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CONTENTS.

Physics. — *"Energy and mass"*. By J. D. VAN DER WAALS Jr. (Communicated by Prof. J. D. VAN DER WAALS).

§ 1. *Introduction.* In classical mechanics the mass of the bodies was considered to be constant and the force was defined as the time derivative of the momentum; in consequence of this the law of conservation of momentum, the law: the action is equal to the reaction, and the law of the uniform motion of the centre of inertia of an isolated system were considered to be three different ways to state the same law of nature. At present many physicists consider this not to be the case. They assume that the law of conservation of momentum holds good in nature. For this assumption, however, it is necessary to generalize the notion of momentum, so that we ascribe also a quantity of momentum to the electromagnetic field. The law action = reaction however, most physicists consider not to be fulfilled. It is easy to generalize also the notion "force" in such a way that this law is satisfied. The only thing that is required for this purpose is that also the time derivative of the electromagnetic momentum is called "force".

To this the objection has been raised that we must consider the ether to be stagnant, and that it is therefore, meaningless to speak of a force which is exerted upon it. But we speak of the momentum of this stagnant ether and I do not see why we cannot as well speak of the force which acts on it. Moreover we may avoid both expressions and attribute the momentum not to the ether, but to the electromagnetic energy, and in the same way we may take the force to be exerted on this energy. Thus we also attribute the entropy not to the vacuum or to the ether, but to the radiating energy.

These are after all mere questions of nomenclature. More important is the question, whether the motion of the centre of inertia of an isolated system is really uniform. It is evident that we may assume this to be the case if we conceive the electromagnetic momentum to consist of a mass which is in motion. As an instance we will consider a stationary body with a mass M and a ray of light which is absorbed by it. The ray represents a quantum of momentum which we will denote by mc, the radiation propagating with the velocity c. When the ray is absorbed, the total momentum must remain constant. Now we can make two different assumptions. In the first place that of POINCARÉ [1]), who assumed that the mass M obtained a velocity v, so that $Mv = mc$. The uniformity, however, of the centre of inertia required the following rather startling assumption about the mass m: when the radiation is absorbed the mass m is stopped, it is however not annihilated, but becomes stationary at that place where the energy has been absorbed. The body which has absorbed the energy however moves in the mean time away from that place. POINCARÉ himself declares that a physical meaning cannot be ascribed to this theory.

Another possible assumption was proposed by the present writer also in 1900 in defending his theses on the occasion of his promotion to the degree of doctor. This assumption consists in this, that we imagine the mass m to remain in the body which has absorbed the energy. This body would then obtain a velocity v', which is determined by the equation $(M + m)\, v' = mc$. This assumption involves a hypothesis with a very decided physical meaning, namely that the mass of a body depends on its energy. In 1900 however there seemed to

1) H. POINCARÉ. Livre Jubilaire dédié à H. A. LORENTZ p. 252 Anno 1900.

be little reason to doubt the constancy of the mass of the bodies. I therefore felt obliged to reject this hypothesis and with it the law of the uniformity of the motion of the centre of inertia, and the law action = reaction.

The electron theory, however, has since that time given rise to doubts as to the constancy of the mass of the bodies, and EINSTEIN [1]) has moreover shown that LORENTZ's theory of relativity requires in some cases that we ascribe to the bodies a mass varying with their energy. It seemed therefore desirable to return to the idea rejected in 1900 and to ascribe a mass to the energy, and that as well with POINCARÉ for the case that the energy moves in the electromagnetic field [2]) as with EINSTEIN for the case that it occurs in ponderable bodies. LAUE [3]) has given a general theory for this latter case. Although my results partially coincide with those of LAUE the following considerations will perhaps not be superfluous.

§ 2. In the first place we may deduce from the formulae:

$$\text{The current of energy} = \mathfrak{S}$$

$$\text{The momentum per unit volume} = \frac{1}{c_2}\,\mathfrak{S}$$

that the mass of a quantity of energy ε is equal to $\frac{1}{c_2}\,\varepsilon$. The velocity of this mass in the electromagnetic field may be assumed to be $\mathfrak{w} = \frac{\mathfrak{S}}{W}$. $W = \frac{1}{2}(\mathfrak{E}^2 + \mathfrak{H}^2)$ representing the density of the energy. I say it *may* be assumed to have that value, for we may also introduce another supposition namely that at a point several parts of the energy have different velocities. And in connection with § 5 this assumption has some advantages.

So if a ray of light moves in an electrostatic field, the above value of $\mathfrak{w}$ is different from c. We may however also assume that the light energy moves with the velocity c, and the other quantities of energy move with different velocities or are stationary. It is however immaterial which supposition is introduced, if only care be taken

[1]) A. EINSTEIN. Ann. d. Phys XVIII p. 639, 1905, and XXIII p. 371, 1907. Compare also G. NORDSTRÖM and M. ABRAHAM Phys. Zeitschr. X and XI Anno (1909 and 1910) and H. A. LORENTZ. Versl. Kon. Akad. Amst. Juni 1911, p. 87. (Still to be published in these Proceedings).

