Mathematics. — "*Continuous one-one transformations of surfaces in themselves.*" (4^{th} communication [1])). By Dr. L. E. J. Brouwer. (Communicated by Prof. D. J. Korteweg).

(Communicated in the meeting of May 27, 1911).

In this communication as in the preceding one we shall occupy ourselves with continuous one-one transformations with invariant indicatrix of a two-sided surface in itself.

If for such a transformation there is an invariant arc of simple curve, it contains at least *one* invariant point; more than one invariant point need not appear.

If, however, each of its two sides is invariant, then the arc contains at least *two* invariant points; more than two invariant points need not appear.

Of the former of these two evident theorems we have shown in § 2 of the third communication that it can be extended to the most general circular continuum (of which the arc of simple curve can be regarded as the simplest type); to the latter theorem we shall give the same extension in the following.

A segment of the circumference formed by the accessible points of a circular continuum will be called a *complete circumference segment,* if the set of its limiting points is identical to the circular continuum itself.

As the generalization of the arc of simple curve with two invariant sides we can consider a circular continuum φ' whose circumference can be divided by two "Schnitte" into two complete circumference segments, both invariant for the transformation.

Of φ' together with a certain vicinity ψ' we construct a continuous one-one representation on a finite region of a Cartesian plane, where they pass successively into φ and ψ, and we draw in that Cartesian plane a simple closed curve $\varkappa$ lying together with its image and its counterimage in ψ, whilst its inner domain contains φ.

All figures to be constructed in the following and likewise their images and their counterimages we suppose to lie in ψ.

According to the third communication φ possesses a point I invariant for the transformation; we shall suppose that this point I is the only invariant point of φ.

The two Schnitte determining on φ the two invariant complete circumference segments o_1 and o_2, we shall represent by S_1 and S_2.

1) See these Proceedings Vol. XI, p. 788, Vol. XII, p. 286, Vol. XIII, p. 767.

An arc of simple curve joining two points of the circumference of φ, and for the rest not meeting φ, will be called a *skeleton arc.*

We surround φ by a fundamental series of polygons $\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \ldots$ approximating φ at distances $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots \ldots \left(\varepsilon_{k+1} < \frac{1}{8}\,\varepsilon_k\right)$. The side of the largest square whose inner domain lies between $\mathfrak{P}_n$ and φ, we represent by e_n; for indefinitely increasing n we find that e_n converges to zero.

Each polygon $\mathfrak{P}_k$ we divide into segments in which the distance of the endpoints lies between $4\varepsilon_k$ and $12\varepsilon_k$, and the distance of two arbitrary points does not exceed $24\varepsilon_k$, and we draw from the points which separate these segments, to φ paths $< 2\varepsilon_k$ not intersecting each other, and cutting each polygon $\mathfrak{P}_n\,(n > k)$ only once. Each two of these paths which immediately succeed each other, form together with the segment of $\mathfrak{P}_k$ connecting them a skeleton arc.

We first suppose that the Schnitt S_1 is *not* determined by an accessible point, and we choose on a fundamental series of polygons $\mathfrak{P}_{\alpha_1}, \mathfrak{P}_{\alpha_2}, \ldots$ a fundamental series of skeleton arcs $s_{\alpha_1}, s_{\alpha_2}, \ldots$, not intersecting each other, converging to a single point P, and all containing between their endpoints the Schnitt S_1. The arc of $\mathfrak{P}_{\alpha_p}$ belonging to s_{α_p} we shall represent by q_{α_p}.

We then construct an arc of simple curve b ending in P, intersecting each element s_{τ_p} of a certain fundamental series $s_{\tau_1}, s_{\tau_2}, \ldots$ (contained in the series of the s_{α_p}) once and only once in a point P_{τ_p} of q_{τ_p}, and passing there from the outside of s_{τ_p} to its inner side. The part of b contained between $P_{\tau_{p-1}}$ and P_{τ_p} we represent by b_{τ_p}, the part of $\mathfrak{P}_{\tau_p}$ preceding resp. following q_{τ_p}, and lying inside $s_{\tau_{p-1}}$, by t_{τ_p} resp. v_{τ_p}. Then it is impossible that as well the part of t_{τ_p} lying to the right of b_{τ_p}, as the part of v_{τ_p} lying to the left of b_{τ_p}, converge to zero; for, in that case P would be an accessible point.

