Citation:

Vries, H. de, On the conoids belonging to an arbitrary surface. (1st part), in:
KNAW, Proceedings, 14 I, 1911, Amsterdam, 1911, pp. 481-495

Cadmium, Magnesium, Calcium and Mercury. Moreover the principle of combination appears to point here to summational and differential vibrations of the intensest (first) lines of the already known series, so that we can account for the new lines *without making use of a spectral formula.* In PASCHEN's recent paper on the systems of series in the spectra of Zinc, Cadmium and Mercury it is particularly the *very intense* lines Zn 2138,6, Cd 2288,1 and Hg 1849, which occur in combinations; they must be considered as first line of a prinçipal series, lying in the ultra-violet. This principal series is indicated [1]) by 1,5 S—mP, a second subordinate series being indicated by 2 P—mS. The series 2,5 S—mP is a differential vibration of the lines of the principal series with the *first* line of the 2nd S.S.

$$2{,}5\,S-mP = \{(1{,}5\,S-mP) - (1{,}5\,S-2P)\} - (2P-2{,}5\,S)$$

$$= m^{th} \text{ line P.S.} - 1^{st} \text{ line P.S.} - 1^{st} \text{ line II S.S.} =$$

$$= m^{th} \text{ line D.S.} - 1^{st} \text{ line II S.S.}$$

In this I have called 1st line II S.S. ($m = 2{,}5$), what is considered the 2nd line by RITZ.

Mathematics. — *"On the conoids belonging to an arbitrary surface."* By Prof. HK. DE VRIES. (1st part).

§ 1. Among the examples current in Descriptive Geometry of non-developable scrolls we meet the so-called *right sphere conoids*, formed by all the lines which intersect a given directrix, run parallel to a plane perpendicular on that directrix, and touch a given sphere; it is a surface of order four, which has the given directrix as well as the line at infinity of the director plane as nodal lines, and the points of intersection of these two straight lines with the sphere as cuspidal points; the generatrices passing through these points coincide namely in so-called torsal lines, distinguished from the other generatrices on account of the tangential planes coinciding in all their points.

If we substitute for the sphere an arbitrary surface of order n, then the right conoid appears belonging to this arbitrary surface, which conoid seen from a mathematical point of view does not differ from the scroll formed by all the lines intersecting two arbitrary directrices r_1, r_2, crossing each other, and touching a surface Φ^n of order n; on this surface some observations follow.

§ 2. We suppose the surface Φ^n to be point general. A plane

[1]) F. PASCHEN, Ann. d. Phys. 35, 1911, p. 863.

brought through a point A_1 of r_1 and through r_2, cuts out of Φ^n a curve k^n of order n and class $n(n-1)$, from which ensues that *the two directrices r_1, r_2 are $n(n-1)$ fold lines of the scroll Ω under examination.*

A plane through r_1 contains the $n(n-1)$ fold line r_1, likewise the $n(n-1)$ single generatrices through the point of intersection of that plane with r_2: *so Ω is a surface of order $2n(n-1)$.*

Let S_1 be a point of intersection of r_1 and Φ. The plane S_1r_2 now cuts Φ according to a k^n, containing the point S_1 itself, from which ensues that two of the $n(n-1)$ generatrices of Ω through S_1 coincide with the tangent in S_1 to k^n; *through each of the n points S_1 passes therefore a torsal line of Ω, and the tangential plane belonging to it, which for convenience' sake we shall call "torsal plane", is evidently the plane S_1r_2.* The same holds of course for r_2.

There are however more cuspidal points on r_1. If namely we imagine a tangential plane through r_2 to Φ, then it will intersect Φ in a k^n with a node in the point of contact; the line connecting this point of contact with the point of intersection C_1 of the indicated tangential plane and r_1 counts for two coinciding generatrices of Ω through C_1 and is thus likewise a torsal line; *so the points C_1 are also cuspidal points of Ω.* Their number is equal to the class of Φ, thus $n(n-1)^2$, and the corresponding torsal planes are the planes C_1r_2. The same holds of course for r_2.

Other cuspidal points on r_1 or r_2 are not possible. For, if for a point A_1 of r_1 two tangents to the curve k^n lying in the plane A_1r_2 are to coincide, then this is only possible either in one of the manners described just now or because an inflectional tangent or a double tangent of k^n passes through A_1. These last cases appear in reality (comp. §§ 4, 6), however, they evidently do not lead to torsal lines, but to cuspidal edges and nodal generatrices. *The complete number of cuspidal points on r_1 (or r_2) amounts therefore to*

$$n + n(n-1)^2 = n(n^2-2n+2).$$

§ 3. As each generatrix of Ω is a tangent of Φ the scroll Ω and the surface Φ will touch each other along a certain curve, whilst both surfaces will possess in general a proper curve of intersection besides; for, of the n points of intersection of a generatrix of Ω with Φ only two (coinciding ones) belong to the curve of contact, the remaining $n-2$ to the curve of intersection.