[2]) H. POINCARÉ l c. Compare also A. EINSTEIN. Ann. d. Phys. XX p. 627. 1906 and M. PLANCK. Ann. d. Phys. XXVI p. 1. 1908 and Phys. Zeitschr. IX p. 828. 1908.

[3]) M. LAUE. "Das Relativitätsprinzip." VIEWEG und SOHN. Braunschweig 1911. Also Ann. d. Phys. XXXV p. 524. 1911.

that $\Sigma W\mathfrak{w} = \mathfrak{S}$. The momentum has then also the right amount $\frac{1}{c_2}\mathfrak{S}$. In this paragraph I will assume for simplicity that at a given point all electromagnetic energy has the same velocity.

It is important to point out that c is the maximum value which can be assumed by $\mathfrak{w}$. This value is reached if $\mathfrak{E}$ and $\mathfrak{H}$ are equal and perpendicular to one another. In all other cases $\mathfrak{w}$ is smaller than c.

Finally I point out that the essential property of mass according to our considerations is that $m\mathfrak{v} =$ momentum. That moving mass involves a quantity of kinetic energy is not to be considered as essential [1]). It is not even generally true. When a body radiates energy in all directions, the energy (and therefore a part of the mass) which was originally in rest, is set in motion. This motion, however, is not connected with any kinetic energy, for the energy is constant. The fact that the motion of a body is accompanied with kinetic energy must therefore be considered as a secondary phenomenon. The energy is then not only set in motion, but also augmented. So the energy of an electrically charged conductor is augmented with an amount of magnetic energy when this conductor is moved, and something of the same kind must happen in other cases where kinetic energy occurs [2]).

We will now put the question: what are the forces which are exerted on the electromagnetic energy? We will start from the well known equation:

$$-\Delta V\, Div.\, p_x = \frac{d(\varrho \Delta V . \mathfrak{v}_x)}{dt} + \Delta V \frac{1}{c^2}\frac{\partial \mathfrak{S}_x}{\partial t} \quad . \quad . \quad . \quad (1)\ ^{3})$$

Here ΔV is an element of volume, ϱ the density of the material mass, $\mathfrak{v}$ its velocity and $\mathfrak{S}$ the Poynting vector. The symbol $\frac{\partial}{\partial t}$ indicates a partial differentiation with respect to the time with constant value of the coordinates, and

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathfrak{v}_x \frac{\partial}{\partial x} + \mathfrak{v}_y \frac{\partial}{\partial y} + \mathfrak{v}_z \frac{\partial}{\partial z}$$

Finally represents

1) This has already been remarked by LAUE.

2) Far from denying the existence of the ether, I should be inclined to account for all the inertia by the action of the medium Cf. Phys. Zeitschr. p. 600. 1911.

3) In accordance with LAUE a pressure is represented by a positive, a tension by a negative value of p.

$$Div.\, p_x = \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{xy}}{\partial y} + \frac{\partial p_{xz}}{\partial z}$$

$$p_{xx} = \tfrac{1}{2}(\mathfrak{E}^2 + \mathfrak{H}^2) - \mathfrak{E}^2_x - \mathfrak{H}^2_x$$

$$p_{xy} = -(\mathfrak{E}_x\, \mathfrak{E}_y + \mathfrak{H}_x\, \mathfrak{H}_y)$$

$$p_{xz} = -(\mathfrak{E}_x\, \mathfrak{E}_z + \mathfrak{H}_x\, \mathfrak{H}_z).$$

In order to introduce the forces which act on the medium energy we will transform the righthand member of equation (1) in such a way that the differential quotient with $\frac{\partial}{\partial t}$ no longer occurs in it, but a differential quotient with $\frac{d'}{dt} = \frac{\partial}{\partial t} + \mathfrak{w}_x \frac{\partial}{\partial_x} + \mathfrak{w}_y \frac{\partial}{\partial_y} + \mathfrak{w}_z \frac{\partial}{\partial_z}$ This can be done as follows. We put

$$\frac{1}{c^2}\,\mathfrak{S} = \varrho' \mathfrak{w}$$

Then equation (1) can be written:

$$-\Delta V\, Div\, p_x = \frac{d(\varrho \mathfrak{v}_x \Delta V)}{dt} + \frac{d'(\varrho' \mathfrak{w}_x \Delta V)}{d't} -$$

$$- \varrho' \mathfrak{w}_x \frac{d'\Delta V}{d't} - \Delta V\left(\mathfrak{w}_x \frac{\partial \varrho' \mathfrak{w}_x}{\partial x} + \mathfrak{w}_y \frac{\partial \varrho' \mathfrak{w}_x}{\partial y} + \mathfrak{w}_z \frac{\partial \varrho' \mathfrak{w}_x}{\partial z} \right);$$

