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changed in consequence of the momentum of the "new mass", which is added to it. The value of $\mathfrak{w}$ being unknown in many cases it will be impossible to perform the separation between force and transport, and we will simply call $\mathfrak{K}$ the force exerted on the body.

In some cases however it will be useful to take the difference between force and transport into account. An electrical condensor e. g. sharing the motion of the earth is suddenly charged, heat is generated in a wire according to the law of JOULE, or a body receives heat from another body. The momentum of these bodies is increased. Is a force required in order to keep the motion of these bodies uniform, and will they suffer a retardation when this force is not applied? The answer to this question will undoubtedly be: If they receive their energy from a stationary source, this will be the case, but not if they receive their energy from a source moving along with the earth.

Mathematics. — *"A bilinear congruence of quartic twisted curves of the first species."* By Prof. JAN DE VRIES.

1. If we allow each quadric Q^2 of a pencil (Q^2) to bisect each surface of a second pencil $(Q^2)'$, a congruence Γ is formed of biquadratic twisted curves, ϱ^4, of *order one*; for through an arbitrary point P passes *one* ϱ^4, the intersection of the two Q^2, which is determined by P in the two pencils.

An arbitrary line l is cut by the pencils into two quadratic involutions, which have, in general, *one* pair in common; the congruence Γ is thus of *class one* (an arbitrary line is bisecant of *one* curve).

2. The base-curves β^4 and β'^4 of the pencils are *singular curves*; each of their points bears ∞^1 curves ϱ^4. As β^4 and ϱ^4 lie on a Q^2, they cut each other in eight points. So we can determine Γ also as the system of the ϱ^4 cutting each of two given biquadratic twisted curves in eight points.

Each bisecant b of β^4 is a *singular line*. For the surface Q^2 determined by a point of b contains b and the pencil $(Q^2)'$ cuts b in the pairs of an involution, so that b is bisecant of ∞^1 curves ϱ^4.

3. Besides the two congruences (2,6) of singular bisecants determined by β^4 and β'^4, the congruence Γ has a congruence of singular bisecants on which (Q^2) and $(Q^2)'$ describe *the same* involution.

The double points of such an involution are harmonically separated by two surfaces chosen arbitrarily out of (Q^2) and two surfaces taken arbitrarily out of $(Q^2)'$. The locus of those double points is therefore the surface of JACOBI belonging to those four surfaces. If $a^2_x = 0$, $b^2_x = 0$, $c^2_x = 0$, $d^2_x = 0$ are the equations of the indicated surfaces, then for a pair of double points X, Y we have

$$a_x a_y = 0, \quad b_x b_y = 0, \quad c_x c_y = 0, \quad d_x d_y = 0, \quad . \quad . \quad . \quad (1)$$

so that the surface of JACOBI is represented by

$$\begin{vmatrix} a_1 a_x & b_1 b_x & c_1 c_x & d_1 d_x \\ a_2 a_x & b_2 b_x & c_2 c_x & d_2 d_x \\ a_3 a_x & b_3 b_x & c_3 c_x & d_3 d_x \\ a_4 a_x & b_4 b_x & c_4 c_x & d_4 d_x \end{vmatrix} = 0 .$$

To find the number of pairs X, Y lying in a plane we put in (1) $x_4 = 0$, $y_4 = 0$. By elimination of y_1, y_2, y_3 we then find the conditions

$$\left\| \begin{matrix} a_1 a_x & b_1 b_x & c_1 c_x & d_1 d_x \\ a_2 a_x & b_2 b_x & c_2 c_x & d_2 d_x \\ a_3 a_x & b_3 b_x & c_3 c_x & d_3 d_x \end{matrix} \right\| = 0 \quad . \quad . \quad . \quad . \quad (2)$$

The determinants arising from this matrix, if one omits the third or the fourth column, disappear for the points of intersection of two twisted curves; to these belong the three points, for which the matrix of the first two columns disappears. The four twisted curves indicated by (2) have thus six points in common forming three pairs of X, Y. In an arbitrary plane lie therefore *three singular bisecants* of the second species.

