Steinerian pair of order three.

So we find for the locus of

A_1: the line $D_2\ B_9$;

A_2: a C_3 with a double point in D_1, not passing through D_2 and B_9 but cointaining $D_3, D_4 \ldots, D_8$;

A_3: a C_7 with an ordinary point in D_1,
a threefold point in D_2,
double points in $D_3, D_4 \ldots, D_8$,
a fourfold point in B_9;

A_4: a C_{12} with a sixfold point in D_1,
a threefold point in D_2,
fourfold points in $D_3, D_4 \ldots, D_8$,
a double point in B_9;

A_5; a C_{19} with a fourfold point in D_1,
a sevenfold point in D_2,
sixfold points in $D_3, D_4 \ldots, D_8$,
a ninefold point in B_9;

A_6 : a C_{27} with a twelvefold point in D_1,
an eightfold point in D_2,
ninefold points in D_3. D_4 . . ., D_8,
a sixfold point in B_9;

A_{2n} : a C_{3n^2} with a $n(n+1)$-fold point in D_1,
a (n^2-1)-fold point in D_2,
n^2-fold points in D_3, D_4 . . ., D_8,
an $n(n-1)$-fold point in B_9;

A_{2n+1} : a C_{3n^2+3n+1} with an n^2-fold point in D_1,
an (n^2+n-1)-fold point in D_2,
(n^2+n)-fold points in D_3, D_4 . . ., D_8.
an $(n+1)$-fold point in B_9.

We prove this as follows. It goes without saying that the locus of A_1 is the line D_2B_9. Through any point A_1 of this line one curve u_3 of (β') passes and this curve is cut by A_1D_1 for the third time in A_2. In D_1 we draw the tangent to u_3 and we indicate by A_1' the point common to this tangent and D_2B_9. Now if u_3 describes the pencil (β') it will happen twice that A_1 and A_1' coincide; in each of these two cases A_2 coincides with D_1, so that D_1 is a double point of the locus of A_2. This point A_2 describes a rational cubic curve, to be indicated henceforth by α_3, any line through D_1 having only one more point in common with this curve. It contains the points $D_3, D_4, \ldots, D_8$, as D_2B_9 cuts each of the lines D_1D_3, $D_1D_4, \ldots, D_1D_8$ in one point.

Let us now consider the locus of A_3. It is immediately evident that D_1 is an ordinary and D_2 a threefold point of this locus; for α_3 is cut by B_9D_1 in only one, by B_9D_2 in three points; in the same manner we prove $D_3, D_4, \ldots, D_8$ to be double points. So we have still to investigate how many times A_3 coincides with B_9. Let A_2 be once more an arbitrary point of α_3 and u_3 the curve of (β') through A_2; then the tangent of u_3 in B_9 cuts α_3 in three points A'_2. So the points A_2 and A'_2 generate a correspondence (1,3) furnishing — α_3 being rational — 4 coincidences. Any coincidence of A_2 and A_2' gives a coincidence of A_3 and B_9; so A_3 describes a curve of order seven, to be indicated henceforth by α_7, any line through B_9 containing three points more of this locus.

We can prove that B_9 is a fourfold point of α_7 also as follows. In case A_2 coincides with one of the points A_2', A_2 is at the same time the tangential point of B_9 on the curve out of (β') through A_2. So the number of points common to α_3 and the *tangential curve* of B_9 — i.e. the locus of the tangential point of B_9 on any curve of (β') — amounts to four, the common points coinciding with the base points of (β') disregarded; for, the tangential curve is of order four and admits B_9 as threefold point whilst it passes only once through $D_1, D_2, \ldots D_8$. So B_9 is a fourfold point of α_7 and this curve is of order seven. From the number of the double points we deduce that α_7 is rational; this is right, for it corresponds point by point with the line D_2B_9.

As to the locus of A_4 it is immediately clear that B_9 is a double point and D_2 a threefold point on it, while it passes four times through $D_2, D_3, \ldots, D_8$.

The tangential curve of D_1 is cut by α_7 besides in the base points in 6 points more, which implies that D_1 is a sixfold point on the locus of A_4 and that this curve is of order twelve, any line through D_1 containing six more points of it. In the same manner we determine the loci of the points A_5, A_6, A_7, etc. and then the loci of A_{2n} and A_{2n+1} can be found by the Bernoullian method. Provisionally we only still wish to remark, that the locus of A_8 has an eightfold point in D_2, for this proves that the points B_9 and D_1 form two Steinerian points of order three on 8 curves of (β').

§ 9. Let us return to the point we started from. We have seen that the curve j_x under discussion — the locus of the ninth threefold point — is at the same time the locus of the points each of which forms with B_9 a Steinerian pair of the third order. On each curve of (β') lie besides the base points eight points more of j_x; moreover $D_1, D_2, \ldots, D_8$ are eightfold points of j_x:

We have now to investigate whether B_9 lies on j_x or not. This can only happen if on a curve v_3 of (β') the point B_9 coincides with one of the eight points each of which forms with it a Steinerian pair of the third order. However it is easy to prove that a suchlike coincidence of two Steinerian points can only present itself in a node; for the group of the nine inflexions this is immediately evident and for the other groups of Steinerian points of the third order it can be deduced from this by projection. Now B_9 is not a node of a curve out of (β'); so it does not lie on j_x.

