position was not taken by a single leaf. Of this species there are two specimens more in the Hortus Bog. under Number 1 and Number 1*A* in garden-bed I H.

§ 5. Adenanthera microsperma TEYSM. & BINN. — Tree grown under this name in Hortus Bogor. in garden-bed IB, Number 49. On 5 February at 9 a.m. I found that the leaves (in this species bipinnate), at least the younger ones, performed rather distinct but only feeble irritation movements after vigorous shaking of the branches. These movements reminded of those typical for *Poincianía*, but in *Adenanthera* the movement is much slighter. The determination of the species has been verified by me and, in as much as the material at hand allowed, found correct.

§ 6. Tetrapleura Thonningii BENTH. — A young tree about 1 M. high, grown under the said name in the Hortus under Number 14 in garden-bed I.G. The leaves are bipinnate and remind of Adenanthera. At 7 a.m. on 5 February, the plant was vigorously shaken. Within few minutes the younger leaves which before, like the older ones, were quite expanded, plainly showed irritation movements similar to the Poinciania type, but much less vivid.

§ 7. Schrankia hamata HB. & BPL. — Undershrub kept in the Buitenzorg Hortus under that name, which was verified and found right, in garden-bed A XXV of the Leguminosae herbs division, under Number 2, with bipinnate leaves and very narrow leaflets. Not only in the forenoon but also in the afternoon, all the branches when mechanically irritated (by shaking) reacted almost as quickly as Mimosa pudica.

Buitenzorg, February 12, 1911.

Mathematics. — *"On the structure of perfect sets of points"* (second communication [1])). By Dr. L. E. J. BROUWER. (Communicated by Prof. KORTEWEG).

(Communicated in the meeting of April 28, 1911).

§ 1.

A further extension of Cantor's fundamental theorem.

The proof of CANTOR's fundamental theorem and of its Schoenflies extension, given in § 2 of the first communication, holds also for the following property :

[1]) For the first communication see these Proceedings, Vol. XII, p. 785.

THEOREM 1. *A well-ordered set of points in Sp_n each point of which possesses a finite distance from the set formed by all the following points, is denumerable.*

Out of this is obtained in the following form a generalization of CANTOR's fundamental theorem, probably the widest one of which it is capable·

When a closed set of points is replaced by a closed set contained in it, we shall say that the first set is *lopped.*

A fundamental series of closed sets of points will be called a *lopping series,* if each following set is contained in the preceding one. The greatest common part of the terms of such a series is a closed set, which we shall call the *limiting set* of the lopping series.

By an *inductible property of closed sets of points* we shall understand a property which, when possessed by each term of a lopping series, holds also for the limiting set of that series.

From theorem 1 now follows:

THEOREM 2. *Let μ be a closed set of points of Sp_n possessing the inductible property α; we can reduce it by a denumerable number of loppings of a definite kind β to a closed set of points μ_i possessing still the property α, but losing it by any new lopping of kind β.*

This theorem can be specialized in many directions.

If we choose as property α the simple property of being closed, and as lopping of kind β the destruction of an isolated point resp. of an isolated piece, then CANTOR's fundamental theorem resp. its Schoenflies extension appears.

An other special case is obtained in the following way:

After ZORETTI[1]) a continuum C is called *irreductible between P and Q,* if the pair of points (P,Q) belongs to C, but to no other continuum contained in C, and JANISZEWSKI[2]) and MAZURKIEWICZ[3]) have proved the following theorem:

Let C be an arbitrary continuum and P and Q two of its points, then in C is contained a continuum irreductible between P and Q.

This property appears likewise as a special case of theorem 2, namely by choosing as property α the property of containing P and Q and being continuous, and as lopping of kind β the most general lopping.

§ 2.

The structure of closed sets of pieces.

In § 3 of the first communication it has been proved that all perfect

[1]) Annales de l'École Normale, 1909, p. 485.

[2]) Comptes Rendus, t. 151, p. 198.

[3]) ibid., p. 296.

sets of pieces possess the same geometric type of order, namely the common type of order of linear, perfect, punctual sets of points. An analogous theorem exists for closed sets of pieces.