The order of the curve of contact we can find in the following way. A plane through r_2 and a point A_1 of r_1 intersects Φ in a curve k^n, and the points of contact of the tangents drawn out of

A_1 to this curve, are the points of intersection of k^n with the first polar curve p_1^{n-1} of A_1 with respect to k^n. The locus of all these curves p_1^{n-1} is a surface, which for convenience' sake we shall call "first polar surface of r_1 with respect to Φ and r_2"; the intersection of this surface and Φ is the curve of contact to be found.

It is easy to see that the first polar surface of r_1 with respect to Φ and r_2 is of order n and contains the line r_2 as single line. A plane through r_2 namely contains the first polar curve p_1^{n-1} of the point of intersection A_1 of that plane with r_1; if now the plane rotates round r_2, then the points of intersection of p_1^{n-1} and r_2 will travel in general along the line r_2, from which ensues that r_2 itself lies on the polar surface to be found; so the question is only how many different polar curves p_1^{n-1} pass through an arbitrary point of r_2. We choose as this point one of the points of intersection S_2 of r_2 and Φ. If the first polar curve p_1^{n-1} of a certain point A_1 of r_1 is to pass through S_2, then one of the tangents drawn in the plane A_1r_2 to the curve k^n lying in that plane must have its point of contact in S_2, and it must therefore touch the surface Φ in S_2. Now the tangential plane in S_2 to Φ intersects the line r_1 only in *one* point; so only *one* curve p_1^{n-1} passes through S_2, and so also through an arbitrary other point of r_2.

Each plane A_1r_2 contains thus of the surface to be found a curve p_1^{n-1} and the single line r_2; *the surface is thus of order n.* We shall indicate it by the symbol Π_1^n. It intersects Φ in a curve of order n^2, and *this is the required curve of contact c^{n^2} of Ω and Φ.* Also r_2 possesses of course a first polar surface, Π_2^n, but now with respect to Φ and r_1; it intersects Φ according to the same curve c^{n^2}. It is clear that c^{n^2} contains the n points of intersection S_1 of r_1 and Φ as well as the n points of intersection S_2 of r_2 and Φ; the torsal lines through these points touch here c^{n^2}, because they touch Φ as well as Π_1 and Π_2. In a point S_1 namely the torsal line touches a curve k^n, thus Φ, and a curve p_1^{n-1}, thus Π_1, and therefore also the section c^{n^2} of these two surfaces.

We control these results analytically. Let r_1 coincide with the edge A_3A_4 $(x_1 = x_2 = 0)$, and r_2 with the edge A_1A_2 $(x_3 = x_4 = 0)$ of the fundamental tetrahedron, and let Φ be a homogeneous polynomium of order n in $x_1, \ldots x_4$, and let $\Phi = 0$ be the equation of the surface Φ.

For a plane through $r_2 = A_1A_2$ the two homogeneous coordinates ξ_1 and are zero, so the equation runs:

$$\xi_3x_3 + \xi_4x_4 = 0 ;$$

if this plane is to pass through a point (x'_3, x'_4) of A_3A_4, then we find

$$\xi_3 x'_3 + \xi_4 x'_4 = 0,$$

so that finally the equation of this plane runs:

$$x'_4 x_3 - x'_3 x_4 = 0.$$

If we now take of the point (x'_3, x'_4) the first polar surface

$$x'_3 \frac{\partial \Phi}{\partial x_3} + x'_4 \frac{\partial \Phi}{\partial x_4} = 0$$

with respect to Φ, then the section of this surface with the plane $x'_4 x_3 - x'_3 x_4 = 0$ is the polar curve p_1^{n-1}; the locus of these, hence the surface Π_1^n, we find by elimination of x'_3 and x'_4 out of both equations; so the equation runs:

$$\Pi_1 = x_3 \frac{\partial \Phi}{\partial x_3} + x_4 \frac{\partial \Phi}{\partial x_4} = 0,$$

really a surface of order n containing the line $r_2 (x_3 = x_4 = 0)$ as a single line.

The equation of Φ can be written in the form

$$\sum_{i=1}^{4} x_i \frac{\partial \Phi}{\partial x_i} = 0;$$

so the coordinates of the points of intersection with $r_1 (x_1 = x_2 = 0)$. satisfy

$$x_3 \frac{\partial \Phi}{\partial x_3} + x_4 \frac{\partial \Phi}{\partial x_4} = 0,$$

i.e. the equation of Π_1.

In the case of the right sphere conoid one of the two polar surfaces is a parabolic cylinder, the other a cylinder of revolution. Let us call the director line r_1, the line at infinity of the director plane $r_{2\infty}$, then each plane through a point A_1 of r_1 and through $r_{2\infty}$ intersects the sphere according to a circle, so that the first polar curve of A_1 becomes a line normal to the plane through r_1 and the centre of the sphere; this line as well as $r_{2\infty}$ form the complete intersection of the considered plane with Π_1. If however we consider in particular the plane at infinity we have to take the polar line of the point of r_1 at infinity with respect to the absolute circle, which coincides with $r_{2\infty}$; so Π_1 is indeed a parabolic cylinder whose generatrices are normal to the plane through r_1 and the centre of the sphere. In the planes through r_1 on the other hand we have to take the vertical diameters of the circles of intersection with the sphere lying in that plane, from which ensues immediately that Π_2 becomes a quadratic cylinder with vertical generatrices. The points of intersection of $r_{2\infty}$ with the sphere are isotropic points; the circle lying in the plane through such a point and r_1 passes itself through that point and touches here the absolute circle, so that the polar line of that point becomes a tangent to the absolute circle; so the cylinder touches the absolute circle twice and is therefore a cylinder of rotation. The sphere and these two cylinders intersect each other according to a twisted curve of order 4 and the 1st species, containing among others the isotropic points of intersection of $r_{2\infty}$ with the sphere; on the plane through r_1 and the centre of the sphere it projects itself as a parabola, on a horizontal plane as a circle.