If we now put:

$$p'_{xx} = \varrho' \mathfrak{w}_x^2 = \frac{\mathfrak{S}_x^2}{c^4\, \varrho'} = \frac{\mathfrak{S}_x^2}{c^2\, W}$$

$$p'_{xy} = \varrho'\, \mathfrak{w}_x\, \mathfrak{w}_y = \frac{\mathfrak{S}_x\, \mathfrak{S}_y}{c^4\, \varrho'} = \frac{\mathfrak{S}_x\, \mathfrak{S}_y}{c^2\, W}$$

$$p'_{xz} = \varrho'\, \mathfrak{w}_x\, \mathfrak{w}_z = \frac{\mathfrak{S}_x\, \mathfrak{S}_z}{c^4\, \varrho'} = \frac{\mathfrak{S}_x\, \mathfrak{S}_z}{c^2\, W}$$

and $p - p' = t$ (i. e. $p_{xx} - p'_{xx} = t_{xx}$ etc.), and we add $\Delta V\, Div\, p'_x$ to both members of the equation, then we get:

$$-\Delta V\, Div\, t_x = \frac{d(\varrho \mathfrak{v}_x \Delta V)}{dt} + \frac{d'(\varrho' \mathfrak{w}_x \Delta V)}{d't} -$$

$$- \varrho'\, \mathfrak{w}_x \frac{d'\,\Delta V}{d't} + \varrho'\, \mathfrak{w}_x\, \Delta V\, Div\, \mathfrak{w}.$$

As however $\frac{d'\Delta V}{d't} = \Delta V\, Div\, \mathfrak{w}$, the last two terms cancel each other. So we get an equation the left hand member of which may be interpreted to represent the force which is exerted by the tensions t on the volume element ΔV, whereas the righthand member represents the increase of the momentum of the masses $\varrho \Delta V$ and $\varrho' \Delta V$.

Thus we are led to consider the elements of t as the quantities which determine the tensions in the medium. In the electrostatic and in the purely magnetic field these expressions agree with those given by MAXWELL. But in the general case they differ from those values. In a plane wave for instance the tension in the direction of propagation becomes zero. At first sight this may seem strange. For MAXWELL deduced the existence of the pressure of the radiation from his value of the tensions, and it appears that in putting $t_{xx} = 0$ (x being the direction of propagation) we deny the existence of that pressure. Yet this is not the case. For often we deduce the existence of the pressure of radiation from the momentum of the electromagnetic field without making use of the tensions. Properly speaking these two explanations of the pressure are contradictory; or at least one of them is superfluous. If both the tension and the momentum existed in the medium, the effect of these two causes ought to be added and we should find the double value for the pressure.

This difficulty does not exist if we ascribe the above values to the tensions. According to them a force exerted on a body is to be ascribed either to a tension or to the momentum of the medium. And if they both exist, their effect must be added. Let us consider a ray of light reflected on a perfect mirror. In the ray we do not assume any longitudinal tension, but at the surface of the mirror the normal component of $\mathfrak{S}$ is zero, and our expression for the tension is by no means zero, but coincides with that of MAXWELL. The effect of a ray of light on a mirror is therefore quite analogous to the effect of a jet of water on a surface by which it is thrown back. In the jet there need not be any pressure, but on the surface where the water is thrown back, a pressure does exist.

The tensions t, which we introduced, are therefore quite analogous to elastic tensions in bodies; the tensions of MAXWELL on the other hand are analogous to the *absolute* tensions, as LAUE calls them, i. e. of those quantities whose divergence is equal to the change in momentum of a *stationary* element of volume. This change is occasioned by two causes: 1st the tensions t, 2nd the transport of momentum through the surfaces of the volume element.

The result of our general considerations is this, that we — it is true — deny the existence of bodies with a constant mass, and that our assumptions differ in this respect from those of classical mechanics. But on the other hand the law of conservation of energy warrants that the total amount of mass is constant, so that the only difference is that we assume that the mass can be transferred with the energy from one body to another. Moreover we have reassumed the law action = reaction and the law of the uniform motion of the centre of inertia. Comparing these assumptions with those of the older theory of electrons, where the total mass was a variable quantity, it appears that we by no means deviate farther from classical mechanics, but rather that we return to it.

