round R_1, the latter counted $n(n-1)$ times, form the complete apparent circuit, indeed a degenerated curve of class $2n(n-1)$.

For the vertical projection the centre Y_∞ lies on $r_{2\infty}$; the apparent circuit on the vertical plane consists therefore of $n(n-1)$ pencils round points on the projection of r_1, and of a pencil whose vertex is the point at infinity on the x-axis and which pencil must be counted $n(n-1)$ times.

Mathematics. — *"Surfaces, twisted curves and groups of points as loci of vertices of certain systems of cones"* by Prof. P. H. SCHOUTE[1]). First paper.

1. We consider as given $(n+2)_2$ pairs of straight lines crossing each other, (a_i, a'_i), (b,b'), where i assumes successively the values $1, 2, \ldots, \frac{1}{2}n(n+3)$. We represent by t_{a_i} a transversal of (a_i, a'_i), by t_b a transversal of (b, b'). The points P emitting $(n+2)_2$ transversals t_{a_i}, t_b lying on a cone C^n of order n form a surface (P) of which the order is to be determined.

However we remark first, that the $2(n+2)_2$ given lines (a_i, a'_i), (b, b') are lines of multiplicity n on (P). For, the cone C^n with an arbitrary point P of b as vertex and the transversals t_{a_i} emitted by this point as edges, cuts the line b' in n points and is therefore to be counted n times among the considered system of cones C^n, i.e. once for each of these points of intersection.

Moreover it is immediately evident, that each point of each of the two common transversals $t_{i,k}$ and $t'_{i,k}$ of the pairs (a_i, a'_i) and (a_k, a'_k) is vertex of a cone of the system, as we find for this point $(n+2)_2-1$ edges only. So these lines, $6(n+3)_4$ in number, are single lines of (P).

2. In order to determine the order of (P) we try to find the number of points P satisfying the conditions of the problem lying on an arbitrary transversal t_b, by means of a figure lying in an arbitrarily chosen plane π connected with our figure in space in the following way.

We consider the transversals t_{a_i} emitted by the points P of t_b and remark that they form a regulus (t_b, a_i, a'_i) of which t_b, a_i, a'_i

[1]) Suggested by the last communication of Prof. JAN DE VRIES (These Proceedings, XIV, p. 259).

are directrices. The quadric bearing this regulus cuts π in a conic c_i^2 and on this conic the regulus itself marks a series of points, in projective correspondence with the series of points P on t_i. So we get in π a system of $(n+2)_2-1$ series of points on conics, in mutual projective correspondence, with the particularity that the point of intersection B of t_b and π is a common corresponding point of all. As often as a point P of t_b different from B furnishes $(n+2)_2-1$ points P_i of these series on conics lying on a curve c^n of order n passing through B, as often t_b cuts the surface (P) in a point not lying on one of the lines b, b'; in other words, if the first number is p, the order of (P) is $p+2n$. Now the number p can be easily determined. If we assume in π a triangle of coordinates of which B is the vertex $x_2=0$, $x_3=0$, the $(n+2)_2-1$ series of points can be represented by

$$x_1^{(i)} = f_{1,i}\lambda^2 + g_{1,i}\lambda + h_{1,i}\,,\quad x_2^{(i)} = f_{2,i}\lambda^2 + g_{2,i}\lambda + h_{2,i}\,,\quad x_3^{(i)} = f_{3,i}\lambda^2 + g_{3,i}\lambda + h_{3,i}$$

where $h_{2,i}$ and $h_{3,i}$ will have to disappear for all the values of i if we stipulate that $\lambda=0$ corresponds to the common point B. So the equation of the curve c_j^n through the $(n+2)_2-1$ points P_i corresponding to λ is obtained by putting a determinant of order $(n+2)_2$ equal to zero, of which

$$x_1^n\,,\; x_1^{n-1}x_2\,,\; x_1^{n-1}x_3\,,\; x_1^{n-2}x_2^2\,,\; x_1^{n-2}x_2x_3\,,\; \dots,\; x_3^n$$