§ 4. We again imagine a point A_1 of r_1, then a plane A_1r_2, and the section with Π_1 lying in this plane and consisting of the curve p_1^{n-1} and the line r_2. We take this system as a curve of order n and we determine the first polar curve q_1^{n-1} for the pole A_1, which is of order $n-1$, and contains the $n-1$ points of intersection of p_1^{n-1} and r_2, but moreover the points of contact of the $(n-1)(n-2)$ tangents which can be drawn out of A_1 to p_1^{n-1}. We now look for the locus of the curves q_1^{n-1} and show that this is again a surface of order n, having r_2 as a single line. The first polar surface of the point (x'_3, x'_4) with respect to $\Pi_1 = 0$ has for equation

$$x'_3 \frac{\partial \Pi_1}{\partial x_3} + x'_4 \frac{\partial \Pi_1}{\partial x_4} = 0,$$

hence (see § 3):

$$x'_3 \frac{\partial \Phi}{\partial x_3} + x'_3 x_3 \frac{\partial^2 \Phi}{\partial x^2_3} + (x'_3 x_4 + x'_4 x_3) \frac{\partial^2 \Phi}{\partial x_3 \partial x_4} + x'_4 \frac{\partial \Phi}{\partial x_4} + x'_4 x_4 \frac{\partial^2 \Phi}{\partial x_4^2} = 0,$$

a surface of order $n-1$ and which, cut by the plane $x'_4 x_3 - x'_3 x_4 = 0$, furnishes the curve q_1^{n-1}. The locus of this curve, found by elimination of x'_3 and x'_4 out of the last two equations, is therefore the surface

$$K_1 = x_3 \frac{\partial \Phi}{\partial x_3} + x_3^2 \frac{\partial^2 \Phi}{\partial x_3^2} + 2 x_3 x_4 \frac{\partial^2 \Phi}{\partial x_3 \partial x_4} + x_4 \frac{\partial \Phi}{\partial x_4} + x_4^2 \frac{\partial^2 \Phi}{\partial x_4^2} = 0;$$

it is indeed of order n and contains r_2 $(x_3 = x_4 = 0)$ as a single line, just as Π_1. The section with Π_1 is therefore a curve of order n^2, of which r_2 forms a part; it is however easy to show that r_2 must be counted twice, so that there remains a residual section of order n^2-2. The section of Π_1 and K_1 lies namely evidently also on the surface:

$$K_1{}^* = x_3^2 \frac{\partial^2 \Phi}{\partial x_3^2} + 2 x_3 x_4 \frac{\partial^2 \Phi}{\partial x_3 \partial x_4} + x_4^2 \frac{\partial^2 \Phi}{\partial x_4^2} = 0,$$

which has evidently the line A_1A_2 as a double line. For the section of Π_1 and $K_1{}^*$, or K_1 and $K_1{}^*$, the director r_2 counts double; thus it must also count double for the section of Π_1 and K_1, with which is shown that these two surfaces have in each point of r_2 the same tangential plane.

The equation of Φ can not only be written in the form $\Sigma x_i \frac{\partial \Phi}{\partial x_i} = 0$, but also in the symbolic form $\left\{ \Sigma x_i \frac{\partial \Phi}{\partial x_i} \right\}^2 = 0$. Let us put in it $x_1 = x_2 = 0$ in order to determine the n points of intersection S_1 with r_1, then exactly the equation $K_1{}^* = 0$ remains, from which follows that the n points S_1 lie at the same time on $K_1{}^*$ and therefore also on K_1; it is even easy to show that each of these points counts double among the number of points of intersection of the three surfaces Φ, Π_1, $K_1{}^*$. In a plane $S_1 r_2$ lie namely, as intersection with Φ, a curve k^n, as intersection with Π_1 the first polar curve of these, $p_1{}^{n-1}$, and these curves touch each other in S_1. Now however the curve $q_1{}^{n-1}$ is again the first polar curve of S_1 with respect to the curve of order n, consisting of $p_1{}^{n-1}$ and r_2; so $q_1{}^{n-1}$ touches in S_1 the two other curves. The tangential planes in S_1 to the three mentioned surfaces intersect each other according to the same line, namely the torsal line of Ω through S_1 (§ 2); each of these points counts thus indeed for two points of intersection of the three surfaces. Now outside r_2 (see above) lie $n(n^2 - 2)$ of these points; if moreover we subtract still the $2n$ points S_1 then $n(n^2 - 4)$ points remain, lying neither on r_1 nor on r_2. If we suppose a plane through such a point P and r_2, which is intersected in A_1 with r_1, then the curves k^n, $p_1{}^{n-1}$, $q_1{}^{n-1}$ lying in this plane (and therefore also the second polar curve $p_1{}^{n-2}$ of A_1) all pass through P, from which ensues that P is for k^n an inflectional point and therefore A_1P one of the two principal tangents (osculating tangents) of Φ in P. With this we have shown, that *in the congruence of the principal tangents of the general surface of the n^{th} order $n(n^2 - 4)$ of these lines rest on two arbitrary lines, or in other words, that the principal tangents intersecting an arbitrary line form a scroll of order $n(n^2 - 4)$.*