§ 3. Let us now consider a special case: a body is set in motion by a force $\mathfrak{K}$. We will assume

$$\frac{dm}{dt} = \frac{1}{c^2}\mathfrak{v}\mathfrak{K} \text{ and } \mathfrak{K} = m\frac{d\mathfrak{v}}{dt} + \mathfrak{v}\frac{dm}{dt}$$

From this assumption follows:

$$\frac{dm}{m} = \frac{\frac{1}{c^2}\,\mathfrak{v}\,d\mathfrak{v}}{1-\frac{\mathfrak{v}^2}{c^2}}$$

or

$$l(m) = -\tfrac{1}{2}\,l\left(1 - \frac{\mathfrak{v}^2}{c^2}\right) + C$$

Writing $l(m_0)$ for c, we find:

$$m = \frac{m_0}{\sqrt{1 - \frac{\mathfrak{v}^2}{c^2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Without making use of the theory of relativity we find therefore the well-known expression derived by LORENTZ [1]) in his ingenious paper in which he drew up that theory. Perhaps we may be astonished to find this relation without introducing the LORENTZ contraction, whereas LORENTZ derived it for bodies which do undergo this contraction. In order to explain this fact we observe that the above deduction is always applicable, if the force $\mathfrak{K}$ represents the only change of the energy of the body. And this is the case, 1st if the shape of the body is invariable, 2nd if the body undergoes the LORENTZ contraction according to the theory of relativity. For according to this theory the contracted form is the form of equilibrium for the moving body. A virtual change of form, therefore, does not require any work, and if a body is accelerated quasistationarily, no work is expended for the change of form. For an electrically charged body e. g. the negative work done by the electrical forces when the body contracts will be compensated by positive work of other forces (which we will call elastic forces).

[1]) H. A. LORENTZ. These proceedings VI, p. 809, Anno 1904.

This observation throws a new light on the signification of the well-known experiments of KAUFMANN (BUCHERER, HUPKA). These experiments are carried out with a purpose to investigate whether the electrons contract when moving. We here see however that, even if the accuracy of formula (2) is perfectly confirmed by experiments of this kind, this by no means proves the existence of the contraction. What really can be decided by these experiments is whether we have rightly attributed mass to the energy.

In order to deduce equation (2) we have assumed that the increment of the mass is equal to $\frac{1}{c^2}$ the increment of the energy. We are now inclined to ask whether also $m_0 = \frac{1}{c^2}\varepsilon_0$ ($\varepsilon_0 =$ the energy of the body when its velocity is zero). Specially we will put this question for electrons with surface charge. For the electromagnetic energy and the electromagnetic momentum we find respectively:

$$\varepsilon' = \frac{c^2 + \frac{1}{3}\mathfrak{v}^2}{c^2\sqrt{1-\frac{\mathfrak{v}^2}{c^2}}}\,\varepsilon'_0 \quad \text{and} \quad \mathfrak{G}' = \frac{4}{3}\,\frac{\mathfrak{v}}{c^2\sqrt{1-\frac{\mathfrak{v}^2}{c^2}}}\,\varepsilon'_0$$

ε'_0 representing the electrostatic energy of the stationary electron. These values do not agree with the formulae:

$$\varepsilon = \frac{\varepsilon_0}{\sqrt{1-\frac{\mathfrak{v}^2}{c^2}}} \quad \text{and} \quad \mathfrak{G} = \frac{1}{c^2}\,\varepsilon\mathfrak{v}\,.$$

but it does not follow that these formulae would not be satisfied if we had taken the total energy and the total momentum instead of ε' and $\mathfrak{G}'$. It is namely known that an electron has besides its electromagnetic energy, still energy of another kind [1]) (elastic energy), in consequence of which its mass and its momentum must be augmented by a positive term. But there is another reason why $\mathfrak{G}'$ must be diminished by a certain amount in order to find the total momentum. For inside the electron is an amount of momentum whose direction is opposite to the direction of the motion of the electron. To prove this we will investigate the vector of POYNTING when the electron moves in the direction of the positive X-axis. At the half of the electron turned towards the positive X-axis this vector is directed inward, at the half directed towards $-X$ it is directed outward;

[1]) Comp. i. a. H. A. LORENTZ. The theory of electrons p. 113 and 114, where also the remarks of POINCARÉ and ABRAHAM referring to this are discussed.

in the electron it is zero. The continuity of the motion of the energy requires that in the electron the transport of energy in the direction $-X$ takes place, and that therefore also an amount of momentum in that direction exists. This transport of energy in the electron is occasioned by the elastic forces. In the electron exists namely a tension, and this is always accompanied with a transport of energy opposite to the motion of the body, in the same way as a pressure is accompanied with a transport in the direction of the motion.

So we see that $\mathfrak{G}'$ and ε' must be augmented by several amounts which are at present unknown. It is therefore impossible to decide whether the equation $m_0 = \frac{1}{c^2}\,\varepsilon_0$ is satisfied. It is not even certain that this question has a real meaning. For in mechanics the energy is never perfectly determined, but contains an arbitrary constant. And though for some kinds of energy no reasonable doubt can exist as to the absolute amount, as for the kinetic, the electric, and the magnetic energy, it is by no means certain that for all kinds of energy we have a sufficient reason for the determination of the zero of energy. So we must content ourselves with observing that it is certainly also impossible to prove that the equation does not hold good.

