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of Γ form a system ∞^3 of quadrics; the corresponding cones have their vertices on the surface of JACOBI of the system. This surface contains ten lines, which are double lines of as many pairs of planes belonging to the system. From this ensues that Γ contains *ten* figures consisting each of two conics cutting each other twice.

Mathematics. — "*A quartic surface with twelve straight lines.*" By Prof. JAN DE VRIES.

1. We regard as given the three pairs of straight lines a, a'; b, b'; c, c'. Let t_a denote a transversal of a and a'; and let t_b and t_c have an analogous meaning. The points P sending out three transversals t_a, t_b, t_c lying in a plane, form a surface (P) of which we intend to determine the order.

First we notice that the six given lines belong to (P). For, if P is a point of c and Q the point of intersection of c' with the plane through the transversals t_a, t_b, the transversal $t_c \equiv PQ$ lies with t_a, t_b in a plane.

We can designate six other lines lying on (P), viz: the two transversals t_{ab}, t'_{ab} of the pairs a, a'; b, b' and the analogous lines t_{bc}, t'_{bc}; t_{ac}, t'_{ac}. For, t_b coincides with t_a for a point P on t_{ab}, so that t_a, t_b and t_c are complanar.

Let t_c be an arbitrary transversal of c, c', in a plane τ through t_c. The lines t_a and t_b lying in τ determine on t_c two points A and B which describe projective series of points when τ revolves; the two coincidences A and B are evidently points of (P). The points of intersection of t_c with c and c' also belonging to (P) the locus to be found is a *quartic surface.*

If we allow t_c to describe a pencil, whose vertex C lies on c, then the above mentioned coincidences describe a curve of order three; for, if C' is the point of intersection of c' with the plane of the lines t_a, t_b through C, then one of the coincidences $A \equiv B$ or $t_c \equiv CC'$ lies in C.

2. The surface is entirely determined by the ten lines a, a'; b, b'; c, c'; t_{ab}, t'_{ab}; t_{ac}, t'_{ac}. For, if on each one of the first six lines we take arbitrarily five points and on each of the remaining four lines *one* point then the quartic surface determined by those thirty-four points will contain the ten lines mentioned.

Being moreover as locus of the point P entirely determined *a quartic surface through the above-mentioned ten lines must contain two other lines* (*viz*: t_{bc}, t'_{bc}).

Such a surface can be regarded in *two ways* as surface (P). For the six lines a,a'; b,b'; c,c' can be found out of the six t_{bc}, t'_{bc}; t_{ac}, t'_{ac}; t_{ab}, t'_{ab}, in the same way as the second six out of the first. For, a,a' are the two transversals of t_{ab}, t'_{ab}; t_{ac}, t'_{ac}, etc. So the surface is at the same time the locus of the points P, which send out three complanar transversals to the pairs t_{ab}, t'_{ab}; t_{bc}, t'_{bc}; t_{ac}, t'_{ac}.

The points of intersection and the connecting planes of the 12 lines form evidently a configuration $(24_7, 24_7)$. Each of those planes intersects (P) in a conic; so the surface contains 24 conics.

3. The plane π of the transversals t_a, t_b, t_c envelops a surface of class four containing the same twelve lines. For, each plane through c contains a t_a and a t_b; the line connecting the point $t_a t_b$ with the point in which c' meets the plane is the corresponding transversal t_c. For a plane π through t_{ab} the corresponding point P lies in the point of intersection of t_{ab} with t_c lying in π.

The following confirms the fact that π is *a surface of class four*. If we let a plane ν revolve about the line l, the point of intersection N of the lines t_a, t_b lying in ν describe a twisted cubic which, considered as the intersection of the hyperboloids (laa') and (lbb'), has l as bisecant.

The transversal t_c lying in ν describes a hyperboloid passing through the points of intersection of the above mentioned twisted cubic with the bisecant l. In each of the remaining four common points three complanar lines t_a, t_b, t_c meet; hence the line l bears four planes π.

4. We now regard four pairs of lines a, a'; b, b'; c, c'; d, d' and we determine the locus of the points P for which the four transversals t_a, t_b, t_c, t_d lie in *one* plane.