Through an arbitrary point of space pass $n(n-1)(n-2)$ of those lines[1]; for we have but to take the points of intersection of the surface itself with the first and the second polar surface of the chosen point; *the surface just found has thus the right line on which all generatrices rest, as an $n(n-1)(n-2)$-fold line.*

A plane through this line contains, besides the $n(n-1)(n-2)$-fold line, a curve of intersection of order $n(n^2-4) - n(n-1)(n-2) = 3n(n-2)$,

[1]) CREMONA—CURTZE: "Grundzüge einer allgemeinen Theorie der Oberflächen", p. 64, or SALMON—FIEDLER: "Anal. Geom. des Raumes", II. Theil, S. 24.

of which it is easy to show that it consists of $3n(n-2)$ lines; for, through an arbitrary point of this section a principal tangent of the surface must pass resting on the multiple line, therefore lying entirely in the plane. The $3n(n-2)$ lines are evidently the inflectional tangents of the section of the plane under consideration with the surface of order n.

An ordinary point of contact of a generatrix of Ω with Φ is a single point of the curve of contact c^{n^2} (§ 3), in each of the $n(n^2-4)$ points P just now found, however, the generatrix A_1P has with Φ a three point contact, with Π_1 a two point one, and therefore also with c^{n^2} a two point one; so *there are* $n(n^2-4)$ *generatrices of* Ω *touching* c^{n^2}.

§ 5. A generatrix of Ω touches Φ, and has thus, besides the point of contact, still $(n-2)$ points in common with this surface; in a plane A_1r_2 lie therefore $n(n-1)(n-2)$ such points, namely on each of the $n(n-1)$ generatrices in this plane every time $n-2$. All these points lie on a curve of order $(n-1)(n-2)$, the satellite curve of the first polar curve p_1^{n-1} of A_1 with respect to k^n. If the plane revolves around r_2, the satellite curve will generate a surface which we shall call "the satellite surface" of r_1 with respect to Φ and r_2, and which will evidently cut out of Φ the residual intersection of Ω with Φ.

The intersection of the satellite surface Σ_1 with a plane A_1r_2 consists of a satellite curve s_1 of order $(n-1)(n-2)$, and of the line r_2; the question is how many different satellite curves pass through an arbitrary point of r_2. In order to answer this question we shall consider again in particular a point of intersection S_2 of r_2 and Φ. If the curve s_1 lying in a plane A_1r_2 is to pass through S_2, then A_1S_2 must be a tangent to Φ without the point of contact coinciding with S_2. Now the plane r_1S_2 cuts Φ in a curve of order n containing the point S_2 itself and to which $n(n-1)-2$ tangents can be drawn out of S_2, not touching in S_2 itself; in the planes through these tangents and r_2 the curves s_1 will pass through S_2. *So we find for the satellite surface* Σ_1 *a surface of order* $(n-1)(n-2)+n(n-1)-2=2n(n-2)$, *with an* $\{n(n-1-2\}$*-fold line* r_2. *The satellite curve of* c^{n^2}, *the intersection of* Φ *and* Σ_1, *is thus a curve of order* $2n^2(n-2)$, *with* $\{n(n-1)-2\}$*-fold points in the* n *points of intersection* S_2 *of* Φ *and* r_2.

Now however it is clear, that just as there is only *one* curve of contact c^{n^2}, immaterial whether we start from the polar surface of r_1 or of r_2, there is also only *one* satellite curve; for the curve of

33*

contact is simply the locus of the points of contact of the generatrices of Ω with Φ, and the satellite curve is the locus of the points of intersection of the same generatrices with Φ. However, if we start from r_2, we find as satellite surface Σ_2 a surface of order $2n(n-2)$ with an $\{n(n-1)-2\}$-fold line r_1, from which ensues that the satellite curve of c^{n^2} has also $\{n(n-1)-2\}$-fold points in the n points of intersection S_1 of r_1 and Φ. This result is also easy to control with the aid of Σ_1; this Σ_1 namely does *not* contain the line r_1, but it does the points S_1, and it has in these points a contact with Φ of higher order, and inversely Σ_2 does *not* contain the line r_2, but it does the points S_2, and it has likewise in these points a contact of higher order with Φ.