The explanation of the relation between the energy, the mass, and the momentum of a moving body given here differs from that of EINSTEIN [1]), who assumes that the energy of a moving body varies if a system of equal and opposite forces is applied, although they influence neither the velocity nor the shape of the body and accordingly do not change the energy when evaluated from a system of coordinates which shares the motion of the body. According to LORENTZ [2]) these forces bring about also a variation of the momentum.

It appears however to me that this view cannot be maintained. In the first place the existence of a rigid body is assumed, and the existence of such a body would be at variance with the fundamental hypothesis of the theory of relativity [3]). But the increase of energy and momentum would not be found even if we assumed the existence of such a body.

For a body cannot be rigid with respect to every system of coordinates. If it has that quality with respect to a coordinate system which shares its motion, it cannot have it with respect to other coordinate systems. A disturbance which propagates with infinite velocity when evaluated from a system which shares the motion of

1) A. EINSTEIN, Ann. d. Phys. XXIII, p. 371. 1907.

2) H. A. LORENTZ. Versl. Kon. Ak. Amst. Juni 1911, p. 95.

3) M. LAUE. Phys. Zeitschr. 12, p. 48, Anno 1911.

the body, will propagate with other velocities when evaluated from a system relative to which the body moves with a velocity $\mathfrak{v}$. Evaluated from such a system the velocity of propagation in the direction of the motion is $\frac{c^2}{\mathfrak{v}}$, in the opposite direction it is $-\frac{c^2}{\mathfrak{v}}$.

Let us now take a rod whose ends will be called A and B. In A and B two equal and opposite forces are applied. These forces are applied at the same time when evaluated from a system relative to which the rod is in rest. An observer relative to whom the rod moves in the direction from A towards B will find that the force in A is applied earlier than that in B. Call t' the moment in which the force in A is applied, then he will find that the force in B is applied at the moment $t' + \frac{\mathfrak{v}}{c^2} x'$. The energy and the momentum calculated by EINSTEIN and by LORENTZ are those quantities imparted to the body by the force A during the interval $\frac{\mathfrak{v}}{c_2} x'$, during which the force in B was not yet applied and could not cancel it. We have here however not yet taken into account that the effect of the force in B propagates in the rod with a negative velocity and that it is felt in A before it is applied in B. During the interval $\frac{\mathfrak{v}}{c^2} x'$, during which it is not yet applied, the force in B imparts notwithstanding energy and momentum to the body which exactly cancel those amounts which are imparted by the force in A.

We see here again that the assumption of the existence of rigid bodies leads in the theory of relativity to unacceptable conceptions. We are therefore induced to assume that every body is elastically compressible and that in such a way that the same law which holds for the propagation of light in moving media also applies to the propagation of elastic disturbances.

Let us apply to a body a system of equal and opposite forces, by which it is compressed, and if we then set it in motion, in consequence of which it contracts farther, then the forces will again do a certain amount of work when this contraction takes place. This is perfectly analogous to the case that we apply first a set of forces A, which compress a body, and afterwards another set B, which compress it still further. At this second compression the set A will again do some work. So it proves to be true that a set of equal and opposite forces changes the energy of a moving (and also of a stationary) body, but this energy is exclusively the consequence of the contraction and change of form of the body.

So it appears to me that the circumstances are much simpler than we should conclude from the rather startling conclusion of EINSTEIN, that forces which do not impart any change of velocity or shape to a body, yet would change its energy.

I lay some stress upon this point because it appears to me that everything in the theory of relativity may be interpreted in a much more rational and intelligible way than many people imagine. So the fact that according to the theory of relativity two velocities cannot be added in the ordinary way by means of the parallellogram is often thought to necessitate a new doctrine of kinematics. We must, however, take into account that velocities, measured by the same observer, may be added in the usual way. Only for velocities evaluated from coordinate systems moving with different velocities this is not the case. Those velocities, however, are measured with different units of length and time. And velocities measured with different units cannot directly be added. This was already the case according to the old doctrine of kinematics. For that reason we do not want a new one.

Neither is the LORENTZ contraction a sufficient reason to speak of a new doctrine of kinematics. It appears to me that the best way to formulate the discovery of LORENTZ is to say, that when a body is set in motion, it experiences forces which try to make it contract in the wellknown manner. It is however possible that those forces are cancelled by other forces, and then the contraction does not take place. So the contraction cannot take place when a body rotates; a beginning contraction is in this case opposed by elastic forces.

In the same way we formulate the law of NEWTON by saying, that two masses at a distance r attract each other with a force $f\frac{m_1 m_2}{r^2}$. Whether they will obtain the corresponding accelerations depends upon the possible existence of other forces which perhaps cancel the Newtonian force. So it appears to me that the law of LORENTZ concerning the contraction no more belongs to the region of kinematics than the law of NEWTON concerning gravitation.