So out of the series of the τ_p we can select such a fundamental series $\beta_1, \beta_2, \ldots$ (preceded in the series of the τ_p successively by the elements $\gamma_1, \gamma_2, \ldots$), and determine to that series such a quantity c that for each β_p is attained on e. g. the part of t_{β_p} lying to the right of b_{β_p} a maximum distance $> 32c$ from P by a certain point Q_{β_p}, whilst neither s_{γ_p}, nor s_{β_p}, nor b_{β_p} reach a distance $> c$ from P, and ε_{γ_p} as well as e_{γ_p} are $< c$.

Then on v_{β_p} lies a point R_{β_p} which can be joined with Q_{β_p} inside

21*

$\mathfrak{P}_{\beta_p}$ by a path $\leqq e_{\gamma_p}\sqrt{2}$, whilst farthermore Q_{β_p} and R_{β_p} may be connected with φ by paths $Q_{\beta_p}\, H_{\beta_p}$ and $R_{\beta_p}\, K_{\beta_p} < \frac{3}{2}\varepsilon_{\beta_p}$, lying outside $\mathfrak{P}_{\beta_p}$, and not cutting s_{β_p}, thus containing S_1 between them. These three paths form a skeleton arc $H_{\beta_p}\, Q_{\beta_p}\, R_{\beta_p}\, K_{\beta_p}$ whose size for indefinitely increasing p converges to zero, and which we shall represent by σ_{β_p}

So out of the series of the β_p we can select a fundamental series $r_1, r_2, \ldots$, in such a way that *for indefinitely increasing p the skeleton arc σ_{r_p} converges to a single point V not identical to P.*

We shall now suppose that the Schnitt S_1 is determined by an accessible point P. Let in that case w be a path leading to P, and let $s_1, s_2, \ldots$ be a fundamental series of skeleton arcs separating S_1 from $\varkappa$, and whose size converges to zero. Then as soon as p has exceeded a certain value, all s_p must cut w, and that in points which for indefinitely increasing p uniformly converge to P, so that s_p converges *for indefinitely increasing p uniformly to P.*

So if S_1 resp. S_2 is not determined by an accessible point coinciding with I, we can construct a skeleton arc $U_1 V_1$ resp. $U_2 V_2$ as small as we like, separating S_1 resp. S_2 from $\varkappa$, and not cutting its image $U'_1 V'_1$ resp. $U'_2 V'_2$, so that either the circumference segment $U_1 V_1$ resp. $U_2 V_2$ is a part of the circumference segment $U'_1 V'_1$ resp. $U'_2 V'_2$, or the circumference segment $U'_1 V'_1$ resp. $U'_2 V'_2$ is a part of the circumference segment $U_1 V_1$ resp. $U_2 V_2$.

Farthermore it is impossible that S_1 and S_2 are determined by accessible points coinciding with each other, for, in that case the derived sets of o_1 and o_2 would have only that *one* point in common, so that o_1 and o_2 would not be complete circumference segments.

On o_1 we choose a point P not coinciding with I; the image of P we represent by P', the image of P' by P'', the counterimage of P by P_i. From $\varkappa$ we draw to P, P', P'', P_i paths w, z, u, v not meeting each other, and containing such endsegments e, e', e'', e_i that e' is the image of e, e'' the image of e', e_i the counterimage of e, and we construct an arc of simple curve k starting in P, not passing through I, cutting o_2, and not meeting w; the image of k we represent by k', the image of k' by k'', the counterimage of k by k_i, the size of k, k', k'', k_i successively by g, g', g'', g_i, the largest resp. smallest one of the latter four quantities by g_h resp. g_l. We describe circles a, a', a'', a_i containing in their inner domains j, j', j'', j_i at a distance g_h successively the arcs k, k', k'', k_i, and we take care to choose k

so small that two arbitrary ones of the sets of points $w+j$, $z+j'$, $u+j''$, $v+j_i$ possess a distance $> 8\,g_h$ from each other, that the parts of w, z, u, v contained in j, j', j'', j_i belong entirely to e, e', e'', e_i, and that k cannot contain a skeleton arc separating a Schnitt S_1 or S_2 determined by an accessible point coinciding with I, from the infinite.

Either k or k' contains a point Q of o_2 accessible from $\varkappa$ along a path not cutting $\varphi + k + k'$. In the following we shall assume Q to belong to k; if it were to belong to k', we might consider instead of the given transformation its inverse, and then follow the reasoning of the text.

