## Huygens Institute - Royal Netherlands Academy of Arts and Sciences (KNAW)

## Citation:

Barrau, J.A, The surfaces of revolution ofquadratic cylinders of non-Euclidean space, in: KNAW, Proceedings, 14 I, 1911, Amsterdam, 1911, pp. 148-153

This PDF was made on 24 September 2010, from the 'Digital Library' of the Dutch History of Science Web Center (www.dwc.knaw.nl) > 'Digital Library > Proceedings of the Royal Netherlands Academy of Arts and Sciences (KNAW), http://www.digitallibrary.nl'

Mathematics. - "The surfaces of revolution or quadiatic cylinders of non-Euclidean space". By Prof. J. A. Bakrau. (Communicated by Prof. J. Cardinaat).
(Communicated in the meeting of April 28, 1911).
In hyperbolic space each quadratic surface whose intersection with the absolute fuadratic surface $\Omega$ degenerates into two conics (casuquo still farther) is surface of revolution as well as cylinder, in such respect that the line of intersection of the planes of the products of degeneration is cylinder axis, its reciprocal polar line with respect to $\Omega$ axis of revolution.

In one consideration the surface is generated as locus of a conic revolving round one of its axes ${ }^{1}$ ); in the other as locus of an invariable conic, of which one of the centres describes a right line, to which its plane always remains perpendicular, while its points describe plane curves. It is clear, that by assuming this definition of cylinder, that one as cone with vertex at infinite distance, which coincides with it in Euclidean geometry, is abandoned.

Now according to the axis of revolution being metrically real (i.e. having a real part within $\Omega$ ) and therefore the cylinder axis ideal, or the reverse, it will be more natural to regard the surface as surface of revolution than as cylinder (classes A and B), whilst a transition class C is formed by the cases in which both axes are conjugated tangents to $\Omega$.

If the surface is projectively real (i. e. metrically real or ideal) then the planes of the degenerations are (projectively) either real or conjugated complex; both axes are thus in any case projectively real. Each plane through the cylinder axis cuts the surface along a conic in double contact with $S$, that is along a circle after hyperbolical measure.

If this intersection is metrically real, then it is a (finite) circle, a limiting circle (or circular parabola) or a line of distance, according to the surface being ranged in class A, C or $\mathrm{B}^{2} \%$. So we have a first system of circular sections for each surface.

But the general quadratic surface possesses four systems of circular sections, namely the tangential planes to the four focal cones (cones in the pencil determined by $\Omega$ and the surface). Of these four systems two are absorbed in our case by the above-

[^0]mentioned first system, so in general two are still to be expected, which can however be ideal or imaginary (in which case we shall not describe them) or they can coincide.

We stall now give an enumeration of the possible types of these surfaces, including the cones of revolution, excluding however the - purely ideal forms. ${ }^{1}$ )

## A. Surfaces or revolution Proper.

Cylinder axis ideal (or indetinite), first system of (finite) circles.
I. Both planes of degeneration metrically real.

1. Cone of revolution with real vertex and real axis.
2. Cone of revolution with ideal vertex: and real axis.

Has a gorge-circle with centre in the vertex of 1 and a system of distance lines in tangential planes to cone 1.
3. Hyperboloid of revolution first lind.

Two-sheeted, non rectilinear ${ }^{\circ}$ ) surface falling between 1 and the planes of degeneration. Divides space (inside $\boldsymbol{\Omega}$ ) into one outer domain (in the ordinary projective sense) and two inner domains.
4. Hyperboloid of revolution second kind.

Two-sheeted, non rectilinear surface falling outside the planes of degeneration. One inner domain, two outer domains.
5. Hyperboloid of revolution third kind.

One-sheeted, rectilinear surface between 1 and 2. Is generated by revolution of a real right line around a real axis. Has a gorge-circle and a system of distance lines in tangential planes to 1.
6. Hyperboloid of revolution fourth kind.

One-sheeted, non rectilinear surface, outside 2 . Has a gorge-circle and two systems of distonce lines, resp. in langential planes to 1 and 2.
II. One plane of degeneration metrically real $\left(D_{1}\right)$ onetouching $\boldsymbol{\Omega}$ (e.g. in $P$ ).
7. Limiting cone of revolution (vertex in $\Omega$, real axis).
8. Hyperbolic paraboloid of revolution furst kind.

One-sheeted, non rectilinear surface between 7 and $D_{1}$. Right lines out of $P$ (within 7) intersect first the surface, then $D_{1}$.
9. Hyperbolic paraboloid of revolution second kind.