§ 4. *Mutual Mass.* Let us imagine two electrons with equal charges e but of opposite sign, and both with a total mass m. Their distance be r. Then we may distinguish three masses: m in the centre of each molecule, and a mass $m_{12} = \frac{e^2}{4\pi r}$, which really resides in the field, but which for many purposes may be thought to be concentrated in its centre of inertia i. e. in the point halfway between the two electrons. If we impart to this system a velocity $\mathfrak{v}$, then the momentum will be $(2m + m_{12})\,\mathfrak{v}$ [1]. If one of the electrons is set in motion and the other remains in rest, the momentum will be $(m + \frac{1}{2} m_{12})\,\mathfrak{v}$, for the mass m_{12}, remaining halfway between the two electrons, moves with a velocity $\frac{1}{2}\,\mathfrak{v}$. This however is only true for quasi stationary motion, and we must keep in view that the requirements for quasi stationary motion are in this case by no means so easy to be fulfilled as in the case of a single electron. If e. g. the electron vibrates with a wavelength $< r$, then the mass residing in the field and contributing to m_{12} cannot be assumed to have everywhere the velocity $\frac{1}{2}\,\mathfrak{v}$. This mass therefore may not be thought to be concentrated in the centre of inertia and the mass of the electron may not be augmented with $\frac{1}{2}\,m_{12}$.

Let us consider electrons on the sun. They have a greater potential energy than those on earth. Are we justified in ascribing a greater mass to them and in expecting that the period with which they vibrate will accordingly be greater? [2] In order to answer this question we must investigate whether this potential energy shares the motion of the electrons or not. If we assume that gravity propagates with infinite velocity, we shall have to assume that the gravitational energy moves with the electron, and then the mass of electrons on the sun would really be greater than that on earth. If on the other hand gravitation propagates with the velocity of light this conclusion would not be justified.

If the shifting of the spectral lines in the light of the sun as expected by EINSTEIN therefore does not occur, this fact does not prove that we are wrong in ascribing a mass to the energy. But it proves that gravitation propagates with finite velocity. If on the other hand the effect did occur, it would show that gravitation propagates with infinite velocity or at least with a velocity which is very great compared with that of light. The effect would therefore be in direct contradiction to the hypothesis of relativity.

§ 5. We will still consider the following special case. A rod of 1 cm² cross section experiences a pressure t_{xx} in the direction of its length. We will call the ends of the rod A and B and choose the direction from A to B as positive X-axis. The rod moves with a velocity $\mathfrak{v}$ in this direction. W be the density of the energy of the rod. The amount of energy which passes through a stationary plane of unit

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[1]) This agrees with the calculations of L. SILBERSTEIN, Phys. Zeitschr. XII, p. 87, 1911.

[2]) A. EINSTEIN, Jahrbuch der Radioakt. u. Elektr. IV, p. 459.

area perpendicular to the X-axis, would according to classical mechanics be:

$$\mathfrak{S}_x = (W + t_{xx})\, \mathfrak{v}.$$

According to our considerations the question would be a little less simple. We shall have to conceive W to be separated into three parts: W_1 moving along with the rod with the velocity $\mathfrak{v}$, W_2 moving with the velocity $\mathfrak{w}_2$ in the $+X$ and W_3 moving with the velocity $\mathfrak{w}_3$ in the $-X$-direction. We are inclined to suppose that $W_2 + W_2$ will be the elastic energy which is a consequence of the compression, and that $\mathfrak{w}_2$ and $\mathfrak{w}_3$ are the velocities with which a perturbation propagates in the moving rod according to a stationary observer. If we put $\mathfrak{v} = 0$, we get $\mathfrak{w}_2 = \mathfrak{w}_3$ and the assumption, which I introduce here is, that also in this case the elastic energy is not in rest, but that we cannot ascertain its motion because two equal currents of energy move in opposite directions. If we again impart the velocity $\mathfrak{v}$, both currents will be changed, but in a different degree, in consequence of which a current of energy in a definite direction can be ascertained. These considerations are confirmed by the fact that the energy transported by the tension through the moving rod cannot move with a velocity $\mathfrak{v}$. So it cannot be transformed into rest together with the rod.

For our purpose however it is not necessary to determine the values of W_2, W_3, $\mathfrak{w}_2$, and $\mathfrak{w}_3$. We certainly may put:

$$\mathfrak{S}_x = W_1 \mathfrak{v} + W_2 \mathfrak{w}_2 - W_3 \mathfrak{w}_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The force exerted by the rod on a body against which its end B rests, may not simply be put equal to t_{xx}. For we must take into account that the rod contains two quantities of momentum: a quantity with a density $\frac{1}{c^2} W_2 \mathfrak{w}_2$ moving with the relative velocity $\mathfrak{w}_2 - \mathfrak{v}$ towards the end B, and a quantity with a density $\frac{1}{c^2} W_3 \mathfrak{w}_3$ moving with a relative velocity $\mathfrak{w}_3 + \mathfrak{v}$ away from it. The force exerted on the end of the rod is therefore:

$$\tau_{xx}{}^{1)} = t_{xx} + \frac{1}{c^2} (\mathfrak{w}_2 - \mathfrak{v})\, W_2 \mathfrak{w}_2 + \frac{1}{c^2} (\mathfrak{w}_3 + \mathfrak{v})\, W_3 \mathfrak{w}_3 \,.$$

[1]) It is obvious that in principle t_{xx} has the closest analogy to what is ordinarily called elastic tension. The quantity t_{xx}, however cannot be measured and in so far τ_{xx}, which represents the force as it is measured, is a more important quantity. An easy calculation shows that t_{xx} is the same quantity as the quantity t_{xx} of LAUE. The tensor t is symmetrical, whereas τ (t in the notation of LAUE) is an asymmetrical tensor.