is the first row, whilst the other rows can be deduced from this one by substituting for x_1, x_2, x_3 successively the quadratic forms in λ of $x_1^{(i)}, x_2^{(i)}, x_3^{(i)}$ corresponding to the different values of i. Substitution of $1, 0, 0$ for x_1, x_2, x_3 in the first row furnishes then the equation of condition determining λ. If Δ is the minor of the determinant with respect to x_1^n, the equation of condition is $\Delta=0$, the substitution of $1, 0, 0$ in the first row annulling all the elements of this row with exception of the first. The order of this minor in λ would be $\frac{1}{2}n(n+3)$ times $2n$, or $n^2(n+3)$, if $h_{2,i}$ and $h_{3,i}$ did not disappear for all values of i. But on this account the order has to be lessened, as we can divide the elements of columns 1 and 2 of the minor by λ, those of the columns 3, 4 and 5 by λ^2, those of the columns 6, 7, 8 and 9 by λ^3, etc. and those of the last $n+1$ columns by λ, whilst the value zero of λ, corresponding to the point of coincidence B of the series, has to be discarded. So we have to diminish $n^2(n+3)$ by

$$2.1 + 3.2 + 4.3 + \ldots + (n+1)\,n = \frac{1}{3}\,n\,(n+1)\,(n+2) = 2\,(n+2)_3$$

and find for p the value $n^2\,(n+3) - 2\,(n+2)_3$ and therefore for the order $p + 2n$ of (P)

$$n^2\,(n+3) - \frac{1}{3}\,n\,(n+1)\,(n+2) + 2n = \frac{2}{3}\,n\,(n+1)\,(n+2) = 4\,(n+2)_3$$

So we have got [1]):

THEOREM I. "The locus of the point P emitting transversals lying on a cone C^n to $(n+2)_2$ arbitrarily given pairs of lines is a surface (P) of order $4\,(n+2)_3$, of which the given lines are lines of multiplicity n and the pairs of transversals of the given pairs taken by two single lines."

For $n = 1$ this result is contained in the paper quoted above; for $n = 2$ it admits of a simple check. In the special case of six pairs of *intersecting* lines any quadratic cone of the system must fulfil with respect to the combination of the point of intersection A_i and the connecting plane α_i of each pair (a_i, a'_i) one of two conditions, i. e. either pass through A_i or have a vertex lying in α_i, in which latter case the cone is to be counted twice, once for each of the two edges lying in α_i. So we find in this case the generally known surface of the vertices of the cones passing through six given points A_i and besides this surface O^4 with 25 straight lines the six planes α_i counted twice, i. e. an O^{16} as the theorem requires.

Inversely we find by means of the corresponding case for an arbitrary n, i. e. of the case of $(n+2)_2$ pairs of intersecting lines:

THEOREM II. "The locus of the vertices of the cones C^n passing through $(n+2)_2$ arbitrarily given points is a surface of order $(n+2)_3$ of which the given points are points of multiplicity n."

In the special case of $(n+2)_2$ pairs of intersecting lines the $O^{4(n+2)_3}$ of the vertices of cones C^n consists of the $(n+2)_2$ connecting planes α_i counting n times and of the surface of the second theorem. So the order of this surface is

$$4\,(n+2)_3 - n\,(n+2)_2 = 4\,(n+2)_3 - 3\,(n+2)_3 = (n+2)_3\,.$$

As the lines connecting the $(n+2)_2$ points of intersection A_i by two lie on $O^{(n+2)_3}$ each of these points must be an n-fold point of this surface.

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1) We remark that the number of points of the locus lying on an arbitrary line can be found quite as easily by means of the method used above: in that case the determinant itself, with its $(n+2)_2$ rows each of order $2n$ in λ, would have been of order $6\,(n+2)_3$ in λ, and diminution with $2\,(n+2)_3$ would have given the same result $4\,(n+2)_3$. This confirms that the given lines are n-fold lines of the locus.