As the number of points common to j_x and u_3 amounts to 72 we find:

"*The curve j_x is of order twenty-four; it has $D_1, D_2, \ldots, D_8$ as eightfold points.*"

§ 10. We will enumerate some points of j_x, which curve will be denoted furthermore by j_{24}. It is cut by the line D_1D_2 in eight more points; any point P of these eight determines with $D_1, D_2, \ldots, D_8$ a pencil of curves c_9 with threefold points in these nine points. Any other point Q of D_1D_2 determines a curve c_9 of this pencil having ten points in common with that line and breaking up therefore into that line and a curve c_8 with double points in D_1, D_2, P and threefold points in $D_3, D_4, \ldots, D_8$. So any point of intersection P of $D_1 D_2$ and j_{24} is at the same time a node of a c_8 forming with $D_1 D_2$ a c_9 with nine threefold points. At first this result may seem astonishing; for we can indicate *eleven* points on D_1D_2 each of which forms with $D_1, D_2, \ldots, D_8$ a set of nine threefold points of a c_9, and of these eleven points we find back eight only. But the three other ones prove to determine a c_9 (and therefore a pencil of curves c_9) excluded from the beginning.

To prove this we consider the net $[\delta]$ of curves c_8 determined by the six threefold points $D_3, D_4, \ldots, D_8$ and the double points D_1, D_2; the curve of JACOBI of this $[\delta]$ is of order twenty-one and, as it passes five times through D_1, D_2, it is cut by the line D_1D_2 in 11 points more. So D_1D_2 contains 11 points each of which is a node of a c_8 belonging to $[\delta]$.

Now let us consider the curve c_5 passing through D_1, D_2 and admitting $D_3, D_4, \ldots, D_8$ as nodes; this completely determined curve cuts $D_1 D_2$ in three points E, F, G more. Each of these points lies on the curve of JACOBI of $[\delta]$, for c_5 forms with the curve c_3 of (β') passing through E a c_8 of $[\delta]$, of which the point E is a node; likewise these two curves form with the line $D_1 D_2$ a curve c_9 of which $D_1, D_2, \ldots, D_8$ and E are threefold points. However E does not lie on j_{24}, for this c_9 can be considered as the combination of a c_3 of (β') and a c_6 and this combination has been excluded beforehand (§ 5). But it is evident that E, F, G do lie on the curve j_9 quoted in § 4.

The eight remaining points of intersection of line D_1D_2 and the curve of JACOBI of $[\delta]$ do lie on j_{24}; so on each of the 28 lines D_iD_k can be indicated eight points of j_{24}.

Moreover j_{24} is cut by the conic $D_1, D_2, \ldots, D_5$ in eight more points. These lie at the same time on the curve of JACOBI of the net $[\varepsilon]$ of curves of order seven passing twice through $D_1, D_2, \ldots, D_5$ and thrice through D_6, D_7, D_8. This curve of JACOBI of order eighteen is cut by the conic $D_1, D_2, \ldots, D_5$ in eleven more points; of these however once more three do not lie on j_{24}, i.e. the points common to this conic and the curve c_4 passing once through $D_1, D_2, \ldots, D_5$ and twice through D_6, D_7, D_8.

So on each of the 56 conics $D_i\ D_k\ D_l\ D_m\ D_n$ can be indicated eight points of j_{24}.

§ 11. We now treat in a summary way the general case:
ϱ is an arbitrary number.

Once more the arbitrarily chosen points $D_1, D_2, \ldots, D_8$ are given and the question is to determine the locus of the point forming with these given points a set of nine ϱ-fold points of a non degenerated curve of order 3ϱ. In the same way as we have used the results obtained for $\varrho = 2$ in the solution of the problem for $\varrho = 3$, we can solve the successive cases $\varrho = 4,5, \ldots$ by using every time the results obtained in the immediately preceding case. So we consider for $\varrho = 4$ at first a variable c_{12} with fourfold points in $D_1, D_2, \ldots, D_8$ and touching a curve u_3 of pencil (β') in a point X; then we determine the third point of intersection of u_3 with the line connecting the last two points of intersection of c_{12} and u_3, which point is independent of the choice of c_{12}, etc.

But before we state our results more in detail we wish to make a remark. We find, that any point D_9 which can present itself as ninth ϱ-fold point of a non degenerated $c_{3\varrho}$ must coincide with one of the points forming with B_9 a Steinerian point of order ϱ. The locus of the latter points is a curve $c_{3(\varrho^2-1)}$ with (ϱ^2-1)-fold points in $D_1, D_2, \ldots . D_8$. Now however it is evident that this curve degenerates in several cases. So, if e.g. we consider the case $\varrho = 6$, we shall find among the points forming with B_9 on a curve of (β') Steinerian pairs of order six also the points which form with B_9 Steinerian pairs of order two and of order three. So the curve $c_{3(\varrho^2-1)}$, here of order 105, must break up into j_9, j_{24} and a curve of order 72 passing 24 times through $D_1, D_2, \ldots, D_8$. Now the latter curve forms the locus proper of the ninth sixfold point of a non degenerated curve c_{18}. So the two curves of which the first is the locus of the ninth ϱ-fold point, the second that of the point forming with B_9 a Steinerian pair of order ϱ, coincide completely if ϱ is a prime number; if ϱ is no prime number the first curve is a part of the second. So we have found:

"*The locus of the ninth ϱ-fold point coincides completely or partially with that of the points forming with B_9 on the curves of (β') Steinerian pairs of order ϱ. The latter curve cuts any curve of (β') besides in the base points in (ϱ^2-1) more points, has the points $D_1, D_2, \ldots, D_8$ for (ϱ^2-1)-fold points and is therefore of order $3(\varrho^2-1)$. The former coincides completely with this curve, if ϱ is a prime; in the opposite case its order and the multiplicity of the base points on it can be easily deduced from the corresponding numbers of the second curve.*"