From this ensues that the quadruple involution in which Γ is cut by any plane contains fifteen singular lines; this corresponds to a result obtained by me in another research [1]).

4. We consider the bisecants sent out by the curves ϱ^4 through a given point P and we determine the surface Π on which their points of intersection lie. The ϱ^4 passing through P is projected out of P by a cubic cone of which the generatrices touch Π in P; so P is a three fold point (triconic point) of Π. An arbitrary line through P is bisecant of *one* ϱ^4; so Π is a *surface of order five* with *triconic* point P.

The cone of contact for P can have with Π^5, besides the ϱ^4 through P, only straight lines in common. Hence through P pass *eleven singular bisecants.*

1) See my paper: "A quadruple involution in the plane and a triple involution connected with it" (Proc. of Amst. Vol. III p. 84).

To these belong *two* bisecants of β^4 and *two* of β'^4; the remaining *seven* are singular bisecants of the second species.

So the *singular bisecants* of Γ form *two congruences* (2,6) and *one congruence* (7,3).

5. The section of Π^5 with a plane through P is a curve with a triple point, hence of class fourteen sending eight tangents through P. The *tangents* of the ϱ^4 form therefore a *complex of order eight.*

The points of contact of the tangents drawn out of P lie evidently on a twisted curve of order *nine.*

This can be confirmed as follows. The points of contact of the tangents out of P to the surfaces Q^2 lie on a cubic surface, the "*polar-surface*" of P with respect to the pencil (Q^2). A second cubic surface contains the points of contact of the tangents out of P to the surfaces of $(Q^2)'$. Each point of intersection of the two polar surfaces determines a ϱ^4, of which the tangent passes through P; so the points of contact of the tangents drawn out of P lie on a ϱ^9.

6. The quadrics (Q^2) and $(Q^2)'$ are arranged in a correspondence (2,2) when two surfaces intersecting each other on the line l are made to correspond. This causes the points of a line m to be arranged in a correspondence (4,4); so m contains eight points each bearing two surfaces intersecting each other on l. From this it is clear that the curves ϱ^4 intersecting l form a surface Λ^8.

On a line intersecting β^4 the (4,4) is replaced by a (2,4); we conclude from this that β^4 and β'^4 are *nodal curves* of Λ^8.

The ϱ^4, too, having l as bisecant is a *nodal curve* of Λ^8.

7. A plane λ through l cuts Λ^8 still according to a curve λ^7 passing through the points of intersection of the nodal curve ϱ^4, having l as bisecant, with this line. In each of the remaining five points of intersection of l with λ^7 the plane λ is touched by a ϱ^4. The locus of the points of intersection of a given plane with curves of Γ is therefore a *curve of order five,* λ^5.

Evidently λ^5 is the curve of coincidences of the quadruple involution determined by Γ and it passes through the eight points, in which β^4 and β'^4 are intersected by λ [1]).

This involution containing fifteen quadruples in which three points coincide [2]), an arbitrary plane is osculated by fifteen curves of Γ.

Furtheron four quadruples consist each of two coincidences [2]); so each plane is *bitangential* plane for *four* curves of Γ.

1) See loc. page 82. 2) See loc. page 83.

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If we allow λ to revolve round l, then the curve λ^5 describes a surface on which l is a single line. For through each point of l passes *one* ϱ^4, and one of the planes through the tangent of that point contains l. From this ensues that the locus of the points in which curves ϱ^4 can be touched by planes of a pencil with axis l is a *surface* of *order six*.

8. The curves ϱ^4, touching the plane λ in the points of the curve λ^5, intersect λ each in two points; the locus of those pairs of points is a curve of order fourteen [1]) (branchcurve) with quadruple points in the points of intersection of the base-curves β^4 and β'^4. The surface containing the curves ϱ^4 meant here, has thus with λ an intersection of order twenty-four, is therefore a surface A^{24} with quadrisecant curves β^4 and β'^4.