3. Have we been able to deduce until now theorems holding for an arbitrary value of n, in proceeding to the determination of the twisted curve ϱ forming the locus of the point P, emitting transversals lying on a cone C^n to $\frac{1}{2}n(n+3)+2$ pairs of lines (a_i, a'_i), (b, b'), (c, c') we are obliged to treat the cases $n=2$, $n=3$, etc. separately. We will indicate first what is the cause of this and restrict ourselves then in this communication to the case $n=2$.

The surfaces $(P)_b$ and $(P)_c$, corresponding in the manner indicated in theorem I to the systems (a_i, a'_i), (b, b') and (a_i, a_i), (c, c'), admit as such the $\frac{1}{2}n(n+3)$ pairs of lines a_i, a'_i as common lines of multiplicity n and the $\frac{1}{2}n(n+3)\left[\frac{1}{2}n(n+3)-1\right]$ transversals cutting these pairs by two as common single lines. So these surfaces intersect each other still in a curve of order

$$\frac{4}{9}n^2(n+1)^2(n+2)^2 - n^3(n+3) - \frac{1}{4}n(n+3)(n^2+3n-2)$$

$$=\frac{1}{36}n(n+1)(16n^4+80n^3+83n^2-53n+54).$$

If now we had the certainty that each point P of this completing intersection was the vertex of a cone C^n with the transversals emitted by this point to the $\frac{1}{2}n(n+3)+2$ pairs of given lines as edges, the number indicated just now would represent the order of the curve ϱ under discussion. This however is only the case for $n=1$ where the obtained result passes into a ϱ^{10}, as it ought to do (see the paper quoted). For in the case of higher values of n the completing intersection found above consists of two or more parts, one or more of which do not belong to the locus. In order to show this we must treat the two cases $n=2$ and $n>2$ separately.

For $n=2$ the two surfaces O_b^{16} and O_c^{16} have still in common besides the ten common double lines and the twenty common single lines the five twisted curves ϱ^{10} — as we shall see immediately not connected with solutions of the problem — which form the loci of the point emitting complanar transversals to four of the five pairs (a_i, a'_i). Let P_1 be a point emitting to the four pairs (a_2, a'_2), (a_3, a'_3), (a_4, a'_4), (a_5, a'_5) four transversals lying in the plane α_1 and let β_1 and γ_1 represent the planes of the pairs of transversals from P_1 to (a_1, a_1'), (b, b') and (a_1, a'_1), (c, c'); then (α_1, β_1) and (α_1, γ_1) represent quadratic cones degenerated into pairs of planes with respect to the two sextriples of pairs of lines (a_i, a'_i), (b,b') and (a_i, a'_i), (c,c') and therefore P_1 lies on O_b^{16} and O_c^{16} without being vertex of a quadratic cone with the transversals to the *seven* pairs (a_i, a'_i), (b,b'), (c, c') as edges. So each of the quadruples out of the five pairs of lines (a_i, a'_i) furnishes a ϱ_i^{10} common to O_b^{16} and O_c^{16} but not corresponding to solutions of the problem; so the curve ϱ^{196} found above consists of these five curves ϱ_i^{10} which are to be discarded and the locus proper ϱ^{146}.

The result ϱ^{146} is easily checked as follows. Starting from seven pairs of *intersecting* lines for which A_i and α_i, $(i = 1,2,\ldots,7)$ represent the seven points of intersection and connecting planes, the locus consists of:

1. the locus ϱ^6 of the vertices of the cones contained in the net of surfaces O^2 through the seven points A_i,

2. the section c^4 of any of the seven planes α_i with the surface O^4 forming the locus of the cones through the six points A with a subscript different from i, counted *twice*,

3. the lines of intersection of the seven planes α_i by two, counted *four* times.

So we find $6 + 7 \cdot 4 \cdot 2 + 21 \cdot 1 \cdot 4 = 146$.