In § 3 of the first communication the set of pieces μ, taken there as perfect, was broken up into two such closed sets μ_0 and μ_2, that $\delta(\mu_h) \leqq \delta(\mu)$ and $a(\mu_0, \mu_2) \geqq \delta(\mu)$; then each μ_h into two such closed sets μ_{h0} and μ_{h2}, that $\delta(\mu_{hk}) \leqq \delta(\mu_h)$ and $a(\mu_{h0}, \mu_{h2}) \geqq \delta(\mu_h)$; and so on. In this way the μ_F's converged for indefinite accrescence of the rows of indices F uniformly to the pieces of μ, and we could construct a continuous one-one correspondence between the pieces of μ and a nowhere dense perfect set of real numbers between 0 and 1, where for each of those numbers the row of figures in the numerical system of base 3 was identical to the row of indices of the corresponding piece of μ.

If, however, μ is a closed, not perfect set of pieces, then the breaking up of an arbitrary μ_F into μ_{F0} and μ_{F2} can take place in the same way with the only exception that a μ_F consisting of a single piece also appears as μ_{F0}, whilst μ_{F2} falls out. Then too the μ_F's converge for indefinite accrescence of the rows of indices F uniformly to the pieces of μ, and we can construct a continuous one-one correspondence between the pieces of μ and a nowhere dense closed set of real numbers between 0 and 1, where for each of those numbers the row of figures in the numerical system of base 3 is identical to the row of indices of the corresponding piece of μ.

So we have proved:

THEOREM 3. *Each closed set of pieces in Sp_n possesses the geometric type of order of a linear, closed, punctual set of points.*

§ 3.

The division of the plane into more than two regions with a common boundary.

On a former occasion [1]) I constructed a division of the plane into three regions with a common boundary, and I communicated at the same time that by a suitable modification of the method followed there a division into an arbitrary finite number, and even into an infinite number of regions with a common boundary can be obtained. That modified method I shall now explain.

---

[1]) Compare "*Zur Analysis Situs*", Mathem. Annalen, Vol. 68, p. 422–434.