The surfaces $(P)_{abc}$ and $(P)_{abd}$ have evidently the six lines a, a'; b, b'; t_{ab}, t'_{ab} in common.

For an arbitrary point of a the transversals t_b, t_c and t_d are not complanar; this *is* the case for the four points of intersection of (a) and $(P)_{bcd}$. Consequently a, a'; b, b' are quadrisecants of the twisted curve ϱ^{10} which $(P)_{abc}$ and $(P)_{abd}$ have still in common.

Moreover, t_{ab}, t'_{ab} are bisecants of ϱ^{10}; for, on each line, hence also on t_{ab}, lie two points for which the plane $t_c t_d$ passes through that line (see § 1).

Hence we may conclude that the locus of the points bearing four complanar transversals is *a twisted curve of order six* having the four given pairs of lines as *quadrisecants* and their six pairs of transversals as *bisecants*.

5. We finally regard five pairs of lines and determine how many points give five complanar transversals.

The surface $(P)_{abc}$ has forty points in common with the twisted curve ϱ_{abcd} found above. Sixteen of these lie on the four quadrisecants a, a'; b, b'; in each of those points t_a, t_b, t_c, t_d are complanar, but their plane does not contain t_e.

Then to those forty points belong the four points of intersection of the bisecants t_{ac}, t'_{ab} of ϱ with that curve; in such a point the plane $t_c t_d$ passes through the bisecant, but not through t_e.

Hence there are *twenty* points for which the *five* transversals lie in *one* plane.

This result can be confirmed as follows by applying the law of the permanency of the number.

If we substitute for each of the five pairs of lines a pair of intersecting lines and if A, B, C, D, E are the five points of intersection, $\alpha, \beta, \gamma, \delta, \varepsilon$ the five connecting planes, we then find one of the points P in the point of intersection of the plane ABC with the line $\delta\varepsilon$; for the lines PA, PB, PC are to be considered as transversals t_a, t_b, t_c, the traces of δ and ε as transversals t_d, t_e. Analogously the point $\alpha\beta$ satisfies the question; t_d and t_e connect it with D and E; t_a, t_b, t_c are the intersections of α, β, γ with the plane through $\alpha\beta\gamma$, D and E. In all we evidently find twenty points P.

6. In connection with § 2 we have still to notice that we can bring a quartic surface through six arbitrarily chosen lines and four of the thirty quadrisecants which they possess four by four. But such a surface will contain in general not more than these ten lines.

We can determine quartic surfaces also passing through a *bisextuple of a cubic surface*. For, each O^4 through the thirty points of intersection of the two sextuples must contain the twelve lines, as each line contains five points of O^4. Thus through a bisextuple pass ∞^4 surfaces O^4.

So we can find surfaces with thirteen lines; the thirteenth line must then intersect one of the lines of the bisextuple.

An O^4 with fourteen lines is found by drawing two lines, each of which rests on three of the twelve given lines and by making the surface to pass still through four points, two of which lie on each of those transversals.

If the lines of the bisextuple in wellknown notation are indicated with a_k, b_k and if l is a line in the plane $(a_1 b_2)$ cutting b_1, then an O^4 through two arbitrary points of l will contain not only this line, but moreover a fourteenth line complanar to l, a_1 and b_2

and intersecting a_2. As we can let O^4 pass still through two arbitrary points, there is a possibility of bringing through the bisextuple an O^4 with *sixteen* lines. To this end we have but to repeat the above consideration for e.g. the lines a_3, b_4, b_3.

7. An O^4 through a *hyperboloidical quadruple* contains a second quadruple consisting of four quadrisecants of the former. For, through an arbitrary point of the intersection of O^4 with the hyperboloid containing the given quadruple we can draw a line of the second system of the hyperboloid, which then contains five points of O^4 and lies therefore on O^4; the intersection of the two surfaces consists then of two hyperboloidical quadruples.