The necessity of discarding a part of the completing intersection, on account of the existence of a locus of points P for which the cones C_b^n and C_c^n corresponding to the systems (a_i, a'_i), (b, b') and (a_i, a'_i), (c, c') break up into a common part C^p and two different completing parts C_b^{n-p} and C_c^{n-p}, presents itself in the case $n = 2$ only. For this locus puts in its appearance under the condition

$$[\tfrac{1}{2}p(p+3)+2] + \tfrac{1}{2}(n-p)(n-p+3) = [\tfrac{1}{2}n(n+3)+1]$$

only; for then the locus of the point P for which $\tfrac{1}{2}p(p+3) + 2$ transversals lie on a cone C^p furnishes a curve common to O_b^{16} and O_c^{16} the points of which do not satisfy the conditions of the problem. As this equation reduces itself to $p(n-p)=1$ the only possible case is $p=1$, $n=2$.

We now pass to a consideration of the cases $n > 2$ and take $n = 3$ as example. Here the two surfaces $(P)_b$ and $(P)_c$ corresponding to the systems (a_i, a'_i), (b,b') and (a_i, a'_i), (c,c'), where i goes from one to nine included, admit besides the 18 common threefold lines and the 72 common single lines a common twisted curve not connected with solutions of the problem, i.e. the curve forming with the two groups of 18 and 72 lines the locus of the point P emitting to the pairs (a_i, a'_i) nine transversals forming the base edges of a pencil of cones C^3 instead of determining a single cubic cone. This particularity presents itself also for larger values of n. So we can say in general that for $n > 2$ a twisted curve occurs forming with the two groups of lines corresponding to the value of n the locus of the point P for which the transversals to the $(n+2)_2 - 1$ pairs (a_i, a'_i) determine a pencil of cones C^n. If the order of this twisted curve is found we also know the order of the locus ϱ of the point emitting transversals lying on a cone C^n to $(n+2)_2 + 1$ given pairs of lines. Though the theoretical determination of the order of the first curve implies no difficulties the practical execution requires more room than we have at our disposal here; this is the cause why we restrict ourselves now to the case $n = 2$.

4. Now that we have experienced that the curve ϱ^{10} of the case $n = 1$ plays a part in the investigation of the case $n = 2$ we may conjecture that the twenty points with complanar quintuples of transversals (see the paper quoted) will do likewise.

Let D be a point emitting five transversals t_{a_i} lying in a plane δ to the five pairs (a_i, a_i'), and let l be a line through D not lying in δ. Then the method indicated in art. 2 furnishes in δ with the aid of the six reguli (l, a_i, a'_i), (l, b, b') six series of points in mutual projective correspondence of which five lie on lines r_i and the sixth on a conic passing through D. So the number of points common to l and O_b^{16} and different from D is equal to the order of the equation

$$\begin{vmatrix} (u_{1,i}\lambda + v_{1,i})^2, & (u_{1,i}\lambda + v_{1,i})(u_{2,i}\lambda + v_{2,i}), & \ldots, & (u_{3,i}\lambda + v_{3,i})^2 \\ (f_1\lambda^2 + g_1\lambda + h_1)^2, & (f_1\lambda^2 + g_1\lambda + h_1)(f_2\lambda^2 + g_2\lambda + h_2), & \ldots, & (f_3\lambda^2 + g_3\lambda + h_3)^2 \end{vmatrix} = 0$$

in λ, i. e. 14. So D is a node [1]) of O_b^{16}. As the five curves ϱ_i^{10} corresponding with four of the five pairs (a_i, a'_i) pass through D the tangential cone of O_b^{16} in D is determined by the tangents in D to these five curves; so not only the point D itself but also the tangential cone of O_b^{16} in D is entirely independent of the sixth pair (b, b'). So the surfaces O_b^{16} and O_c^{16} admit in the common node D a common tangential cone But this implies that the complete intersection of O_b^{16} and O_c^{16} passes through D with six branches. For, if D is the origin and t_k a homogeneous form in x, y, z of order k, the equations of the two surfaces assume the form