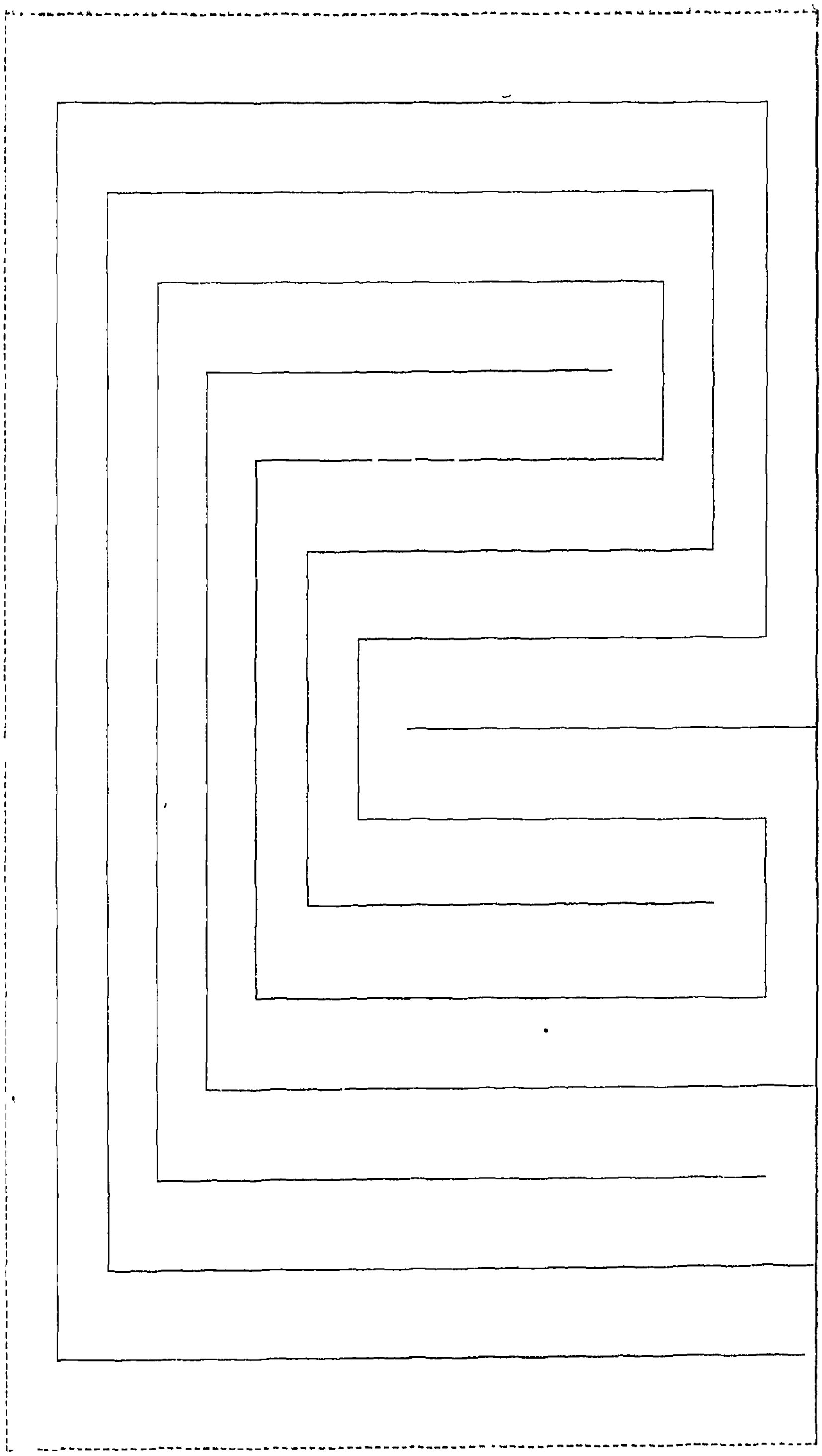

Fig. 1.

Our starting-point is figure 2 (Plate I) explained l. c. p. 423 and 424 of Vol. 68 of the Mathem. Annalen, which figure in the following will be called the *primitive figure.*

We first simplify the primitive figure by leaving out the red band, and by reducing the breadth of the black bands to zero. These contracted black bands we shall call *"supporting threads"*, and we draw each of them through the middle of the white band determined by all the preceding supporting threads, as is executed here in figure 1 for the first four supporting threads (this figure is to be looked at in the position indicated by the subscription: Fig. 1).

The rectangular circumference of figure 1 we shall indicate by k, the circumference together with its inner domain by F. The circumference together with the supporting threads we shall call the *skeleton* of the figure. Two arbitrary points of the skeleton possess the property of being contained in a perfect coherent part of the skeleton.

We now consider a horizontal section l of figure 1 cutting all the vertical line segments of the supporting threads, and we determine the points of l by their *abscis,* i. e. their distance from the left endpoint of l. The length of l we choose as unity of length.

Then the abscis of the point of intersection of l with the first supporting thread is $\frac{1}{2}$; the abscissae of the points of intersection of l with the second supporting thread are $\frac{1}{4}$ and $\frac{3}{4}$; those with the third supporting thread are $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$ and $\frac{7}{8}$; and so on.

So the set of points determined on l by the system of supporting threads possesses as their abscissae the set of dual fractions between 0 and 1.

Two points of F will be called *directly coherent,* if they are contained in a perfect coherent part of F having no point in common with the skeleton. Two points directly coherent with a third point are also directly coherent with each other. The points directly coherent with a given point form a set which will be called a *coherence thread.*

The abscissae of the points of intersection of l with a coherence thread form a set of numbers to be called a *directly coherent set of numbers.* Two abscissae then and only then belong to the same directly coherent set of numbers, if either their sum or their difference is a dual fraction.

The set of coherence threads possesses the power of infinity of the continuum.

Of this figure 1 we shall now construct a generalization; in the white band determined by the first n supporting threads we shall namely draw the $(n+1)^{\text{th}}$ supporting thread *not through the middle*; thereby each supporting thread segment gets an arbitrary distance from the corresponding white band edge segments; however we take care firstly that the new supporting thread penetrates into each segment of the corresponding white band, and secondly that each vertical supporting thread segment cuts the line l.