Let us suppose an O^4 to be laid through six lines a_k of which a_1, a_2, a_3, a_4 and at the same time a_1, a_2, a_5, a_6 lie hyperboloidically. The hyperboloids bearing these quadruples have still two lines t and t' in common which are evidently intersected by the six lines a and are therefore situated on O^4.

Besides these two transversals O^4 contains still two transversals of the first quadruple and two of the second. In all O^4 contains therefore twelve lines; they form a configuration in which the six transversals appear in the same manner as the six lines a. For, the six transversals form two hyperboloidical quadruples with a_1, a_2 as transversals to six lines.

It is evident that again ∞^4 surfaces O^4 can be made to pass through this configuration of twelve lines. So we can obtain an O^4 with fourteen lines by drawing a transversal of a_1, a_3, a_5 and a transversal of a_2, a_4, a_6, and by assuming on each of these lines two points through which we make O^4 pass.

The six lines a_k can be chosen also in such a way that they form *three* hyperboloidical quadruples. Let a_1, a_2, a_3, a_4 be such a quadruple, a_5 an arbitrary line. The hyperboloids $(a_1\, a_2\, a_3)$ and $(a_1\, a_2\, a_5)$ have still two lines t and t' in common resting on the five lines a. The hyperboloids $(a_1\, a_2\, a_5)$ and $(a_3\, a_4\, a_5)$ have now the lines a_5, t and t', therefore one line a_6 more, in common, resting on t, t'. Consequently also the quadruples a_1, a_2, a_5, a_6 and a_3, a_4, a_5, a_6 lie hyperboloidically.

Each surface O^4 containing this sextuple of lines passes at the same time through the two transversals t, t' and through the three pair of quadrisecants belonging respectively to the three quadruples; the surface contains therefore at least *fourteen* lines.

If we do not take t, t' into consideration we have a configuration of twelve lines, showing the same structure as the configuration of § 2.

But in consequence of the special position of the lines a, the locus (P) now consists of the three hyperboloids $(a_1\, a_2\, a_3\, a_4)$, $(a_1\, a_2\, a_5\, a_6)$ $(a_3\, a_4\, a_5\, a_6)$.

8. Two triplets of planes $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3$ determine a pencil of cubic surfaces of which the nine lines $(\alpha_k\, \beta_l)$ form the basis. If these surfaces are conjugated projectively to the planes through an arbitrary line l, the surface O^4, generated by the two pencils contains besides the already mentioned ten lines six lines more of which each of the given six planes furnishes one.

These *sixteen* lines form a configuration, in which each line is intersected by six others; it is identical to the figure which is generated when four arbitrary planes α_k are intersected by four other planes β_l. For, the planes through l, conjugated to the figures $(\alpha_1, \alpha_2, \alpha_3)$ and $(\beta_1, \beta_2, \beta_3)$ can be called successively β_4 and α_4.

Let t be a transversal of the lines $l, (\alpha_1\, \beta_1), (\alpha_2\, \beta_2), (\alpha_3\, \beta_3)$. The projectivity indicated above can be arranged in such a way that the plane (lt) is conjugated to the cubic surface passing through a point of t, hence containing t. In an analogous manner we can deal with two other lines, each of which rests on l and on three not intersecting lines $(\alpha_k\, \beta_l)$. Then the projectivity is determined and the surface O^4 generated in this way evidently now contains *nineteen* lines.

We finally note that E. TRAYNARD (*Bull.Soc. Mat. de France*, vol. 38, p. 280) has described an O^4 with *thirty* lines.

Chemistry. — "*The application of the new theory of allotropy to the system sulphur.*" By Prof. A. SMITS. (Communicated by Prof. A. F. HOLLEMAN).

Those who have been occupied with the sulphur problem up to now, have always thought the pseudo system to be binary, i.e. they assumed that they had to deal with two pseudo components or two kinds of molecules, which can be converted into each other, and one of which gives rise to the formation of the well-known crystallized modification, the monoclinic and the rhombic sulphur, whereas the other would produce the amorphous sulphur, called so because attempts to make this form of sulphur crystallize have not been successful as yet.

Though in my opinion the above view is not the correct one, I will begin with treating sulphur as a pseudo-binary system, and show