From $\varkappa$ to Q we lay a path m not cutting $\varphi + k + k' + w$.

The part of k contained between P and Q we represent by r, its image by r', the image of r' by r''. If we then approximate $\varphi + r$ at a sufficiently small distance by a polygon $\mathfrak{P}$, this polygon $\mathfrak{P}$ contains two arcs p_1 and p_2 both connecting w and m, and having no point in common. Together with certain parts of $w + r + m$ these arcs p_1 and p_2 form two polygons $\mathfrak{P}_1$ and $\mathfrak{P}_2$ whose inner domains have no point in common, so that the inner domain of e.g. $\mathfrak{P}_1$ does *not* contain the point I. We then determine the positive sense of circuit of the circumference of φ by a circuit from P to Q inside $\mathfrak{P}_1$.

The circumference segment PQ contains *one* and not more than *one* of the two Schnitte S_1 and S_2: we may assume the Schnitt S_1 to belong to the circumference segment PQ.

Then S_1 cannot be determined by an accessible point coinciding with I; for, in that case r could not contain a skeleton arc separating S_1 from the infinite, so that the point I would be accessible inside $\mathfrak{P}_1$, which is impossible, I lying outside $\mathfrak{P}_1$.

We represent the image of Q by Q', and according to the manner of succession of the points P, P', Q, Q' for a positive sense of circuit we distinguish four cases.

First case: P' precedes P, and Q' precedes Q.

In this case r contains a skeleton arc d separating Q' from the infinite, and accessible from the infinite without a crossing of $\varphi + r + r'$. Let M be the endpoint of d preceding Q' on the circumference of φ, t a segment of d containing M, c the part of r that remains after destroying in r all skeleton arcs separating Q' from the infinite.

Between the image w' of w and t we construct a polygonal line $\mathfrak{P}'_3$, and between t and the image m' of m a polygonal line $\mathfrak{P}'_4$ which both approximate $\varphi + c + r' + r''$ at a distance ε.

The segment cut off from w' resp. t by $\mathfrak{P}'_3$ we represent by

$\digamma'$ resp. τ'_1; the segment cut off from t resp. m' by $\mathfrak{P}'_4$ we represent by τ'_4 resp. μ'; the part of t contained between the endpoints of $\mathfrak{P}'_1$ and $\mathfrak{P}'_4$ we represent by τ'. The arcs r', $\digamma'$, $\mathfrak{P}'_3$, τ' $\mathfrak{P}'_4$, μ' form together a polygon $\mathfrak{P}'$; I lies outside this polygon. For the lengths of the transformation vector and of the inverse transformation vector inside $\mathfrak{P}'$ there exists a certain minimum i_{ε}. Let f be a quantity smaller than g_l and smaller than $\frac{1}{8}i_{\varepsilon}$; then we take care to choose ε so small that

$$\varepsilon < \frac{1}{32}f,\ \digamma' < \frac{1}{32}f,\ \mu' < \frac{1}{32}f,\ \tau'_3 < \frac{1}{32}f,\ \tau'_4 < \frac{1}{32}f.$$

We divide $\mathfrak{P}'_3$ and $\mathfrak{P}'_4$ into segments in which the distance of the endpoints lies between $\frac{1}{8}f$ and $\frac{3}{8}f$, and the distance of two arbitrary points is smaller than $\frac{3}{4}f$. From the points separating these segments we draw to $q+c+r'+r''$ rectilinear paths whose lengths lie between $\frac{1}{2}\varepsilon$ and $\frac{3}{2}\varepsilon$, but among these paths we retain only those whose endpoints do not lie on r, r' or r''. These remaining paths determine together with w', m', τ'_3, and τ'_4 skeleton arcs lying against $\mathfrak{P}'_1$ and $\mathfrak{P}'_4$, and not meeting their counterimage skeleton arcs, whilst these counterimage skeleton arcs can meet neither r nor r'.

The last point of intersection with $\mathfrak{P}'_3$ of the counterimage skeleton arc s separating Q' from the infinite, we represent by L; the image of L we represent by L', the image of s by s', the first point of intersection of r with $\mathfrak{P}'$ by E, the image of E by E'.

