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only in large crystals, and exclusively after the colourless minerals.

In the aegirinenephelinesyeniteporphyries from Olivenfontein (145), which are very rich in nepheline, the first crystallized mineral is the apatite; it was followed by small crystals of nepheline and sodalite, still later by larger crystals of perforated nepheline, sodalite and felspar, simultaneously with the enclosed small needles of aegirine; finally the perforated aegirines could still be formed in large crystals.

On account of the tardy crystallization of the larger crystals the order of succession of the crystallizations can be studied more easily in these rocks than in their normalgrained equivalents.

The sieve structures described above, can be distinguished from those of the contactrocks and crystalline schists by the occurrence of exclusively idiomorphic or rounded inclusions, according to their relative age. From the real phenocrysts of the porphyric rocks the larger crystals here described differ in this respect that the inclusions are not ranged after the laws of crystallization of the enclosing crystal.

As the perforated crystals usually show a perfectly idiomorphic form, we see that the rule according to which the relative age of the crystals in igneous rocks is proportional to their idiomorphism, does not hold good here.

Mathematics. — "*An extension of the integral theorem of* FOURIER."

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As is known, for an extensive class of functions $f(x)$ the equation

$$f(x)=\int_0^\infty d\alpha \int_a^b \psi(x, y, \alpha) f(y)\, dy$$

becomes an identity in x, if we put $b=-a=\infty$ and $\psi(x,y,\alpha)= = \cos \alpha(x-y)$; in this way we find the integral theorem of FOURIER which can be regarded as a limiting case of the series of FOURIER.

In the theory of the integral equations HILBERT and SCHMIDT have proved developments in series of which those of FOURIER are special cases. The following is a theorem which is in such a manner an extension of the integral theorem of FOURIER.

Let $K(x,y)$ be a continuous symmetrical kernel, $\varphi_1(x), \ldots, \varphi_\nu(x), \ldots$ a complete system of normalized orthogonal functions of that kernel and belonging to the limits of integration a and b, and $\lambda_1, \ldots, \lambda_\nu, \ldots$ the corresponding roots ("Eigenwerte").

As

$$K^{(2)}(x, y) = \int_a^b K(x, \xi)\, K(\xi, y)\, d\xi$$

we find

$$K^{(2)}(x, y) = \sum_{\nu=1}^{\infty} \varphi_\nu(y) \int_a^b K^{(2)}(x, z)\, \varphi_\nu(z)\, dz = \sum_{\nu=1}^{\infty} \frac{\varphi_\nu(x)\, \varphi_\nu(y)}{\lambda_\nu^2}$$

and there is with given x and positive ε such a number that for n greater than that number and $a \leqq y \leqq b$ we have

$$\sum_{\nu=n}^{n+m} \left| \frac{\varphi_\nu(x)\, \varphi_\nu(y)}{\lambda_\nu^2} \right| < \varepsilon .$$

If $q \geqq 0$, then as a matter of course

$$\sum_{\nu=n}^{n+m} \left| \frac{\varphi_\nu(x)\, \varphi_\nu(y)}{|\lambda_\nu|^{2+q}} \right| < \varepsilon.$$

Let $\delta > 0$ and $g(\alpha)$ be such a function of α, that for $\alpha \geqq 0$ we have

$$|g(\alpha)| \leqq M \quad \text{and} \quad \left| \frac{g(\alpha)}{\alpha^{1+\delta}} \right| \leqq N,$$

besides

$$\int_0^{\infty} \frac{g(\alpha)}{\alpha^{1+\delta}}\, d\alpha = A;$$

then we find

$$\left| \frac{g\left(\dfrac{\alpha}{|\lambda_\nu|^{2+q}}\right)}{\left(\dfrac{\alpha}{|\lambda_\nu|^{2+q}}\right)^{1+\delta}} \right| \leqq N$$

and consequently the series

$$\psi(x, y, \alpha) = \sum_{\nu=1}^{\infty} \varphi_\nu(x)\, \varphi_\nu(y)\, \frac{g\left(\dfrac{\alpha}{|\lambda_\nu|^{2+q}}\right)}{A\alpha^{1+\delta}}\, |\lambda_\nu|^{(2+q)\delta}$$

converges absolutely and for constant x uniformly in (y, α). So if $f(y)$ is a continuous function of y and if $m > 0$, we find:

$$\int_0^m d\alpha \int_a^b \psi(x,y,\alpha) f(y) dy = \sum_{\nu=1}^{\infty} \varphi_\nu(x) \int_a^b \varphi_\nu(y) f(y) dy \int_0^m \frac{|\lambda_\nu|^{(2+q)\delta}}{A} \cdot \frac{g\left(\dfrac{\alpha}{|\lambda_\nu|^{2+q}}\right)}{\alpha^{1+\delta}}\, d\alpha$$

and therefore

$$\left| \int_0^m d\alpha \int_a^b \psi(x\, y,\alpha)\, f(y) dy - \sum_{\nu=1}^{\infty} \varphi_\nu(x) \int_a^b \varphi_\nu(y)\, f(y) dy \right|$$

$$= \left| \sum_{\nu=1}^{\infty} \left(1 - \int_0^m \frac{|\lambda_\nu|^{(2+q)\delta}}{A} \cdot \frac{g\left(\frac{\alpha}{|\lambda_\nu|^{2+q}}\right)}{\alpha^{1+\delta}} d\alpha \right) \varphi_\nu(x) \int_a^b \varphi_\nu(y) f(y)\, dy \right|, \quad (1)$$