In the more general figure it may happen that some segments of coherence threads have expanded to bands, so that for this figure we shall replace the name of coherence threads by *coherence strips.*

And if each point of F which can be joined to the skeleton by a line segment meeting the skeleton only in its endpoint, is added to the skeleton, then also in the skeleton, just as in the coherence threads, certain segments may expand to bands, so that for the new skeleton we shall replace the name of supporting threads by *supporting strips.*

In the more general figure we assign to the points and intervals in which l is cut by the n^{th} supporting strip as their *coordinates* the same numbers $\frac{2k+1}{2^n}$ which in figure 1 appeared as the abscissae of the corresponding points of intersection of l with the n^{th} supporting thread, and each point or interval determined on l by a coherence strip gets as its coordinate the number corresponding to the Schnitt determined in the coordinates belonging to the supporting strips. Then along l the coordinate is a nowhere decreasing continuous function of the abscis, and like the abscis it has the initial value 0 and the endvalue 1.

Now for a moment we abstract from the figure, and set apart a finite or a denumerable infinite system of directly coherent sets of numbers. The numbers belonging to these sets we shall call *special numbers* and we determine a coordinate function of the just now described kind possessing each special numerical value over a certain interval of abscissae, but each other numerical value (0 and 1 included) only for a single abscis.

Our aim is to construct the generalized figure 1 in such a way that the coherence strips corresponding to the special directly coherent sets of numbers get everywhere a finite breadth, whilst all segments of the other coherence strips and of the skeleton get a breadth zero. Starting from the coordinate function just now constructed on l we succeed in this in the following manner:

The first supporting thread we construct through the point of l with coordinate $\frac{1}{2}$, and each pair of points resp. intervals of l with mean coordinate $\frac{1}{2}$ we join within k and round about the first supporting thread by threads resp. bands twice rectangularly bent and not meeting each other, thereby taking care that the horizontal segments of these threads and bands determine an everywhere dense set of points and intervals on the perpendicular let down from the endpoint of the first supporting thread on the horizontal upper limit of F.

The second supporting thread we construct in such a way through the points of l with coordinates $\frac{1}{4}$ and $\frac{3}{4}$ that it does not cross the threads and bands already constructed, and each pair of points resp. intervals of l with coordinates $> \frac{1}{2}$ and with mean coordinate $\frac{3}{4}$ we join within k and round about the second supporting thread by threads resp. bands twice rectangularly bent and not meeting each other, thereby taking care that the horizontal segments of these threads and bands determine an everywhere dense set of points and intervals on the perpendicular let down from the endpoint of the second supporting thread on the baseline of F.

The third supporting thread we construct in such a way through the points of l with coordinates $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$ and $\frac{7}{8}$, that it does not cross the threads and bands already constructed, and each pair of points resp. intervals of l with coordinates between $\frac{1}{4}$ and $\frac{1}{2}$ and with mean coordinate $\frac{3}{8}$ we join within k and round about the third supporting thread by threads resp. bands twice rectangularly bent and not meeting each other, thereby taking care that the horizontal segments of these threads and bands determine an everywhere dense set of points and intervals on the perpendicular let down from the endpoint of the third supporting thread on the baseline of F.

Continuing in this way we secure that the "special" coherence strips get everywhere a finite breadth and that the other coherence strips and the skeleton get everywhere a breadth zero. The inner domains of the special coherence strips form together a set of points everywhere dense in F, and *all these inner domains have the same boundary*.

If we choose the coordinate function on l in such a way that it

10*

possesses not only each special numerical value but also the value of each dual fraction not identical to 0 or 1 over a certain interval of abscissae, then the just now described construction of the generalized figure 1 can be repeated without modification with the only difference that the supporting threads are replaced by supporting bands. These supporting bands determine together with the rest region of F a region G possessing the same boundary g as the inner domains of the coherence bands.

So this continuum g divides the plane into regions with a common boundary; whether the number of these regions is finite or infinite, depends on the choice of the special directly coherent sets of numbers.

Let us call two points contained in a perfect coherent part of g not identical to g, *directly coherent in* g, and let us call the set formed by the points directly coherent in g with a given point, a *nerve of* g, then the skeleton of the figure and likewise each coherence thread furnishes *one* nerve of g, and each coherence band furnishes *two* nerves of g.

If we choose only *one* special directly coherent set of numbers, then our construction furnishes a closed curve (in the sense of SCHOENFLIES) which can be divided into two improper arcs of curve but not into two proper ones, in which category is included the primitive figure from which we started.

§ 4.

The impossibility of a linear arrangement of the points of an irreductible continuum.

