Mathematics. — *"General considerations on the curves of contact of surfaces with cones, with application to the lines of saturation and binodal lines in ternary systems."* (Communicated by Prof. D. J. KORTEWEG and Prof. F. A. H. SCHREINEMAKERS).

Introduction.

It is a known fact that in the study of the ternary solutions which for given temperature and pressure can be in equilibrium with a solid substance a great part is played by the curve of contact of the tangential cone of the ζ-surface with a given point as vertex.

If namely we project the vertex of the cone and its curve of contact on the horizontal plane, then the projection of the curve of contact represents a ternary line of saturation, namely the series of the solutions, which for assumed temperature and pressure are saturated with the solid substance indicated by the projection of the vertex of the cone.

The form of the line of saturation of a solid substance being thus determined by the form of the curve of contact of a cone, it was our aim to investigate which peculiarities this curve of contact could display in some points of a given surface and in particular of the ζ-surface.

We choose as origin of the system of coordinates a point O of the surface. We assume the X- and Y-axis in the tangential plane of the surface in point O.

For the equation of the surface in the vicinity of point O we can then write:

$$z = c_1x^2 + c_2xy + c_3y^2 + d_1x^3 + d_2x^2y + d_3xy^2 + d_4y^3 + e_1x^4 + e_2x^3y + \ldots \quad (1)$$

The equation of a tangential plane in a point x, y, z of this surface becomes:

$$Z - z = (X-x)\frac{\partial z}{\partial x} + (Y-y)\frac{\partial z}{\partial y}.$$

If we wish to let this tangential plane pass through a point $P(p \, . \, q)$ of the $X \, . \, Y$-plane, then we must have

$$(p-x)\frac{\partial z}{\partial x} + (q-y)\frac{\partial z}{\partial y} + z = 0.$$

If in this equation we substitute the values of $z, \dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ out of (1) we get:

$$(2c_1 p + c_2 q)x + (c_2 p + 2c_3 q)y + (3d_1 p + d_2 q - c_1)x^2 +$$
$$+ (2d_2 p + 2d_3 q - c_2)xy + (d_3 p + 3d_4 q - c_3)y^2 +$$
$$+ (4e_1 p + e_2 q - 2d_1)x^3 + (3e_2 p + 2e_3 q - 2d_2)x^2 y +$$
$$+ (2e_3 p + 3e_4 q - 2d_3)xy^2 + (e_4 p + 4e_5 q - 2d_4)y^3 + \ldots = 0 \quad . \quad (2)$$

The above form (2) is therefore the equation of the curve of contact of a cone touching the surface and having point $P(p \,.\, q)$ as vertex.

We shall now distinguish three cases:

I. O is not a parabolic point.

II. O is a parabolic point.

III. O is a point of osculation.

I. Point O is not a parabolic point.

As O is an elliptic or a hyperbolic point, it follows that $c_1 c_3 - \frac{1}{4} c_2^2 \gtrless 0$. We now assume the line OP as X-axis, so that $q = 0$. We can now distinguish two cases according to OP being an asymptote of the indicatrix or not.

I_A. The line OP is not an asymptote of the indicatrix.

We assume OP as X-axis and the conjugate diameter of the indicatrix as Y-axis; so $q = 0$ and $c_2 = 0$. From (2) follows then:

$$2c_1 px + (3d_1 p - c_1)x^2 + 2d_2 pxy + (d_3 p - c_3)y^2 + \ldots = 0 \ . \quad (3)$$

The curve of contact touches therefore the Y-axis in point O. As the X-axis (the line OP) and the Y-axis are conjugate diameters of the indicatrix, it follows that the line OP, connecting the vertex P of a cone with a point O of its curve of contact, and the tangent in point O to this curve of contact are conjugate diameters of the indicatrix of point O.

In general the curve of contact in the vicinity of point O is of finite curvature and determined by:

$$2c_1 px + (d_3 p - c_3)\, y^2 = 0 \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If p is chosen in such a way that $d_3 p - c_3 = 0$ then the equation is

$$2c_1 px + (e_4 p - 2d_4)\, y^3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

so that the curve of contact has a point of inflection in point O.

Several ternary lines of saturation with one or more points of inflection are known. We find e.g. on the line of saturation of the nitril of ambric acid in the system: water — alcohol — nitril of ambric acid [1]) at 4°.5 two points of inflection.

I_B. The line OP is an asymptote of the indicatrix.

We assume OP as X-axis, the other asymptote as Y-axis so that $q = 0$, $c_1 = 0$ and $c_3 = 0$.

Then the curve of contact is determined by:

$$c_2py + 3d_1px^2 + (2d_2p - c_2)xy + d_3py^2 + \ldots = 0 \quad . \quad . \quad (6)$$

So the generatrix OP of the cone touches the curve of contact in O [2]).

We have here thus the case that through point P we can draw a tangent to the line of saturation of the solid substance represented by P. This point of contact, however, being a hyperbolic point, this case can appear only on the unstable part of the line of saturation.

II. Point O is a parabolic point.

As O is a parabolic point, it follows that $c_1c_3 - \frac{1}{4}c_2^2 = 0$. Point O lies thus on the parabolic or spinodal line of the surface.