if both sums appearing here converge. If we put

$$\beta = \frac{\alpha}{|\lambda_\nu|^{2+q}},$$

we find

$$1 - \int_0^m \frac{|\lambda_\nu|^{(2+q)\delta}}{A} \frac{g\left(\frac{\alpha}{|\lambda_\nu|^{2+q}}\right)}{\alpha^{1+\delta}} d\alpha = 1 - \frac{1}{A} \int_0^{\frac{m}{|\lambda_\nu|^{2+q}}} g(\beta) \frac{d\beta}{\beta^{1+\delta}} =$$

$$= \frac{1}{A} \int_{\frac{m}{|\lambda_\nu|^{2+q}}}^{\infty} \frac{g(\beta)}{\beta^{1+\delta}} d\beta \leqq \frac{M}{A\delta} \cdot \frac{|\lambda_\nu|^{(2+q)\delta}}{m^\delta}.$$

Let p now be an integer, satisfying the condition

$$1 + (2+q)\,\sigma \leqq p < (2+q)\,\sigma + 2 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We find then, if

$$f(x) = \int_a^b K^{(p)}(x,y)\, h(y)\, dy$$

and $h(y)$ is continuous,

$$\int_a^b \varphi_\nu(y) f(y)\, dy = \int_a^b \varphi_\nu(y)\, dy \int_a^b K^{(p)}(y,\xi)\, h(\xi)\, d\xi$$

$$= \int_a^b h(\xi)\, d(\xi) \int_a^b K^{(p)}(y,\xi)\, \varphi_\nu(y)\, dy = \frac{1}{\lambda^p} \int_a^b h(\xi)\, \varphi_\nu(\xi)\, d\xi$$

and so the second member of (1) is equal or smaller than

$$\sum_{\nu=1}^{\infty} \left| \frac{M}{A\sigma m^\delta} \frac{1}{|\lambda_\nu|^{p-(2+q)\delta}} \varphi_\nu(x) \int \varphi_\nu(\xi)\, h(\xi)\, d\xi \right| \qquad (|\lambda| \text{ smallest } |\lambda_\nu|)$$

$$\leqq \frac{1}{A\sigma m^\delta |\lambda|^{p-(2+q)\delta-1}} \sum_{\nu=1}^{\infty} \left| \int_a^b K(x,y)\, \varphi_\nu(y)\, dy \int_a^b \varphi_\nu(\xi)\, h(\xi)\, d\xi \right|.$$

The sum appearing here converges according to § 2 of SCHMIDT's paper on integral equations in the Math. Ann. vol. 63, whilst from the given suppositions about $f(y)$ follows that the sum in the first member of (1) is equal to $f(x)$. For $\lim m = \infty$ follows therefore out of (1)

$$f(x) = \int_0^\infty d\alpha \int_a^b \psi(x,y,\alpha)\, f(y)\, dy \quad . \; . \; . \; . \; . \; . \; . \quad (3)$$

From (2) follows that the smallest value of p is two. If we take e.g. $q=0$, $\sigma=\frac{1}{2}$, $g(\alpha)=\sin^2\alpha$, then $M=1$, $N<\sqrt{\pi}$ and

$$A = \int_0^\infty \frac{\sin^2\alpha}{\alpha^{3/2}}\, d\alpha = \sqrt{\pi}.$$

If therefore $f(x)$ is a continuous function in x, for which the integral equation of the first kind

$$f(x) = \int_a^b K^{(2)}(x,y)\, h(y)\, dy$$

has a continuous solution $h(y)$, then (3) holds if we put

$$\psi(x,y,\alpha) = \frac{1}{\sqrt{\pi}} \sum_{\nu=1}^{\infty} \varphi_\nu(x)\, \varphi_\nu(y)\, |\lambda_\nu|\, \frac{\sin^2(\alpha/\lambda_\nu^2)}{\alpha^{3/2}}.$$

If however we put $q=0$, $\sigma=1$, $g(\alpha)=\sin^2\alpha$, then $M=N=1$ and

$$A = \int_0^\infty \frac{\sin^2\alpha}{\alpha^2}\, d\alpha = \frac{\pi}{2}$$

and therefore

$$\psi = \frac{2}{\pi} \sum_{\nu=1}^{\infty} \lambda_\nu^2\, \varphi_\nu(x)\, \varphi_\nu(y)\, \frac{1}{\alpha^2}\, \sin\; (\alpha/\lambda_\nu^2),$$

whilst $p=3$.

It is easy to see that after a choice of q and σ we can always suffice by taking

$$g(\alpha) = (\sin^2\alpha)^{\frac{1+\sigma}{2}},$$

or also, if r is an integer and

$$1+\sigma \leqq r < 2+\sigma,$$
$$g(\alpha) = \sin^r \alpha.$$

Another example is

$$\psi = \frac{2}{\pi} \sum_{\nu=1}^{\infty} \frac{\varphi_\nu(x)\, \varphi_\nu(y)}{\lambda_\nu^2 + \dfrac{\alpha^2}{\lambda_\nu^2}} \quad , \quad (p=2)$$

which is found for $q=0$ and $\sigma=1$, if we take

$$g(\alpha) = \frac{\alpha^{1+}}{1+\alpha^{1+\sigma}}.$$