By ZORETTI lately a method has been explained of arranging the points of an irreductible continuum linearly, analogously to those of a line segment [1]).

His method is however inapplicable to several continua constructed in the article "*Zur Analysis Situs*" cited above.

This having been pointed out to him, ZORETTI has based a method of more restricted aim on the following theorem [2]):

"*Given an irreductible continuum C and a point c of C, then C can be divided in one definite manner into three sets of points C_1, C_2 and Γ, possessing the following properties: C_1 and C_2 are coherent and have c as their only common point; Γ consists of the common limiting points of C_1 and C_2. Both sets of points $C_1 + \Gamma$ and $C_2 + \Gamma$ are irreductible continua.*"

1) Annales de l'École Normale, 1909, p. 485—497.

2) Comptes Rendus, t. 151, p. 202.

From this theorem would follow that, if not all the points of C, yet a considerable part of them would be capable of linear arrangement, and that it would be possible to crumble C in the same way as a line segment into an indefinitely large number of linearly arranged *"partial arcs"*, two arbitrary ones of which then and only then cohere, if in that linear order they succeed each other immediately.

But neither this theorem can be maintained, if we try to apply it to our primitive figure.

If namely we choose this closed curve as the irreductible continuum C, then either $C_1 + \Gamma$ or $C_2 + \Gamma$ must be identical to C, and either C_2 or C_1 reduces to the single point c, so that the division of C becomes illusory.

It is a priori certain that all attempts to arrange the points of such a continuum linearly by repeated crumblings must fail, the crumbling being practicable only for a single system of directly coherent points, and therefore the linear arrangement being restricted in any case to points of a single *nerve*.

And even of this we are not sure for the most general irreductible continuum. For, in a system of points directly coherent in C again may be contained an irreductible continuum C' breaking up into a set of the power of infinity of the continuum of systems of points directly coherent in C'. And so on.

§ 5.

A generalization of JORDAN's *theorem.*

JORDAN's theorem runs that a continuous one-one image of a circle is a closed curve, i.e. divides the plane into two regions of which it is the common boundary.

The extension lying at hand that a continuous one-one image of a closed curve is again a closed curve, has not yet been proved. However, for a special kind of closed curves a partial result can be arrived at, as we shall explain in the following.

Let C be an arbitrary closed curve, and let us represent by $c\eta_i$ the cyclic type of order of its points accessible from its inner region. A Schnitt s arbitrarily given in $c\eta_i$ determines two *"Schnitt-continua"* σ_l and σ_r, to which $c\eta_i$ converges on the left resp. on the right of s. The points common to σ_l and σ_r form a closed set of points σ, to be called *"the juncture belonging to the Schnitt s"*.

LEMMA. *In the inner region of C we can construct an arc of curve which abroad from its ends is simple, and of which one end reduces to a single point of the inner region of C and the other is contained in the juncture* σ.

PROOF. Let M be a point taken arbitrarily in the inner region of C, and let $a_1, a_2, a_3, \ldots$ be a fundamental series of indefinitely decreasing arcs of simple curve lying abroad from their endpoints in the inner region of C, whilst the endpoints belong to $c\eta_i$, and are separated by the Schnitt s. In this series is contained a series $b_1, b_2, b_3, \ldots$ converging to a single point P of σ, and in which each b_n is separated from M by $C + b_{n-1}$. We can then join M and P by an arc of simple curve z cutting an infinite number of the b_ν in the order of their indices each in one point and passing there from their side turned to M to their side turned to s. The b_α intersected in this way form a series $d_1, d_2, d_3, \ldots$ Let z_n be the part of z enclosed between d_n and d_{n+1}, D_n the point of intersection of z and d_n, A_n resp. B_n the left resp. right endpoint of d_n, φ_n resp. ψ_n the arc of curve determined on C by the Schnitte corresponding to A_n and A_{n+1} resp. to B_n and B_{n+1}, ϱ_n resp. τ_n the part of the inner region of C cut off by the arc of simple curve $A_n D_n D_{n+1} A_{n+1}$ resp. $B_n D_n D_{n+1} B_{n+1}$, u_n resp. v_n the part of φ_n resp. ψ_n lying in τ_n resp. ϱ_n. We then can join D_n and D_{n+1} by an arc of simple curve t_n lying entirely in the part of the inner region of C enclosed between d_n and d_{n+1} and moving away from $z_n + u_n + v_n$ no farther than a certain maximum distance ε_n indefinitely decreasing for indefinitely increasing n. These arcs t_n form together an arc of curve possessing the properties required.