Let us imagine a point S_1 and the section k^n of the plane S_1r_2 with Φ. The point S_1 lies on k^n; so through S_1 pass, besides the tangent in S_1 itself, $n(n-1)-2$ tangents more, from which ensues that the satellite curve s_1 of S_1 has in this point with k^n an $\{n(n-1)-2\}$-pointed contact. If we allow the plane under consideration to revolve a little about r_2 in one sense as well as in the other, then S_1 passes into a point A_1; the tangent in S_1 itself passes in one case into two different real ones, in the other into two conjugate complex ones; on the reality, however, of the other tangents the slight difference in position of the plane will have no influence, and so we see by direct observation that through S_1 pass $n(n-1)-2$ branches of the satellite curve of c^{n^2}. So the points S_1 must lie also on Σ_1; the remaining points of r_1 however lie in general not on it, because the satellite curve s_1 of an arbitrary point A_1 does in general not pass through A_1 itself; so the points S_1 must thus be either singular points of Σ_1, or Σ_1 and Φ must have in those points a contact of higher order. If S_1 were a singular point, thus a multiple point with a tangential cone of order $n(n-1)-2$, then each plane through this point would have to cut Σ_1 according to a curve with an $\{n(n-1)-2\}$-fold point in S_1; we saw, however just now that the plane S_1r_2 cuts the surface Σ_1 according to a curve, which has in S_1 an ordinary point, but with k^n an $\{n(n-1)-2\}$-pointed contact; so S_1 is also an ordinary point of Σ_1, but an $\{n(n-1)-2\}$-fold point for the intersection with Φ.

We control the preceding results in the following way. The complete intersection of Ω and Φ is a curve of order $2n^2(n-1)$; it consists of the curve of contact c^{n^2}, counted double, and of the satellite curve; and $2n^2+2n^2(n-2)$ really furnishes $2n^2(n-1)$.

§ 6. The surface Ω contains in general a certain number of double generatrices, i. e. double tangents of Φ, cutting r_1 and r_2; we determine their number by determining the order of the scroll formed by all the double tangents of Φ which intersect r_1. A plane through r_1 cuts Φ in a k^n and this possesses $\frac{1}{2}n(n-2)(n^2-9)$ double tangents, and through an arbitrary point of r_1 pass $\frac{1}{2}n(n-1)(n-2)(n-3)$ double tangents;[1]) the surface to be found is therefore of order $\frac{1}{2}n(n-2)(n^2-9)+\frac{1}{2}n(n-1)(n-2)(n-3)=(n+1)(n)(n-2)(n-3)$, and it has r_1 as an $\frac{1}{2}n(n-1)(n-2)(n-3)$-fold line. *The number of double generatrices of Ω is equal to the number of points of intersection of this surface with r_2, so equal to* $(n+1)(n)(n-2)(n-3)$.

With the aid of the points of contact of the double generatrices with Φ, likewise of the $n(n^2-4)$ points found in § 4 on principal tangents of Φ, we can now entirely survey the mutual position of the four surfaces Ω, Φ, Π_1, Σ_1, likewise of their intersections. We fix our attention in particular on the curve of contact c^{n^2} and the corresponding satellite curve. According to § 4 there are $n(n^2-4)$ generatrices of Ω touching c^{n^2}; if P is one of the points of contact, A_1 the point of intersection with r_1, then P is an inflectional point for the section k^n with Φ lying in the plane A_1r_2, A_1P the corresponding inflectional tangent, and it counts for two of the $n(n-1)$ tangents which can be drawn out of A_1 to k^n, so that besides the inflectional tangent only $n(n-1)-2$ tangents pass through A_1. Each of these intersects k^n in $n-2$ points, altogether thus in $\{n(n-1)-2\}(n-2)$, whilst the complete number of points of intersection of the satellite curve of p_1^{n-1} with k^n amounts to $n(n-1)(n-2)$; the missing $2(n-2)$ must thus be furnished by the inflectional tangent. Now it is easy to see, that by a slight change of position of A_1 the inflectional tangent would break up into two separate tangents; by attending in this position to the satellite curve and then by returning to the inflectional tangent we convince ourselves that the satellite curve of p_1^{n-1} touches k^n in the $n-3$ points of intersection of the inflectional tangent.

Now but two points are missing and these can lie nowhere else but in P; so the satellite curve of p_1^{n-1} touches in P the curve k^n. Now this satellite curve lies on the satellite surface Σ_1, which intersects Φ according to the satellite curve of c^{n^2}; so this one too must touch in P the line A_1P, just as c^{n^2}, so that the $n(n^2-4)$ points P mentioned above represent $2n(n^2-4)$ points of intersection of c^{n^2} with its satellite curve.