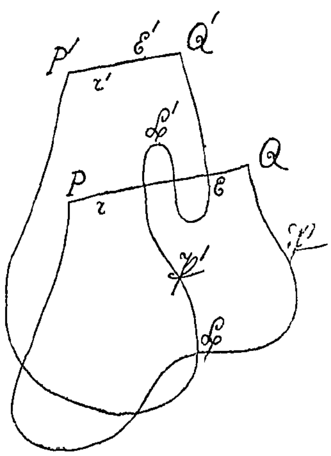

Fig. 1*a*.

A). *s' is separated by s from the infinite.* Our aim is to find the total angular variation ω_1 of the inverse transformation vector for a positive circuit of the polygon $\mathfrak{P}'$, and we represent by χ_1 the total angle described by the inverse transformation vector from P' to L' along $\mathfrak{P}'$; by χ_2 the total angular variation of a nowhere vanishing vector of which the origin runs from P' to L' along $\mathfrak{P}'$, and the endpoint as a continuous function of the origin from P to L along *path arcs* nowhere passing outside $\mathfrak{P}'$, constructed according to § 2 of the third communication[1]); by φ_1 the total angle described by the inverse transformation vector along the segment $L'E'$ of $\mathfrak{P}'$; by φ_2 the total angular variation of a nowhere vanishing vector of which the origin runs from L' to E' along $\mathfrak{P}'$, and the endpoint as a continuous function of the origin from L tot E along a curve p lying inside $\mathfrak{P}'$[2]); by ψ_1 the total angle described by the inverse transformation vector along the segment $E'P'$ of r', by ψ_2 the total angular variation of a nowhere vanishing vector of which the origin runs from E' to P' along r', and the endpoint as a continuous function of the origin from E to P along a curve obtained by replacing in the segment EP of r each part lying outside $\mathfrak{P}'$ by the segment of $\mathfrak{P}'_4$ joining the same endpoints.

Then the following equations hold:

$$\chi_1 = \chi_2 + 2n\pi \; (n \geqq 0)$$
$$\varphi_1 = \varphi_2$$
$$\psi_1 = \psi_2$$
$$\omega_1 = \chi_1 + \varphi_1 + \psi_1 .$$

Now $\chi_2 + \varphi_2 + \psi_2$ represents the total angular variation of a nowhere vanishing vector of which the origin describes the polygon $\mathfrak{P}'$ in a positive sense, and the endpoint as a continuous function of the origin a closed curve nowhere passing outside $\mathfrak{P}'$, so that we have:

$$\chi_2 + \varphi_2 + \psi_2 = 2\pi .$$

Hence:

$$\omega_1 = 2n\pi \; (n \geqq 1),$$

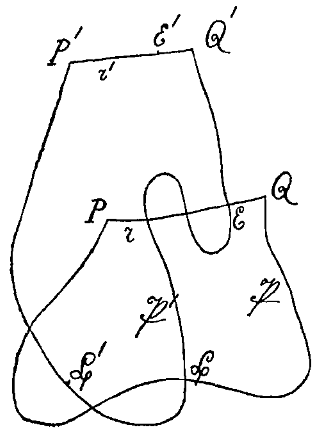

Fig. 1b.

1) See these Proceedings Vol. XIII, p. 770.

2) If L' lies not on $\mathfrak{P}'$, but on one of the paths connecting $\mathfrak{P}'$ and φ, we must take care that p does not meet this path.

so that we arrive at the absurd result that inside $\mathfrak{P}'$ must lie an invariant point.

B). s' is not separated by s from the infinite. Then the two endpoints of s' as well as the two endpoints of s lie on o_2. Defining $\omega_1, \chi_1, \chi_2, \varphi_1, \varphi_2, \psi_1, \psi_2$ in the same way as just now, we arrive here at the following equations:

$\chi_1 = \chi_2 + 2n\pi$ ($n \geqq 1$, because between P' and s' lies the Schnitt S_1)

$\varphi_1 = \varphi_2 - 2\pi$

$\psi_1 = \psi_2$

$\omega_1 = \chi_1 + \varphi_1 + \psi_1$

$\chi_2 + \varphi_2 + \psi_2 = 2\pi.$

Thus again $\omega_1 = 2n\pi$ ($n \geqq 1$), so that inside $\mathfrak{P}'$ there would have to lie an invariant point.

Second case: P' follows P, and Q' precedes Q.

