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P.H. Schoute, The characteristic numbers of the prismotope, in:
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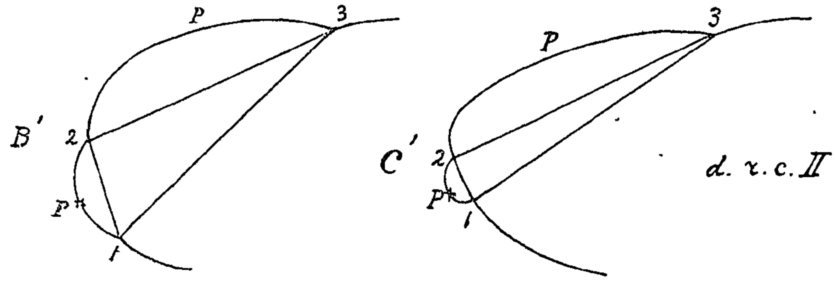
by VAN DER WAALS. The necessity of double retrograde condensation (fourfold intersection with a line // v-axis) is seen in figure D.

In the second supposition (Figs. $B'C'$) 2 passes through the critical end point (B') and immediately after we again obtain the configuration which involves double retrograde condensation with this peculiarity



however, that the first retrograde condensation which occurs on compression is retrograde condensation "of the second kind" owing to the position of the plaitpoint on the vapour side. The phenomenon as a whole might therefore be called *double retrograde condensation of the second kind.* The maximum vapour pressure disappears inside the triangle itself in consequence of the transformations which we know from KORTEWEG's investigations take place inside the plait in this case. Whether the first or the second supposition is the correct one, say for ether and water, I shall not try to decide.

Mathematics. — *"The characteristic numbers of the prismotope."* By Prof. P. H. SCHOUTE.

1. *Mode of generation.* Let $S_{n_1}, S_{n_2}, \ldots, S_{n_p}$ represent a certain number p of spaces respectively of $n_1, n_2, \ldots, n_p$ dimensions having by two no point in common but the point O common to them all. Let us assume in each of these spaces a definite polytope with O as one of its vertices, and let us denote the polytope in S_{n_1} by $(P)_{n_1}$, that in S_{n_2} by $(P)_{n_2}, \ldots$, that in S_{n_p} by $(P)_{n_p}$. Now let us move $(P)_{n_2}$, remaining equipollent to itself, in such a way that the point coinciding originally with O coincides successively with each point within $(P)_{n_1}$; then the locus of all the positions of $(P)_{n_2}$ is a prismotope with two constituents $(P)_{n_1}$, $(P)_{n_2}$ which may be represented by the symbol $(P_{n_1}; P_{n_2})$. Now let us move $(P)_{n_3}$, remaining equipollent to itself, in such a way that the point coinciding originally with O coincides successively with each point within $(P_{n_1}; P_{n_2})$; then the locus of all the positions of $(P)_{n_3}$ is a prismotope with three

constituents $(P)_{n_1}, (P)_{n_2}, (P)_{n_3}$ which may be represented by the symbol $(P_{n_1}; P_{n_2}; P_{n_3})$. In the same way we find by combining $(P)_{n_4}$ and $(P_{n_1}; P_{n_2}; P_{n_3})$ a prismotope $(P_{n_1}; P_{n_2}; P_{n_3}; P_{n_4})$ with four constituents and finally, after having used $(P)_{n_p}$, a prismotope $(P_{n_1}; P_{n_2}; \dots; P_{n_p})$ with p constituents.

It is not difficult to show that the result is independent of the order of succession in which the constituents are introduced in the process mentioned. To that end we have only to demonstrate that the interchange of $(P)_{n_1}$ and $(P)_{n_2}$ in the generation of $(P_{n_1}; P_{n_2})$ does not influence the result. Let P be an arbitrarily chosen point of the position $(P)'_{n_2}$ of $(P)_{n_2}$ in which the point O of $(P)_{n_2}$ coincides with the arbitrarily chosen point O_1 of $(P)_{n_1}$, and let O_2 be determined by the vector equation $O_1P = OO_2$, i. e. let O_2 be the point of $(P)_{n_2}$ corresponding to the point P of $(P)'_{n_2}$. Then OO_1PO_2 is a parallelogram; so P may be considered quite as well as the point of a new position $(P)'_{n_1}$ of $(P)_{n_1}$ corresponding to the point O_1 of $(P)_{n_1}$, this new position $(P)'_{n_1}$ of $(P)_{n_1}$ being obtained by moving $(P)_{n_1}$ equipollent to itself in such a way that the point O of $(P)_{n_1}$ coincides with the point O_2 of $(P)_{n_2}$. Or, in connection with the remark that O_1 and O_2 are arbitrary points of $(P)_{n_1}$ and $(P)_{n_2}$, whilst OP is the resultant of the vectors OO_1 and OO_2: "if O_1 and O_2 are arbitrary points of $(P)_{n_1}$ and $(P)_{n_2}$, the end point P of the resultant OP of OO_1 and OO_2 is an arbitrary point of the prismotope with the two constituents $(P)_{n_1}$ and $(P)_{n_2}$". This can be extended immediately to: "if $O_1, O_2, \dots, O_p$ are arbitrary points of $(P)_{n_1}, (P)_{n_2}, \dots, (P)_{n_p}$, the end point P of the resultant OP of $OO_1, OO_2, \dots, OO_p$ is an arbitrary point of the prismotope $(P_{n_1}; P_{n_2}; \dots; P_{n_p})$".