One-sheeted, non rectilinear surface outside $D_{1}$. Right lines out of $P$ (inside 7) intersect first $D_{1}$ then the surface.

[^1]10. Hyperbolic paraboloid of revolution third kind.

One-sheeted non rectilinear surface outside 7. Has a system of distance lines in tangential planes to 7 .
III. One plane of degeneration metricallyreal, one ideal.
11. Semi-hyperboloid of revolution.

One-sheeted, non rectilinear surface.
IV. Both planes of degeneration ideal.
12. Elongated ellipsoid of revolution.

Closed surface.
V. Planes of degeneration conjugate imaginary.
13. Flattened ellapsoid of revolution.

Closed suxface.
VI. One plane of degeneration ideal, one touching $\boldsymbol{\Omega}$.
14. Elliptic paraboloid of revolution.

One-sheeted, non rectilinear surface.
VII. Both planes of degeneration touching $\Omega$
15. Circular cylinder.

One-sheeted, non rectilinear surface. The curve of intersection with $\Omega$ is degenerated into a skew quadrilateral, the surface belongs also to $B$. Both axes are equivalent, the ideal one bears a pencil of sections along finite circles, the real one along distance lines.
VIII. Planes of degenerationideal, coinciding. 16. Sphere.

Closed surface, $\infty^{2}$ systems of finite circles.
IX. Planes of degeneration touching $\Omega$, coinciding.
17. Limiting-sphere.

One-sheeted, non rectilinear surface, also belonging to $C ; \infty^{2}$ systems of circles, amongst which $\infty^{2}$ systems of limiting circles.
X. Planes of degeneration real, coinciding.
18. Surface of distance.

Locus of points at fixed distance on either side of a plane. Twosheeted, non rectilinear surface ; one inner domain, two outer domains. Belongs also to $B$ and $C ; \infty^{2}$ systems of circles, as well as of distance lines, $\infty^{1}$ systems of limiting circles.

## B. Cylinder-surfaces Propar.

Axis of revolution ideal (or indefinite), first systems of distance lines. The surfaces 15 and 18.
XI. Planes of degeneration real.
19. Cone of revolution with ideal vertex and ideal axis.

Two-sheeted surface, one inner domain, two outer domains, cylinder with directrix degenerated into two right lines.

In the pencil formed with $\Omega$ is a second cone of the same type.
20. Hyperbolic cylinder first kind.

Two-sheeted, non rectilinear surface. One inner domain, two outer domains. Possesses a second system of distance lines in tangential planes to one of the cones 19.
21. Hyperbolic cylinder second kind.

Two-sheeted rectilinear surface. Is generated by revolution of a real right line about an ideal axis.
XII. Planes of degeneration conjugate imaginary, but not touching $\Omega$ (compare VII).
22. Elliptic cylinder.

One-sheeted, non rectilinear surface. In the pencil with $\Omega$ is an ideal cone, whose tangential planes cut the surface along a system of finite circles. These planes make equal angles (on either side) with the plane of the orbit of the great axis of the directing ellipse. ${ }^{1}$ )

## C. Transition Class.

Axes touch $\boldsymbol{\Omega}$, first system of limiling circles.
The surfaces 17 and 18.
XIII. Planes of degenerationreal.
23. Cone of revolution with ideal vertex, axis touching $\Omega$.
24. Limiting hyperbolic paraboloid of revolution, first kind.

One-sheeted, rectilinear surface lying between 23 and the planes of degeneration. Is generated by revolution of a real right line about an axis touching $\Omega$.
25. Limiting hyperbolic paraboloid of revolution second kind.

One-sheeted, non rectilinear surface outside 23 . Has a system of distance lines in tangential planes to 23.
26. Limiting hyperbolic paraboloid of revolution third kind.

Two-sheeted, non rectilineur surface inside 23 , yet outside the planes of degeneration. One inner domain, two outer domains.
XIV. One plane of degeneration real, one touching $\Omega$.
27. Limiting semi-circular paraboloid of revolution.

One-sheeted, non rectilinear surface.
XV. Planes of degeneration conjugate imaginary.
28. Limiting elliptic paraboloid of revolution.

One-sheeted, non rectilinear surface.

[^2]In ellipticspace the same considerations hold, but $\boldsymbol{\Omega}$ is imaginary, in consequence of which the number of cases remains more limited.