With a plane μ the surface A^{24} has a curve μ^{24} in common containing quadruple points in the points of intersection with β^4 and β'^4; these eight points lie on the curve μ^5, which is the locus of the points of contact of μ with curves of Γ. The two curves have $24\times5-8\times4=88$ points in common besides the base-points. So there are 88 curves ϱ^4 *touching two planes*.

9. To Γ belong ∞^1 curves σ^4 which contain a nodal point, because they are the intersections of two surfaces touching each other. According to a wellknown property [2]) the locus of the points of contact of two quadrics belonging to two given pencils is a twisted curve ϱ^{14}, cutting each of the two base-curves in 16 points.

On an arbitrary surface Q^2 lie therefore $2\times14-16=12$ points of intersection with as many surfaces Q'^2. The locus of the curves σ^4 is therefore generated by two quadratic pencils in correspondence (12, 12); consequently it is a surface Δ^{48} on which β^4 and β'^4 are *twelvefold* curves.

10. The intersection of Q^2 and Q'^2 breaks up into a line and a ϱ^3, when they pass through a common bisecant of β^4 and β'^4. The bisecants of these curves forming two congruences (2,6) the number of the common bisecants is $2\times2+6\times6=40$. So to Γ belong orty figures consisting of a cubic curve with one of its bisecants.

Through a ϱ^4 can be laid four cones belonging to the quadratic pencil having ϱ^4 as basis. The pencils determined by the ∞^2 curves

1) C. loc. p. 83.

2) See a.o. *Mineo, Rendiconti del Circolo matematico di Palermo*, XVII, 297.

of Γ form a system ∞^3 of quadrics; the corresponding cones have their vertices on the surface of JACOBI of the system. This surface contains ten lines, which are double lines of as many pairs of planes belonging to the system. From this ensues that Γ contains *ten* figures consisting each of two conics cutting each other twice.

Mathematics. — "*A quartic surface with twelve straight lines.*" By Prof. JAN DE VRIES.

1. We regard as given the three pairs of straight lines a, a'; b, b'; c, c'. Let t_a denote a transversal of a and a'; and let t_b and t_c have an analogous meaning. The points P sending out three transversals t_a, t_b, t_c lying in a plane, form a surface (P) of which we intend to determine the order.

First we notice that the six given lines belong to (P). For, if P is a point of c and Q the point of intersection of c' with the plane through the transversals t_a, t_b, the transversal $t_c \equiv PQ$ lies with t_a, t_b in a plane.

We can designate six other lines lying on (P), viz: the two transversals t_{ab}, t'_{ab} of the pairs a, a'; b, b' and the analogous lines t_{bc}, t'_{bc}; t_{ac}, t'_{ac}. For, t_b coincides with t_a for a point P on t_{ab}, so that t_a, t_b and t_c are complanar.

Let t_c be an arbitrary transversal of c, c', in a plane τ through t_c. The lines t_a and t_b lying in τ determine on t_c two points A and B which describe projective series of points when τ revolves; the two coincidences A and B are evidently points of (P). The points of intersection of t_c with c and c' also belonging to (P) the locus to be found is a *quartic surface.*

If we allow t_c to describe a pencil, whose vertex C lies on c, then the above mentioned coincidences describe a curve of order three; for, if C' is the point of intersection of c' with the plane of the lines t_a, t_b through C, then one of the coincidences $A \equiv B$ or $t_c \equiv CC'$ lies in C.

2. The surface is entirely determined by the ten lines a, a'; b, b'; c, c'; t_{ab}, t'_{ab}; t_{ac}, t'_{ac}. For, if on each one of the first six lines we take arbitrarily five points and on each of the remaining four lines *one* point then the quartic surface determined by those thirty-four points will contain the ten lines mentioned.

Being moreover as locus of the point P entirely determined *a quartic surface through the above-mentioned ten lines must contain two other lines* (*viz*: t_{bc}, t'_{bc}).