[1]) This result could have been predicted by remarking that the quadratic cone with vertex D is indeterminate, as it consists of δ and an *arbitrary* plane through the transversal t_b.

$$t_2 + t_3 + \ldots + t_{16} = 0 \quad , \quad t_2 + t'_3 + \ldots + t'_{16} = 0$$

from which ensues that the total intersection lies on the surface

$$(t_3 - t'_3) + (t_4 - t'_4) + \ldots + (t_{16} - t'_{16}) = 0,$$

admitting a threefold point in the origin. So the completing curve $\sigma^{146}_{b,c}$ must pass once through D, the curves ϱ^{10}_i doing this together five times; moreover the tangent to $\varrho^{146}_{b,c}$ in D lies on the common tangential cone of O^{16}_b and O^{16}_c in D.

Besides the twenty common nodes D each of the two surfaces admits 100 nodes more, which we will represent by E_b and E_c. The 100 points E_b corresponding by 20 to the pair (b, b') and four of the pairs (a_i, a'_i) lie on the curves ϱ^{10}_i and therefore on O^{16}_c; for the same reason O^{16}_b contains the 100 points E_c. So the total intersection of the surfaces O^{16}_b and O^{16}_c passes twice through the 200 points E_b and E_c, these points being nodes of one of the surfaces and ordinary points of the other; as the five curves ϱ^{10}_i pass together once through these points, the completing intersection $\varrho^{146}_{b,c}$ must contain the 200 points[1]).

5. In order to be able to determine the number of points emitting transversals lying on a quadratic cone to eight pairs of lines crossing each other we still want to know how many points the curve $\varrho^{146}_{b,c}$ has in common with each of the 14 given lines (a_i, a'_i), (b, b'), (c, c') and with each of the 42 transversals of these seven pairs by two. Evidently the first number is 16; for the surface O^{16} corresponding to six of the seven pairs is cut by each line of the seventh pair in 16 points. Moreover by means of the method of art. 2 we find for the second number, represented there in general by p, for $n = 2$ the result 12.

6. We now pass to the determination of the number of points P, emitting transversals lying on a quadratic cone to eight given pairs (a_i, a'_i), $i = 1, 2, 3, 4, 5$, and (b, b'), (c, c'), (d, d'). To that end we consider the three systems

1) It is quite natural that the points E_b and E_c lie on $\varrho^{146}_{b,c}$. For if E_b lies on ϱ^{10}_1 the cone with respect to O^{16}_c consists of the plane α through the transversals $t_{a_2}, t_{a_3}, t_{a_4}, t_{a_5}$ and the plane β through the transversals t_{a_1}, t_c, whilst the cone with respect to O^{16}_b consists of α and an *arbitrary* plane through t_{a_1}, for which we can take β as well.

(a_i, a'_i), (b, b') (a_i, a'_i), (c, c') (a, a'_i), (d, d')

and the corresponding surfaces O_b^{16}, O_c^{16}, O_d^{16}, in order to propose the question how many of the 2336 points of intersection of O_d^{16} and $\varrho_{b,c}^{146}$ satisfy the conditions of the problem.

Here we must fix our attention upon the following groups of points which are to be discarded:

a. the twenty common nodes D of O_b^{16}, O_c^{16}, O_d^{16},

b. the hundred nodes E_b of O_b^{16} and the hundred nodes E_c of O_c^{16},

c. the sixteen points F common to $\varrho_{b,c}^{146}$ and any of the ten lines a_i,

d. the twelve points G common to $\varrho_{b,c}^{146}$ and each of the twenty transversals of two of the five pairs (a_i, a'_i),

e. the forty points H common to any of the five surfaces $O^4_{a_i, b, c}$ and the corresponding curve ϱ_i^{10}.