This mode of generation shows clearly the irrelevancy of the order of succession of *all* the constituents.

The prismotope $(P_{n_1}; P_{n_2}; \dots; P_{n_p})$ is a polytope with $\sum_{i=1}^{p} n_i$ dimensions; the space with this number of dimensions containing it is completely determined by the spaces $S_{n_1}, S_{n_2}, \dots, S_{n_p}$ with the common point O.

The aim of this paper is to determine the characteristic numbers of the prismotope $(P_{n_1}; P_{n_2}; \dots; P_{n_p})$.

2. *Notation.* We indicate the numbers of the vertices, edges, faces, ..., limiting polytopes $(l)_{n-1}$ of a polytope P_n by the same letter, say a, with the subscripts $0, 1, 2, \dots, n-1$ and represent

moreover by a_n the unit corresponding to the polytope itself. So, if the polytope belongs to the group of the EULERian polytopes, what we suppose to be the case, the relation holds

$$a_0 - a_1 + a_2 - a_3 + \dots + (-1)^n a_n = 1,$$

what we express by saying that the EULERian form A, i. e. the left hand side of the equation mentioned, is equal to unity.

For the different constituents $(P)_{n_1}, (P)_{n_2}, \dots, (P)_{n_p}$ of the prismotope, all of them supposed to belong to the EULERian group, we introduce for a and A different letters $a, b, \dots, p$ and $A, B, \dots, P$.

3. We now prove the following *lemma*:

"The number of limiting elements $(l)_q$ of q dimensions of the prismotope $(P_{n_1}; P_{n_2}; \dots; P_{n_p})$ is equal to the sum of the terms out of the product $AB \dots P$ of the EULERian forms of the constituents, for which the sum of the subscripts is equal to q, with the positive or the negative sign according to q being even or odd.".

Example. In the case of three pentagons as constituents we find by developing $(a_0 - a_1 + a_2)(b_0 - b_1 + b_2)(c_0 - c_1 + c_2)$ where

$$a_0 = b_0 = c_0 = a_1 = b_1 = c_1 = 5, \qquad a_2 = b_2 = c_2 = 1,$$

if $q_0, q_1, \dots$ refer to the prismotope,

$$q_0 = 125, \ q_1 = 375, \ q_2 = 450, \ q_3 = 275 \ \ q_4 = 90, \ q_5 = 15, \ q_6 = 1$$

and therefore a sixdimensional polytope with the symbol

$$(125, \quad 375, \quad 450, \quad 275, \quad 90, \quad 15)$$

of characteristic numbers; this symbol satisfies the law of EULER.

Proof of the lemma. Let us represent the prismotope $(P_{n_1}; P_{n_2}; \dots; P_{n_p})$ under consideration by Pt for short and let $pt = (\alpha_{s_1}; \beta_{s_2}; \dots; \pi_{s_p})$ represent a new prismotope, the constituents of which are definite limiting elements α_{s_1} of $(P)_{n_1}$, β_{s_2} of $(P)_{n_2}, \dots, \pi_{s_p}$ of $(P)_{n_p}$, where it is allowed to take vertices for some of these limiting elements in which case the corresponding dimension number s_i is zero and the corresponding (P_{n_i}) inactive in the formation of pt. Then pt will be a limit $(l)_q$ of Pt under the condition $\sum_{i=1}^{p} s_i = q$. Reversely each element $(l)_q$ of Pt can be generated in this manner. So the number of limits $(l)_q$ of Pt is equal to the number of the different ways in which we can gather a set of limits $\alpha_{s_1}, \beta_{s_2}, \dots, \pi_{s_p}$, the numbers s_i satisfying the condition $\sum_{i=1}^{p} s_i = q$, which last number is represented evidently by $\Sigma a_{s_1} b_{s_2} \ldots p_{s_p}$ for $\sum_{i=1}^{p} s_i = q$, and in its turn this expression represents the absolute value of the number indicated in the statement of the lemma, where it is affected by the sign $(-1)^q$.

4. If we bring the polytope $(P)_n$ with the symbol $(a_0, a_1, a_2, \ldots, a_{n-1})$ of characteristic numbers in relation with the polynomium

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1} + x^n$$

which may be called the "EULERian function" of the polytope, the connection between the characteristic numbers of the prismotope and those of its constituents can be expressed very simply by:

THEOREM I. "The EULERian function of a prismotope is the product of the EULERian functions of its constituents".