Let us further consider one of the $(n+1)(n)(n-2)(n-3)$ double generatrices of Ω with the points of contact P_1, P_2, and the point

[1]) CREMONA—CURTZE, l. c. p. 64. SALMON—FIEDLER, l. c. p. 25.

of intersection A_1 with r_1. In the plane A_1r_2 now pass also through A_1, besides the double tangent, only $n(n-1)-2$ tangents to k^n, so that now again on the line $A_1P_1P_2$ must lie $2(n-2)$ points of intersection of k^n with the satellite curve of p_1^{n-1}. In the $n-4$ points of intersection of the double tangent with k^n the satellite curve of p_1^{n-1} will again touch k^n; the missing four points must be divided regularly among the two points of contact P_1 and P_2, from which ensues that the satellite curve of p_1^{n-1} touches the double generatrix of Ω in P_1 and P_2. The satellite curve of c^{n^2} will thus also have this property; however as regards c^{n^2} itself, it passes also through P_1 and P_2, but without touching the line $A_1P_1P_2$ in these points; so on all the double generatrices of Ω together lie $2(n+1)(n)(n-2)(n-3)$ points of intersection of c^{n^2} with its satellite curve.

Now c^{n^2} and its satellite curve have more points in common still, but these lie all on r_1 and r_2. The surface Π_1 has r_2 as a single line (§ 3), on the other hand Σ_1 has r_2 as an $\{n(n-1)-2\}$-fold line, so the intersection of the two breaks up into a curve and the line r_2, the latter counted $\{n(n-1)-2\}$ times. The surface Φ cuts r_2 in the n points S_2; so these count for $n\{n(n-1)-2\}$ points of intersection of the three surfaces Φ, Π_1, Σ_1, and therefore for as many points of intersection of c^{n^2} with its satellite curve. We saw further in § 5 that the satellite curve of c^{n^2}, thus the intersection of Φ and Σ_1, has in the n points S_1 on r_1 again $\{n(n-1)-2\}$-fold points; as Φ contains these points also, they count for $n\{n(n-1)-2\}$ points of intersection of c^{n^2} with its satellite curve.

We now add the different amounts found, thus $2n(n^2-4)$, $2(n+1)(n)(n-2)(n-3)$, $2n\{n(n-1)-2\}$ together, and we find $2n^3(n-2)$, just the complete number of points of intersection of the three surfaces Φ, Π_1, Σ_1 of order $n, n, 2n(n-2)$.

§ 7. Through a point A_1 of r_1 pass $n(n-1)$ tangents to the curve k^n lying in the plane A_1r_2 and these intersect r_2 in $n(n-1)$ points A_2; inversely to such a point A_2, $n(n-1)$ points A_1 correspond, from which ensues that we can regard the surface Ω as generated by the lines connecting the corresponding points of two series of points lying on r_1 and r_2, between which there is a $\{n(n-1), n(n-1)\}$-correspondence. If we project these two series out of an arbitrary line l, then two collocal pencils of planes are formed, between which there is likewise an $\{n(n-1), n(n-1)\}$-correspondence; the $2n(n-1)$ coincidences are planes each containing the line connecting two corresponding points, thus a generatrix of Ω, out of which follows $2n(n-1)$ for the order of Ω (§ 2).

On each of the two bearers lie $2n(n-1)\{n(n-1)-1\} = 2n(n^3-2n^2+1)$ branch points[1]), i. e. points of whose corresponding points on the other bearer two coincide, which coinciding points are then called double points; we shall now investigate how in our case the branch points put in an appearance. We consider therefore in the first place the n points of intersection S_1 of r_1 with Φ. In the plane $S_1 r_2$ lies a curve k^n passing through S_1; so through S_1 pass $n(n-1)-2$ tangents which do not touch in S_1, and two coinciding ones which do touch in S_1; so evidently S_1 is a branch point on r_1, and the point of intersection of the torsal line passing through S_1 with r_2 is the corresponding double point. Number n.

Through r_2 pass $n(n-1)^2$ tangential planes of Φ, and each of these cuts Φ in a curve k^n with a node. If the point of intersection of such a plane with r_1 is a point A_1, then out of A_1 start $n(n-1)-2$ proper tangents to k^n, whilst the line connecting A_1 and the node counts for two coinciding ones; so A_1 is also a branch point. Number $n(n-1)^2$.

Further in § 4 we found $n(n^2-4)$ generatrices of Ω which are at the same time principal tangents of Φ. If the point of contact of such a principal line with Φ is P and A_1 the point of intersection of the plane Pr_2 with r_1, then from A_1 start $n(n-1)-2$ ordinary tangents to k^n and moreover the inflectional tangent $A_1 P$ to be counted twice; so A_1 is again a branch point. Number $n(n^2-4)$.

Finally in § 6 we found $(n+1)(n)(n-2)(n-3)$ double generatrices of Ω; it is clear, that also the points of intersection of these with r_1 and r_2 are branch points. Number $(n+1)(n)(n-2)(n-3)$.

Other branch points there are none. If e. g. a point A_1 is to be a branch point, then two of the tangents out of A_1 to k^n must coincide, and that is only possible in one of the four ways described above. If now the four mentioned numbers are added up we do not find the required complete number of branch points $2n(n^3-2n^2+1)$, but only $n(n^3-2n^2-n+4)$, i. e. for very great values of n only half; on the other hand we find the exact number, if we bring the $n(n^2-4)$ points of the third group three times into account, and the $(n+1)(n)(n-2)(n-3)$ of the last twice. The question is how to explain this.