Therefore we find for the energy which passes the stationary plane:

$$\mathfrak{S}_x = (W + \tau_{xx})\,\mathfrak{v}\,.$$

We will introduce in this equation the quantity p_{xx} which is equal to:

$$p_{xx} = t_{xx} + \frac{1}{c^2}\,(W_1\mathfrak{v}^2 + W_2\mathfrak{w}_2{}^2 + W_3\mathfrak{w}_3{}^2)$$

We easily find:

$$\mathfrak{S}_x = (W + p_{xx})\,\mathfrak{v} - (W_1\mathfrak{v} + W_2\mathfrak{w}_2 - W_3\mathfrak{w}_3)\,\frac{\mathfrak{v}^2}{c^2}$$

Taking equation (3) into account and putting $\frac{\mathfrak{v}^2}{c^2} = \beta^2$ we find:

$$\mathfrak{S}_x\,(1+\beta^2) = (W + p_{xx})\,\mathfrak{v}\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It is important to remark that this equation, deduced here without making use ot the theory of relativity, can also be derived from the equations (102) of LAUE [1]):

$$\mathfrak{S}_x = \frac{(1+\beta^2)\,\mathfrak{S}'_x + \mathfrak{v}\,(p'_{xx} + W')}{1-\beta^2}$$

$$W = \frac{W' + \beta^2 p'_{xx} + 2\,\frac{\mathfrak{v}}{c^2}\,\mathfrak{S}'_x}{1-\beta^2}$$

$$p_{xx} = \frac{p'_{xx} + \beta^2 W' + 2\,\frac{\mathfrak{v}}{c^2}\,\mathfrak{S}'_x}{1-\beta^2}$$

If namely we imagine the rod to rest relative to the accentuated system, then $\mathfrak{S}'_x = 0$. We find then equation (4) by eliminating p'_{xx} and W'.

In the same way we can discuss the case that the rod lies parallel to the Y-axis and that a force in the $+X$ direction is applied in the middle of the rod. In the ends of the rod two equal forces act in the X-direction, which together exactly balance the force in the middle. This system moves with a velocity $\mathfrak{v}$ in the X-direction. For this case both ways of calculating yield

$$\mathfrak{S}_{xy} = \mathfrak{v}\,p_{xy}\,.$$

So we see that it is possible to derive several conclusions from the law of the uniform motion of the centre of inertia which usually are derived from the theory of relativity. In principle the two ways of deducing them are equally justified. In both we start from laws which are proved to hold good for some regions of observations and

[1]) M. LAUE. Das Relativitätsprinzip p. 87.

apply them to phenomena belonging to regions for which their applicability has not been experimentally proved. A generalization of this kind is of course hypothetical. The fact that the two deductions here yield the same results will probably be considered as a confirmation of the validity of the hypotheses.

The question suggests itself whether the hypothesis concerning the mass of the energy is not only in the special cases treated above, but with perfect generality in agreement with the theory of relativity.

The most general method to solve this question seems to be that suggested by LAUE. His argument comes in principle to the following. We will take the 16 quantities

$$\begin{array}{cccc} p_{xx} & p_{xy} & p_{xz} & \frac{i}{c}\mathfrak{S}_x \\ p_{yx} & p_{yy} & p_{yz} & \frac{i}{c}\mathfrak{S}_y \\ p_{zx} & p_{zy} & p_{zz} & \frac{i}{c}\mathfrak{S}_z \\ \frac{i}{c}\mathfrak{S}_x & \frac{i}{c}\mathfrak{S}_y & \frac{i}{c}\mathfrak{S}_z & -W \end{array}$$

and differentiate the four quantities of one horizontal row respectively according to x, y, z and $i\,c\,t$ and put the sum of the four terms thus obtained equal to zero [1]). The four horizontal rows yield four equations of this kind; the first three equations determine the increase of the momentum, the fourth equation is an expression of the law of conservation of energy. We have chosen for the elements of the fourth vertical column the same quantities which occur in the forth horizontal row. By making this choice we have introduced the hypothesis of the mass of the energy.