Corollaries. "The EULERian function of the simplex $S(n+1)$ of S_n is

$$(n+1)_1 + (n+1)_2\, x + (n+1)_3\, x^2 + \ldots + (n+1)_1\, x^{n-1} + x^n,$$

for which can be written $\frac{1}{x}\{(x+1)^{n+1} - 1\}$."

"The EULERian function of the simplotope (compare p. 45 of vol. II of my textbook "Mehrdimensionale Geometrie", Leipsic, Göschen, 1905) with the constituents $S(n_i + 1)$, $(i = 1, 2, \ldots, p)$ is represented by

$$\frac{1}{x^p}\{(x+1)^{n_1+1} - 1\}\{(x+1)^{n_2+1} - 1\}\ldots\{(x+1)^{n_p+1} - 1\}."$$

"The EULERian function of a prism of rank k is divisible by $(2+x)^k$. So $(2+x)^n$ is the EULERian function of an n-dimensional parallelotope (l.c., p. 39, vol. II and as for the characteristic numbers line $B_n \ldots$ of p. 245, vol. II)."

So the characteristic numbers of the parallelotope of S_n are connected in a simple way with the digits of the number representing the n^{th} power of 21, if in this evolution the ordinary reduction to higher units is suppressed, i. e. if we write $21^3 = 8\,(12)\,6\,1$, $21^4 = (16)\,(32)\,(24)\,8\,1$, etc.

Example. We consider the sixdimensional prismotope $(tCO; tCO)$ with two tCO (see the last polyhedron of fig. 55, l.c, p. 189, vol. II) for constituents; as the symbol of characteristic numbers of tCO is (48, 72, 26) we find

$$(48+72x+26x^2+x^3)^2 = 2304+6912x+7680x^2+3840x^3+820x^4+52x^5+x^6$$

as EULERian function and therefore (2304, 6912, 7680, 3840, 820, 52) as symbol of characteristic numbers.

Remark. Theorem I is not reversible, i.e. the decomposability of the EULERian function into polynomia which represent EULERian functions of polytopes with smaller numbers of dimensions does not imply that the polytope under consideration is a prismotope. So the polyhedron tO (the last polyhedron of the first row of fig. 55, l.c. p. 189, vol. II) corresponds in symbol (24, 36, 14) of characteristic numbers with the prism P_{12}; therefore its EULERian function $24+36x+14x^2+x^3$ is decomposable into the EULERian functions $12+12x+x^2$ and $2+x$ of the base and the upright edge of P_{12}, though tO is no prism at all.

By putting $x=-1$ we deduce from theorem I that the product of the EULERian forms of the constituents is the EULERian form of the prismotope. So we find:

THEOREM. II. "A prismotope satisfies the law of EULER, if and only if this is the case with all its constituents".

If we denote as "sum of limits" of the polytope $(P)_n$ with the symbol $(a_0, a_1, \ldots; a_{n-1})$ of characteristic numbers the expression $a_0+a_1+\ldots+a_{n-1}+1$, where the last unit corresponds to the polytope itself, the first theorem gives for $x=1$:

THEOREM III. " *The sum of limits of a prismotope is the product of the sums of limits of its constituents*".

Corollaries. "The sum of limits of the simplotope mentioned above is $(2^{n_1+1}-1)(2^{n_2+1}-1)\ldots(2^{n_p+1}-1)$."

"The sum of limits of the n-dimensional parallelotope is 3^n.".

Example. The sum of limits of the prismotope (tCO; tCO) mentioned above is $147^2=21609$.

Physics. — "*The variability of the quantity b in* VAN DER WAALS' *equation of state, also in connection with the critical quantities.*" II. By J. J. VAN LAAR. (Communicated by Prof. H. A. LORENTZ).

For b' we find with $x=1$ according to formula (11):

$$b'=\frac{{}^1\!/_2\beta(1-\beta)\varphi(1+\varphi)}{m}=\frac{0,02162\times 1,227\times 2,227}{1,107}=\frac{0,05908}{1,107}=\underline{\underline{0,0534}}.$$

This value is — as was to be expected (see also I, p. 292) — somewhat lower than that which was formerly calculated with the equation of state *without* RT being multiplied by the factor $1+x\beta$. Before about $\frac{1}{13}$ was found, but now about $\frac{1}{19}$.

For $-vb''$ we calculate with $x=1$ according to (12):

$$-vb''=\frac{v}{v-b}\frac{\varphi}{m^3}\frac{1}{2}\beta(1-\beta)\left[(1+2\varphi)+\frac{1}{2}(\beta^2+2\beta-1)(1+\varphi)^2\right]$$

$$=\frac{2,114}{1,114}\frac{1,227}{1,357}\times 0,02162\left[3,454+0,9104\times(2,227)^2\right]$$

$$=1,898\times 0,9038\times 0,02162\times 7,969$$

$$=1,715\times 0,1723=\underline{\underline{0,295}}.$$