THEOREM 4. *If the closed curve C is divided by the Schnitte s_1 and s_2 of $c\eta_i$ into two proper (i.e. not identical to C) arcs of curve C_1 and C_2, then the points common to C_1 and C_2 form a non-coherent set of points C_{12}.*

PROOF. Let σ_1 resp. σ_2 be the juncture belonging to s_1 resp. s_2, then according to the lemma just now proved we can draw from a point M taken arbitrarily in the inner region of C to ends e_1 and e_2 contained in σ_1 resp. σ_2 two arcs of curve which abroad from their ends are simple, do not meet each other, and lie entirely in the inner region of C. These arcs of curve we represent by $\digamma_1$ and $\digamma_2$ (see the schema in fig. 2), and the largest perfect coherent part of C_{12}, containing e_1 resp. e_2, by p_1 resp. p_2. Let Q_1 resp. Q_2 be a point of $c\eta_i$ belonging to C_1 but not to C_2, resp. to C_2 but not to C_1. Then from M to Q_1 and Q_2 we can draw paths w_1 and w_2 which abroad from their ends lie entirely in the inner region of C, and meet neither each other nor $\digamma_1$ or $\digamma_2$. In the inner region of C these paths w_1 and w_2 are separated by $\digamma_1$ and $\digamma_2$.

About Q_1 as centre we describe a small circle $\varkappa_1$ which together with its inner region has no point in common with $C_2 + \digamma_1 + \digamma_2$, and we draw a path w'_1 joining the infinite with a point Q'_1 of $\varkappa_1$, and abroad from Q'_1 lying entirely in the outer region of C as well

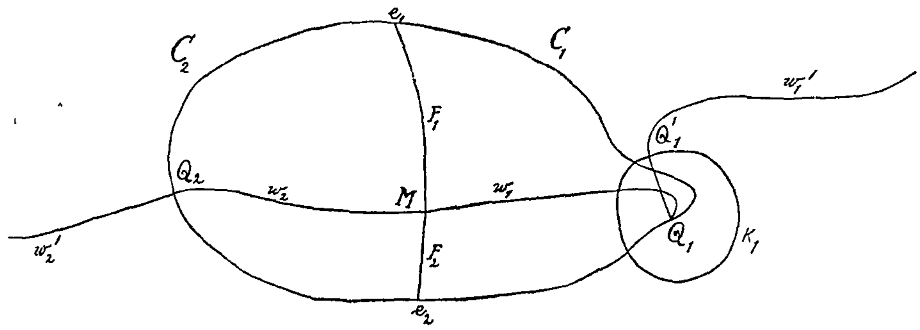

Fig. 2.

as in the outer region of $\varkappa_1$. Then w'_1 forms together with some parts of w_1 and of the radius $Q_1\, Q'_1$ of $\varkappa_1$ a path l_1 joining M with the infinite, and abroad from M not meeting $C_2 + F_1 + F_2$.

In the same way we can construct a path l_2 joining M with the infinite, coinciding for a certain initial part with a part of w_2, and abroad from M not meeting $C_1 + F_1 + F_2$.

As in the vicinity of M the arcs F_1 and F_2 are separated by w_1 and w_2, in the complete plane e_1 and e_2 are separated by $l_1 + l_2$ (whether l_1 and l_2 meet each other abroad from M or not).

So, since $l_1 + l_2$ contains no point of C_{12}, also p_1 and p_2 are separated in the complete plane by $l_1 + l_2$. Hence p_1 and p_2 cannot be identical, and C_{12} cannot be a continuum.

As furthermore two finite continua whose common points form a non-coherent set determine more than one region in the plane [1]), from theorem 4 ensues immediately:

THEOREM 5. *A continuous one-one image of a closed curve divisible into two proper arcs of curve determines in the plane more than one region.*

1) This may be proved by breaking up the boundary of a region determined by the common points of these continua into two closed sets c_1 and c_2 possessing a finite distance from each other, and then applying the reasoning of Mathem. Annalen, Vol. 68, p. 430.