If we bring a plane through an arbitrary point O of space and a generatrix b of Ω, and likewise through an adjacent generatrix b^*, and if we then let b tend to b^* to coincide with it finally, then at the limit the line of intersection OBB^* of the two planes passes

[1]) EMIL WEYR "Beiträge zur Curvenlehre", S. 3.

into an edge of the circumscribed cone of Ω having O as vertex; B becomes the point of contact of that edge with Ω, thus a point of the intersection of Ω with the first polar surface of O. Let us imagine a point A_1 of r_1, lying in the immediate vicinity of a branch point, then from this point among others two generatrices of Ω lying very close together will start; the planes through those generatrices and O are two tangential planes of the circumscribed cone lying very close together, and OA_1 is therefore a line lying in the immediate vicinity of that cone. At the transition to the limit the branch point becomes, just like the point B mentioned above, a point of intersection of Ω with the first polar surface of O. This intersection, however, in our case breaks up into a number of separate parts. Through a double edge of Ω e. g. pass two sheets of Ω and passes *one* sheet of the first polar surface; the double edge forms thus a part of the intersection of the two surfaces, counts however double, and it furnishes therefore in its point of intersection with r_1 two coinciding branch points. Of course likewise for r_2.

Suchlike considerations hold also for the $n(n^2-4)$ cuspidal edges of Ω. Each plane through O cuts Ω according to a curve having cusps on the cuspidal edges, and it is well known that the first polar curve of O with respect to that curve contains the cusps and touches the cuspidal tangents. From this ensues that the first polar surface of O, with respect to Ω, contains the cuspidal edges, and has in each point of such an edge the tangential plane in common with Ω; each cuspidal edge counts thus three times for the intersection and furnishes also three coinciding branch points on r_1 and r_2.

All branch points have been accounted for in this way.

§ 8. The apparent circuit of the surface Ω out of an arbitrary point O of space on a plane e.g. is the section of that plane with the projection (out of O as centre) of the intersection of Ω with the first polar surface of O. This intersection consists however, as we already saw in § 7, of a number of separate parts. For Ω the directors r_1 and r_2 are $n(n-1)$-fold lines, for the polar surface $\{n(n-1)-1\}$-fold lines; for the intersection of both they count $n(n-1)\{n(n-1)-1\}$ times. Each of the $(n+1)(n)(n-2)(n-3)$ double edges counts twice, each of the $n(n^2-4)$ cuspidal edges three times, and as the complete intersection is of order $2n(n-1)\{2n(n-1)-1\}$, there remains a proper curve of intersection of order

$2n(n-1)\{2n(n-1)-1\}-2n(n-1)\{n(n-1)-1\}-2(n+1)(n)(n-2)(n-3)-3n(n^2-4)=$ $2n^4-9n^3+10n^2+10n-12$. This is thus at the same time the order of the projecting cone out of O or of the apparent circuit on a plane, or the class of a plane section of Ω.

For the class of the apparent circuit we must know the number of tangents through an arbitrary point P of the plane of projection. Now OP cuts the surface Ω in $2n(n-1)$ points; through each of these passes a generatrix, and the plane through these and OP is a tangential plane through OP, so the trace of that plane is a tangent to the apparent circuit; the class of the apparent circuit is therefore $2n(n-1)$.

Let us bring a plane through O and a torsal line whose cusp lies on r_1. It cuts Ω according to a curve of order $2n(n-1)-1$, and as the complete intersection, consisting of this curve and r_1, must have $n(n-1)$-fold points on r_1 and r_2, the curve itself has on the directors $\{n(n-1)-1\}$-fold points. These points lie at the same time on the generatrix; the only still missing point of intersection with this generatrix coincides with the cusp, and in projection the apparent circuit touches in this point the torsal line.

A plane through O and a double edge of Ω contains as residual section only a curve of order $2n(n-1)-2$ with $\{n(n-1)-2\}$-fold points on r_1 and r_2, and which thus cuts the double edge in two points more; a plane through a double edge is therefore a double tangential plane and the two points just mentioned are the points of contact. The projection of the double edge is a double tangent of the apparent circuit; the points of contact are the projections of the two points just mentioned on Ω.

In a plane through O and a cuspidal edge the latter counts likewise for two, so that here too remains a residual section of order $2n(n-1)-2$ with $\{n(n-1)-2\}$-fold points on r_1 and r_2; the two missing points of intersection with the cuspidal-edge coincide here and the projection of this edge becomes an inflectional tangent of the apparent circuit.