A). Q' is separated by r from the infinite. We construct the polygonal lines $\mathfrak{P}'_3$ and $\mathfrak{P}'_4$, and the polygon $\mathfrak{P}'$ with its skeleton arcs in the same way as in the first case. Then the counterimage of $\mathfrak{P}'$ is a simple closed curve $\mathfrak{P}$ bearing skeleton arcs which, like those of $\mathfrak{P}'$, cut neither r nor r'. We want to find the total angular variation ϑ_1 of the transformation vector for a positive circuit of $\mathfrak{P}$.

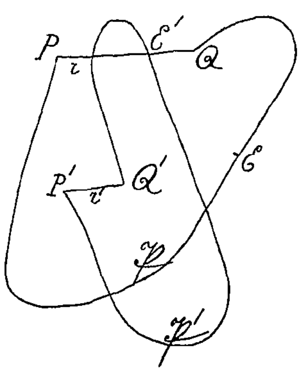

Fig. 2a.

We represent by E' the endpoint of $\mathfrak{P}'_3$ on t; by E the counterimage of E'; by χ_1 the total angle described by the transformation vector along the segment PE of $\mathfrak{P}$; by χ_2 the total angular variation of a nowhere vanishing vector of which the origin runs from P to E along $\mathfrak{P}$, and the endpoint as a continuous function of the origin from P' to E' along path arcs nowhere passing outside $\mathfrak{P}$; by ψ_1 the total angle described by the transformation vector along the segment EP of $\mathfrak{P}$; by ψ_2 the total angular variation of a nowhere vanishing vector of which the origin runs from E to P along $\mathfrak{P}$, and the endpoint as a continuous function of the origin along a curve obtained by replacing in the segment $E'P'$ of $\mathfrak{P}'$ each part lying outside $\mathfrak{P}$ by the segment of r joining the same endpoints.

From the equations

$$\chi_1 = \chi_2 + 2n\pi \ (n \geqq 0)$$
$$\psi_1 = \psi_2$$
$$\vartheta_1 = \chi_1 + \psi_1$$
$$\chi_2 + \psi_2 = 2\pi$$

then ensues $\vartheta_1 = 2n\pi\ (n \geqq 1)$, so that inside $\mathfrak{P}$ there would have to lie an invariant point.

B). *Q′ is not separated by r from the infinite.* We construct between w' and m' a polygonal line approximating $\varphi+r+r'+r''$ at a distance ε, cutting off from w' resp. m' the segment ρ' resp. μ', and forming with ρ', r', and μ' a polygon $\mathfrak{P}'$. The determination of ε, and the construction of the skeleton arcs of $\mathfrak{P}'$ take place in the same way as in the first case. We want to find the total angular variation ϑ_1 of the transformation vector for a positive circuit of the counterimage $\mathfrak{P}$ of $\mathfrak{P}'$, and we understand by ϑ_2 the total angular variation of a nowhere vanishing vector of which the origin describes $\mathfrak{P}$, and the endpoint as a continuous function of the origin runs first from P' to Q' along path arcs nowhere passing outside $\mathfrak{P}$, and finally describes r'.

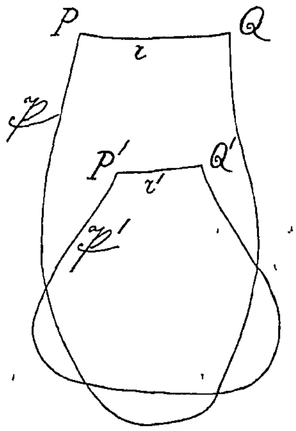

Fig. 2*b*.

Then we have:

$$\vartheta_1 = \vartheta_2 + 2n\pi\ (n \geqq 0)$$
$$\vartheta_2 = 2\pi$$

Hence $\vartheta_1 = 2n\pi\ (n \geqq 1)$, so that inside $\mathfrak{P}$ there would have to lie an invariant point.

Third case: P′ follows P, and Q′ follows Q.

In this case r contains a skeleton arc d separating Q' from the infinite, and accessible from the infinite without a crossing of $\varphi+r+r'$. We determine c, t, and ε, and we construct $\mathfrak{P}'_3$, $\mathfrak{P}'_4$, $\mathfrak{P}'$, $\mathfrak{P}$, and the skeleton arcs of these polygons in the same way as in the second case under *A*).