We consider each of these five groups separately.

a. The 20 points D count *thrice* among the points common to $\varrho_{b,c}^{146}$ and O_d^{16}, for the curve touches in the node of the surface the tangential cone of the surface.

b. Each of the 200 points E_b, E_c counts *once*.

c. A point F common to a_1 and $\varrho_{b,c}^{146}$ counts for *four* points of intersection; for the cone with vertex F corresponding to the six pairs $(a_2, a'_2), \ldots, (a_5, a'_5)$, (b, b'), (c, c') cuts a'_1 twice and F lies on a double line of O_c^{16}.

d. A point G common to $t_{1,2}$ and $\varrho_{b,c}^{146}$ counts *once*.

e. A point H common to $O^4_{a_1, b, c}$ and ϱ_1^{10} lies on $\varrho_{b,c}^{146}$, as it emits three complanar transversals to (a_1, a'_1), (b, b'), (c, c') and four complanar transversals to the other pairs $(a_2, a'_2), \ldots, (a_5, a'_5)$. As the tangential plane in H to O_d^{16} cuts the common tangential plane of O_b^{16} and O_c^{16} in H according to the tangent in H to ϱ_1^{10}, the point H counts for *one* point of intersection.

As all these groups of points admit the property that the cone for O_d^{16} differs from the cone corresponding to O_b^{16} and O_c^{16}, they must be discarded. So the required number is

$$2336 - 3 \cdot 20 - 200 - 4 \cdot 10 \cdot 16 - 20 \cdot 12 - 5 \cdot 40 = 996.$$

We can check easily the obtained result. In the special case of eight pairs of *intersecting* lines, where (A_i, a_i), $i = 1, 2, \ldots, 8$ indicate point of intersection and connecting plane for each pair, we find the following solutions:

1st. the vertices of the *four* cones through the eight points A_i,

2nd. in each of the *eight* planes α_i the vertices of the *six* cones through the seven points A_i not lying in that plane, counted *twice*,

3rd. in each of the *twenty eight* lines of intersection of the eight planes α_i by two the vertices of the *four* cones through the six points A_i not lying in either of the two planes, counted *four* times,

4. each of the *fifty six* points of intersection of the eight planes α_i by three, counted *eight* times. This gives

$$\begin{aligned} 1\,.\,4\,.\,1 &= 4 \\ 8\,.\,6\,.\,2 &= 96 \\ 28\,.\,4\,.\,4 &= 448 \\ 56\,.\,1\,.\,8 &= 448 \\ \hline & \,996 \end{aligned}$$

7. We unite the results found for $n=2$ in:

THEOREM III. "The locus of the point P emitting transversals lying on a quadratic cone to six arbitrarily [1]) given pairs of lines is a surface O^{16}. This surface passes twice through the 12 given lines and once through the 15 pairs of transversals of the six given pairs by two; moreover it contains the 15 twisted curves ϱ^{10} forming the locus of the point for which four of the six transversals are complanar. These 15 curves cut each other by five in 120 points for which five of the six transversals are complanar; each of these points is a node of O^{16} with a tangential cone determined by the tangents of the five curves ϱ^{10} passing through that point."

"The locus of the point P emitting transversals lying on a quadratic cone to seven arbitrarily given pairs of lines is a twisted curve ϱ^{146} cutting each of the 14 given lines in 16 and each of the 42 transversals of the seven pairs by two in 12 points; it passes through the nodes of the surfaces O^{16} corresponding to six of the seven pairs and touches in these points the tangential cones of these surfaces."

"The number of points P emitting transversals lying on a quadratic cone to eight arbitrarily given pairs of lines is 996."

In following communications we hope to extend these considerations to the cases $n=3$, 4, etc. and to give polydimensional generalisations of the problem.

1) We do not wish to enumerate different special cases here. It may only be pointed out that the surface O^{16} becomes indeterminate if in order to obtain six pairs of lines we borrow three pairs of reciprocal polars of each of two linear complexes.