Let us now imagine a plane through O and r_1. Let S_2 be the point of intersection of this plane with r_2, then to this point correspond $n(n-1)$ points on r_1, and the projection of r_1 touches the apparent circuit in the projections of those points; the apparent circuit has therefore the projections of r_1 and r_2 as $n(n-1)$-fold tangents. If we now reduce these multiple tangents to double ones and if we then suppose that the double tangents and the inflectional tangents just now found are the only ones that the curve possesses, and if finally we remember that the class of the curve is $2n(n-1)$, then the Plücker formula to determine the order becomes identical to the formula at the beginning of this paragraph, and so we find for the order the exact number; so we also possess the exact numbers of the double tangents and the inflectional ones, so that only those of the double points and cusps are missing. The PLÜCKER formula $\iota - \varkappa = 3(\nu - \mu)$ furnishes us with $\varkappa = \iota + 3(\mu - \nu)$; if we introduce the values, we find $\varkappa = 6n^4 - 26n^3 + 24n^2 + 32n - 36$. Finally the formula $\nu = \mu(u-1) - 2\delta - 3\varkappa$ furnishes us with the double number of double points: $2\delta = \mu(u-1) - \nu - 3\varkappa$, hence:

$$2\delta = (2n^4-9n^3+10n^2+10n-12)(2n^4-9n^3+10n^2+10n-13) -$$
$$- 2n(n-1) - 3(6n^4-26n^3+24n^2+32n-36).$$

Summing up we have thus found: *the apparent circuit of* Ω *on an arbitrary plane is a curve of order* $2n^4 - 9n^3 + 10n^2 + 10n - 12$, *of class* $2n(n-1)$, *with a number of double points* $= \delta$ (see above), *a number of cusps* $= \varkappa$ (see above), *with* $(n+1)(n)(n-2)(n-3)$ *double tangents, the projections of the double generatrices of* Ω, *with* $n(n^2-4)$ *inflectional tangents, the projections of the cuspidal edges of* Ω, *and with two* $n(n-1)$*-fold tangents, the projections of the two directors* r_1 *and* r_2.

§ 9. If Ω is really a conoid, i. e. if r_2 is the line at infinity of a director plane, then as a rule the latter is chosen as plane of projection, and so the projection of the surface on a plane through one of the two directors becomes of importance. In the numbers mentioned at the end of the preceding § no change takes place; so in the case of the conoid the apparent circuit on a director plane possesses $n(n-1)$ parabolic branches. It is a different thing, however, if the conoid is a right one, i. e. if r_1 is normal to the director plane; if then the latter is horizontal, and if the apparent circuit of Ω is required for the point Z_∞ as centre, then we have to project out of a point of the surface itself, and that one lying on the $n(n-1)$ fold line r_1. It is now immediately clear that the apparent circuit is entirely modified; for a line through Z_∞ cuts Ω besides Z_∞ only in $n(n-1)$ points, and only the generatrices passing through these points give rise, when projected out of Z_∞, to tangents of the apparent circuit; however, they all pass in projection through the point of intersection R_1 of r_1 with the director plane, from which ensues that the pencil round R_1 is discarded and that $n(n-1)$ times.

The plane through Z_∞ and one of the $n(n-1)$ generatrices of Ω (lying entirely at infinity) is indefinite, i. e. each suchlike plane is a tangential plane through Z_∞; of the apparent circuit we have to discard $n(n-1)$ pencils whose vertices are the points of intersection of the generatrices through Z_∞ with $r_{2\infty}$. These pencils and those round R_1, the latter counted $n(n-1)$ times, form the complete apparent circuit, indeed a degenerated curve of class $2n(n-1)$.

For the vertical projection the centre Y_∞ lies on $r_{2\infty}$; the apparent circuit on the vertical plane consists therefore of $n(n-1)$ pencils round points on the projection of r_1, and of a pencil whose vertex is the point at infinity on the x-axis and which pencil must be counted $n(n-1)$ times.

Mathematics. — "*Surfaces, twisted curves and groups of points as loci of vertices of certain systems of cones*" by Prof. P. H. SCHOUTE [1]). First paper.

1. We consider as given $(n+2)_2$ pairs of straight lines crossing each other, $(a_i, a'_i), (b,b')$, where i assumes successively the values $1, 2, \ldots, \frac{1}{2}n(n+3)$. We represent by t_{a_i} a transversal of (a_i, a'_i), by t_b a transversal of (b, b'). The points P emitting $(n+2)_2$ transversals t_{a_i}, t_b lying on a cone C^n of order n form a surface (P) of which the order is to be determined.

However we remark first, that the $2(n+2)_2$ given lines (a_i, a'_i), (b, b') are lines of multiplicity n on (P). For, the cone C^n with an arbitrary point P of b as vertex and the transversals t_{a_i} emitted by this point as edges, cuts the line b' in n points and is therefore to be counted n times among the considered system of cones C^n, i.e. once for each of these points of intersection.

Moreover it is immediately evident, that each point of each of the two common transversals $t_{i,k}$ and $t'_{i,k}$ of the pairs (a_i, a'_i) and (a_k, a'_k) is vertex of a cone of the system, as we find for this point $(n+2)_2-1$ edges only. So these lines, $6(n+3)_4$ in number, are single lines of (P).

2. In order to determine the order of (P) we try to find the number of points P satisfying the conditions of the problem lying on an arbitrary transversal t_b, by means of a figure lying in an arbitrarily chosen plane π connected with our figure in space in the following way.

We consider the transversals t_{a_i} emitted by the points P of t_b and remark that they form a regulus (t_b, a_i, a'_i) of which t_b, a_i, a'_i

[1]) Suggested by the last communication of Prof. JAN DE VRIES (These Proceedings, XIV, p. 259).