The last point of intersection with $\mathfrak{P}$ of the skeleton arc s' of $\mathfrak{P}'_3$ separating Q from the infinite, we represent by L'; the counterimage of L' we represent by L, the counterimage of s' by s, the endpoint of $\mathfrak{P}'_3$ on t by E', the counterimage of E' by E.

A). *s is separated by s′ from the infinite.* Our aim is to find the total angular variation ϑ_1 of the transformation vector for a positive circuit of $\mathfrak{P}$, and we represent by χ_1 the total angle described by the transformation vector from P to L along $\mathfrak{P}$; by χ_2 the total angular variation of a nowhere vanishing vector of which the origin runs from P to L along $\mathfrak{P}$, and the endpoint as a continuous function of the origin from P' to L' along path arcs nowhere passing outside $\mathfrak{P}$; by φ_1 the total angle described by the transformation vector from L to E along $\mathfrak{P}$; by φ_2 the total angular variation of a nowhere vanishing vector of which the origin runs from L to E along $\mathfrak{P}$, and the endpoint as a continuous function of the origin inside $\mathfrak{P}$ from L' to E' along an arc of simple curve p; by ψ_1 the total angle described by the transformation vector from E to P along $\mathfrak{P}$; by ψ_2 the total angular variation of a nowhere vanishing vector of which the origin runs from E to P along $\mathfrak{P}$, and the endpoint as a continuous function of the origin along a curve obtained by replacing in the segment $E'P'$ of $\mathfrak{P}'$ each part lying outside $\mathfrak{P}$ by the segment of r joining the same endpoints.

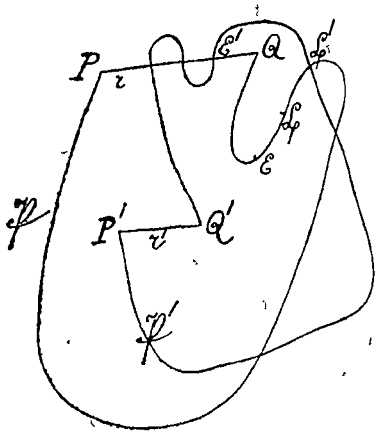

Fig. $3a$.

Then the following equations hold:

$$\chi_1 = \chi_2 + 2n\pi \ (n \geqq 0)$$
$$\varphi_1 = \varphi_2 + 2\pi$$
$$\psi_1 = \psi_2$$
$$\vartheta_1 = \chi_1 + \varphi_1 + \psi_1$$
$$\chi_2 + \varphi_2 + \psi_2 = 2\pi.$$

Hence $\vartheta_1 = 2n\pi \ (n \geqq 2)$, so that inside $\mathfrak{P}$ there would have to lie an invariant point.

B). *s is not separated by s′ from the infinite.* Then the two endpoints of s as well as the two endpoints of s' lie on o_2. Defining $\vartheta_1, \chi_1, \chi_2, \varphi_1, \varphi_2, \psi_1, \psi_2$ in the same way as just now, we arrive here at the following equations:

$$\chi_1 = \chi_2 + 2n\pi \ (n \geqq 1, \text{ because between } P \text{ and } s \text{ lies the Schnitt } S_1)$$
$$\varphi_1 = \varphi_2$$
$$\psi_1 = \psi_2$$
$$\vartheta_1 = \chi_1 + \varphi_1 + \psi_1$$
$$\chi_2 + \varphi_2 + \psi_2 = 2\pi.$$

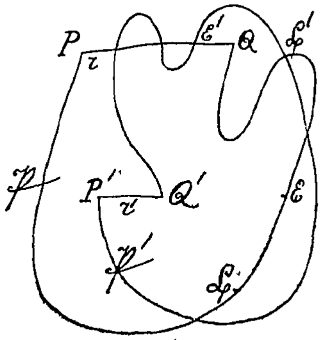

Fig. $3b$.

Thus again $\vartheta_1 = 2n\pi \ (n \geqq 2)$, so that inside $\mathfrak{P}$ there would have to lie an invariant point.

Fourth case: P' precedes P, and Q' follows Q.