In the first place space is now finite - so each surface is closed. Then both axes are always real, so that each surface of revolution is as naturally a cylinder.
Finally there exists only one real type of conic by means of which the surface can be generated: the ellipse. It has three centres (of which one is in the inner domain) and three axes (of which one is in the outer domain). There is also but one type of circle.

The surfaces possess - if nothing further is said - only the first system of circular sections.

We can now distinguish:
I. Planes of degeneration real, differing.

1. Flattened ellipsoid of revolution.

Non rectilinear. Is generated by revolution of the ellipse about that axis cutting it, which measured in the inner domain is the shortest.
II. Planes of degeneration real, coinciding.
2. Sphere.

Non rectilinear. Locus of points at fixed distance of given point, likewise of given plane; $\infty^{2}$ systems of circular sections.
III. Planes of degeneration imaginary, not touching $\Omega$.
3. Cone of revolution.
4. Elongated ellipsoid of revolution.

Non rectilinear, is generated by revolution of the ellipse about the longest axis in the inner domain.
5. Elliptic cylinder.

Rectilinear surface, is generaled by revolution of the ellipse about the outer axis; likewise by revolution of a right line aboul an other right !ine, to which it is not a Clifford parallel. The tangential planes to the cone (of type 3) belonging to the pencil formed with $\Omega$ form a (quadratic) second system of circular sections.

The surface has a gorge and an equator, lying in mutually perpendicular surfaces.
IV. Planes of degeneration imaginary, touching $\Omega$.
6. Circular cylinder.
7. Rectilinear surface. Both axes are equivalent, the surface is generated in two ways by revolution of a circle around the outer axis, likewise in two ways by revolution of a line about an axis to which it is a Clifford parallel. It possesses two systems of circles (in pencils of planes through both axes). The circles of each system
are equal; the surface is in two ways locus of points with fixed distance to a given right line (each of the axes). If those two distances are equal, $\frac{\pi}{4}$ each, the surface divides elliptic space into two congruent parts ${ }^{1}$ ).

Botany. - "On the clistribution of the seeds of certain species of Dischiclia by means of a species of ant: Iridomyrmex mypmecorliae Emery." By Dr. W. Docters van Leeuwen and Mrs. J. Docters van Leeowen-Reynvann. (Communicated by Prof. F. A. F. C. Went).

Schimpre ${ }^{2}$ ) in his well-known work on American epiphytes, has arranged these plants in a number of groups according to the methods by which their seeds are distributed. Obriously it is necessary for these plants, that their seeds should ultimately reach the places, in which the adult plants generally grow. The seeds of epiphytes may be distributed througl the agency of fructivorous animals, and through that of the wind. The representatives of the first group are characterized by the possession of edible portions of the fruit or seed. Various members of this group are known among the orders Rubiaceae, Melastomaceae, Artocarpeae, etc. The wind may distribute the seeds if they are very light, as is the case with Orchids for instance; the spores of epiphytic Lycopodiaceae and Filicinae are also carried from tree to tree by air-currents. Other plants have seeds provided with a floating mechanism such as representatives of Gesneraceac and Asclepiadaceae.

Among well known epiphytes belonging to the last named order are various species of Dischidia, of which D. Rafflesiona has already been dealt with in several works. Since our arrival in Java, we have had repeated opportunities of observing this plant, both in its natural habitat and in our garden. Not only D. Rafflesiana but also $D$. collyris and still more $D$. nummularia are especially abundant in the immediate neighbourhood of our present abode. On the

[^3]
[^0]:    1) Comp. Story: On non-Euclidean Properlies of Gonics (Amer. Journal of Mathematics, vol V, p. 358). His terminology is followed here.
    ${ }^{2}$ ) Some surfaces fall in more than one class.
[^1]:    ${ }^{1}$ ) Of course already well known forms are again included in the place where they fit in this classification.
    ${ }^{\text {g }}$ ) i. e. without real right lines.

[^2]:    ${ }^{1}$ ) In Euclidean geometry this quadratic system degenerates into two linear systems (pencils of parallel planes).

[^3]:    ${ }^{1}$ ) Likewise we find elliptic space $S_{2 n}+1$ of an odd number of dimensions divided into two congruent parts by the quadratic $Q_{2 n}$ containing the points at fixed distance $=\frac{\pi}{4}$ from a given plane $S_{n}$ as well as from its reciprocal polar $S_{n}$ with respect to $\Omega_{2 n}$.
    ${ }^{2}$ ) A. F. W. Schimper. Die epiphytische Vegetation Amcrikas. Bot. Mitt. a. d. Tropen, Jena 1888.