LAUE now postulates that these 16 quantities, when we make use of a moving coordinate system will be transformed as the elements of a fourdimensional tensor, (in this way the equations (102) cited above are found) and so he postulates that the hypothesis concerning the mass of the energy agrees with the hypothesis of relativity. The question must however be put: have we a right to postulate that the quantities will transform in the way given by LAUE? We must keep in view that we are dealing with derived quantities. From the equation $\mathfrak{S}_x = \Sigma \varrho \mathfrak{w}_x$ e. g. it appears that, if we have already assumed in what way ϱ and $\mathfrak{w}_x$ will be transformed, the formula for the

[1]) The choice zero for the righthand member of the equations is an expression of the hypothesis that no "actio in distans" occurs. LAUE does not introduce this hypothesis, his equations therefore have a righthand member differing from zero.

transformation of $\mathfrak{S}_r$ is determined. The considerations of LAUE, therefore, are only justified if he can show that it is possible to attribute to the different kinds of energy a velocity whose value is such that the transformation of ϱ and $\mathfrak{w}$ yields for $\Sigma\varrho\mathfrak{w}$, the same formula as he postulates for the transformation of $\mathfrak{S}$. Considerations of the same kind apply to the quantities p_{ix} and W.

Postscript. Perhaps I have not always been consistent in the use of the words *force* and *tension*. I have thought for a moment that we could do without these notions altogether, and that we could account for every change in the momentum in a volume-element by means of the transport of momentum through its surfaces. But then we are checked by some difficulties. The nomenclature most accurate in principle is of course to use the word *force* only for that change of momentum for which we cannot account by a transport of momentum. But it seems to be impossible to perform the separation between the effect of forces and of transport in an unambiguous way. In § 5 e g. I called τ_{1x} the force exerted on the rod. This is accurate if the quantities of energy W_2 and W_3 are partially reflected at the ends of the rod. If they, however, pass the ends and flow into the other body, t_{x1} would represent the force and $\tau_{x2}-t_{ix}$ would represent a quantity of momentum which is imparted to the rod by means of transport. It seems to be impossible to find good reasons for a choice between those two conceptions. It is after all immaterial to which of these quantities we will apply the name of force.

In the same way we may ask whether we will define the force by the equation

$$\mathfrak{K} = m\frac{d\mathfrak{v}}{dt} + \mathfrak{v}\frac{dm}{dt}$$

or by

$$K = m\frac{d\mathfrak{v}}{dt}.$$

The force adds energy and so also mass to the body, and the value which we ascribed to the force will depend on the momentum which this new mass had, before it was added to the body. If we think that it was then stationary, we shall undoubtedly call $m\frac{d\mathfrak{v}}{dt}$ the force exerted on the "old mass" and $\mathfrak{v}\frac{dm}{dt}$ that exerted on the "new mass". But if this new mass had a velocity $\mathfrak{w}$ before it was absorbed by the body, we shall ascribe another value to the force properly speaking, but say that the momentum of the body is also changed in consequence of the momentum of the "new mass", which is added to it. The value of $\mathfrak{w}$ being unknown in many cases it will be impossible to perform the separation between force and transport, and we will simply call $\mathfrak{K}$ the force exerted on the body.

In some cases however it will be useful to take the difference between force and transport into account. An electrical condensor e. g. sharing the motion of the earth is suddenly charged, heat is generated in a wire according to the law of JOULE, or a body receives heat from another body. The momentum of these bodies is increased. Is a force required in order to keep the motion of these bodies uniform, and will they suffer a retardation when this force is not applied? The answer to this question will undoubtedly be: If they receive their energy from a stationary source, this will be the case, but not if they receive their energy from a source moving along with the earth.

Mathematics. — *"A bilinear congruence of quartic twisted curves of the first species."* By Prof. JAN DE VRIES.

1. If we allow each quadric Q^2 of a pencil (Q^2) to bisect each surface of a second pencil $(Q^2)'$, a congruence Γ is formed of biquadratic twisted curves, ϱ^4, of *order one*; for through an arbitrary point P passes *one* ϱ^4, the intersection of the two Q^2, which is determined by P in the two pencils.

An arbitrary line l is cut by the pencils into two quadratic involutions, which have, in general, *one* pair in common; the congruence Γ is thus of *class one* (an arbitrary line is bisecant of *one* curve).

2. The base-curves β^4 and β'^4 of the pencils are *singular curves*; each of their points bears ∞^1 curves ϱ^4. As β^4 and ϱ^4 lie on a Q^2, they cut each other in eight points. So we can determine Γ also as the system of the ϱ^4 cutting each of two given biquadratic twisted curves in eight points.

Each bisecant b of β^4 is a *singular line*. For the surface Q^2 determined by a point of b contains b and the pencil $(Q^2)'$ cuts b in the pairs of an involution, so that b is bisecant of ∞^1 curves ϱ^4.

3. Besides the two congruences (2,6) of singular bisecants determined by β^4 and β'^4, the congruence Γ has a congruence of singular bisecants on which (Q^2) and $(Q^2)'$ describe *the same* involution.