A). Q' is separated by r from the infinite. We construct the polygon $\mathfrak{P}'$ with its skeleton arcs in the same way as in the third case. We want to find the total angular variation ω_1 of the inverse transformation vector for a positive circuit of $\mathfrak{P}'$, and we represent by χ_1 the total angle described by the inverse transformation vector along the segment $P'Q'$ of $\mathfrak{P}'$; by χ_2 the total angular variation of a nowhere vanishing vector of which the origin runs from P' to Q' along $\mathfrak{P}'$, and the endpoint as a continuous function of the origin from P to Q along path arcs nowhere passing outside $\mathfrak{P}'$; by ψ_1 the total angle described by the inverse transformation vector from Q' to P' along r'; by ψ_2 the total angular variation of a nowhere vanishing vector of which the origin runs from Q' to P' along r', and the endpoint as a continuous function of the origin from Q to P along a curve obtained by replacing in r each part lying outside $\mathfrak{P}'$ by the segment of $\mathfrak{P}'_4$ joining the same endpoints.

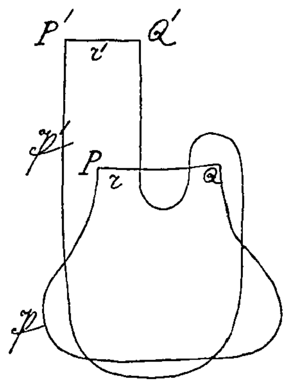

Fig. 4*a*.

From the equations

$$\chi_1 = \chi_2 + 2n\pi \; (n \geqq 0)$$
$$\psi_1 = \psi_2$$
$$\omega_1 = \chi_1 + \psi_1$$
$$\chi_2 + \psi_2 = 2\pi$$

then ensues $\omega_1 = 2n\pi \; (n \geqq 1)$, so that inside $\mathfrak{P}'$ there would have to lie an invariant point.

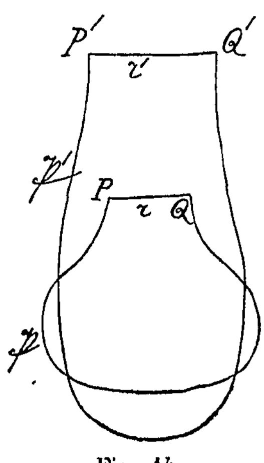

Fig. 4*b*.

B). Q' is not separated by r from the infinite. We construct the polygon $\mathfrak{P}'$ with its skeleton arcs in the same way as in the second case under *B*). We want to find the total angular variation ω_1 of the inverse transformation vector for a positive circuit of $\mathfrak{P}'$, and we understand by ω_2 the total angular variation of a nowhere vanishing vector of which the origin describes $\mathfrak{P}'$, and the endpoint as a continuous function of the origin runs first from P to Q along path arcs nowhere passing outside $\mathfrak{P}'$, and finally describes r.

Then we have:

$$\omega_1 = \omega_2 + 2n\pi \; (n \geqq 0)$$
$$\omega_2 \doteq 2\pi$$

Hence $\omega'_1 \doteq 2n\pi \; (n \geq 1)$, so that inside $\mathfrak{P}'$ there would have to lie an invariant point.

With this we have completely proved the following

THEOREM. *For a continuous one-one transformation with invariant indicatrix of a two-sided surface in itself a circular continuum with two separated invariant complete circumference segments contains at least two invariant points.*

ERRATA.

In the 3rd communication on this subject, these Proceedings Vol. XIII

p. 767, l. 6 from top	for: indicated. but	read: indicated but
l. 20 from top	for: *paraboli*	read: *parabolic*

Physiology. — C. A. PEKELHARING reads a paper on: "*The excretion of creatinin in man under the influence of muscular tonus*", after experiments by Mr. J. HARKINK.

(Communicated in the meeting of September 30, 1911).

Some time ago I reported here on an investigation by Mr. VAN HOOGENHUYZE and myself, proving that in vertebrates the content of creatin in the voluntary muscles increases during the tonus, but not during simple contractions of the muscles. We may therefore expect that by increase of the muscular tonus more creatin passes into the blood than in other circumstances. Moreover a later investigation showed us that creatin, when gradually introduced into the circulating blood, is partly excreted by the kidneys as creatinin [1]). So we may conclude that an increased tonus will lead to a larger excretion of creatinin.

A series of estimations by VAN HOOGENHUYZE and VERPLOEGH showed indeed that less creatinin is excreted per hour during the night when the muscles as a rule are relaxed in sleep, than in the daytime, when the muscles are now in a tighter, now in a less intense tonus. Besides they stated that a smaller amount of creatinin

[1]) Onderzoekingen Physiol. Laborat. Utrecht, 5de R. XI. p. 236.