II_A. The line OP does not coincide with the direction of the axis of the parabola.

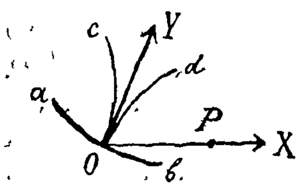

Fig. 1.

In fig. 1 let aOb be the spinodal line, cOd the section of the tangential plane in O with the surface; OY is the tangent in the cusp O of this section and at the same time the direction of the axes.

We now assume OP as X- and OY as Y-axis, so that $q = 0$, $c_2 = 0$ and $c_3 = 0$. Then we find for the equation of the curve of contact:

$$2c_1px + (3d_1p - c_1)x^2 + 2d_2pxy + d_3py^2 + \ldots + 0$$

or:

$$2c_1x + d_3y^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

So the curve of contact touches in O the line OY. The direction of the curve of contact in the vicinity of its point of intersection with the spinodal line is therefore independent of the position of the vertex P of the cone.

1) F. A. H. Schreinemakers. Z. f. Phys. Chem. **27** 114 (1898).

2) See also: H. A. Lorentz. Z. f. Phys. Chem. **22** 523.

We can express this property also as follows: all the lines of saturation passing through a point O of the spinodal line touch each other in this point O.

We have drawn the curve cOd in fig. 1 in such a way that the tangent OY intersects the spinodal line in O. That this is true in general is evident from the following.

The equation of the spinodal line is:

$$\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If now we calculate out of (1) the values of $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$, after having put there $c_2 = 0$ and $c_3 = 0$, we find for (8):

$$(2c_1 + 6d_1 x + 2d_2 y + \ldots)(2d_3 x + 6d_4 y + \ldots) - (2d_2 x + 2d_3 y + \ldots)^2 = 0.$$

As c_1 is not zero, we find by first approximation for the equation of the spinodal line:

$$2d_3 x + 6d_4 y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

From this ensues therefore that the tangent in O to the spinodal line aOb forms an angle with the line OY, unless $d_4 = 0$.

If however $d_4 = 0$, then it follows from (9) that the tangent in O to the spinodal line coincides with the line OY. As then at the same time

$$c_2 = 0, \; c_3 = 0 \text{ and } d_4 = 0$$

point O under consideration is a plaitpoint [1]. Hence: only in a plaitpoint the spinodal line and the curve of contact of a cone can touch each other.

$II_{A.\alpha}$. *Point O is a plaitpoint.*

As $c_2 = 0$, $c_3 = 0$ and $d_4 = 0$ [2]), the equation of the curve of contact becomes

$$2c_1 x + d_3 y^2 + \ldots = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

So all the curves of contact passing through the plaitpoint touch each other there and their curvature is independent of the distance from the vertex P of the cone to the plaitpoint.

That this curvature is also independent of the direction of the line OP and therefore quite independent of the situation of P will soon be evident.

From (8) follows for the equation of the spinodal line:

[1]) D. J. KORTEWEG Arch. Néerl. (1) 24 60 (1891).
[2]) D. J. KORTEWEG. l.c. 63 (1891).

$$(2c_1+6d_1x+2d_2y+..)(2d_3x+2e_3x^2+6e_4xy+12e_5y^2+..)-$$
$$-(2d_2x+2d_3y+3e_2x^2+4e_3xy+3e_4y^2+..)^2=0$$

or at first approximation for the equation of that line in the vicinity of the plaitpoint:

$$4c_1d_3x+(24c_1e_5-4d_3^2)y^2=0 \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The equation of the binodal line in the vicinity of point O is [1]):

$$d_3x+2e_5y^2=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

We now write (10), (11) and (12) in such a way that the coefficient of x is the same for these three; so we find:

for the curve of contact: $2c_1d_3x+d_3^2y^2=0$. . . . . (13)

„ „ spinodal line: $2c_1d_3x+2(6c_1e_5-d_3^2)y^2=0$. (14)

„ „ binodal line: $2c_1d_3x+4c_1e_5y^2=0$ . . . . (15)

We shall now restrict ourselves, as only this is liable to realisation, to a plaitpoint of the first kind [1]), so that

$$4c_1e_5-d_3^2>0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

thus also $c_1e_5>0$ and $6c_1e_5-d_3^2>0$.

From this ensues immediately that in the vicinity of the plaitpoint the curve of contact, the spinodal line, and the binodal line are curved in the same direction.

Out of (16) we can deduce:

$$2(6c_1e_5-d_3^2)>4c_1e_5>d_3^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

If we call the radii of curvature of the spinodal line, the binodal line, and the curve of contact R_s, R_b, and R_r, it follows from (13), (14) and (15):

$$\left.R_s=\frac{c_1d_3}{(12c_1e_5-2d_3^2)\sin\theta}\,,\; R_b=\frac{c_1d_3}{4c_1e_5\sin\theta}\,,\; R_r=\frac{c_1d_3}{d_3^2\sin\theta}\right\} . \quad (18)$$

where θ represents the angle between the line OP and the tangent in the plaitpoint to the binodal line.

In connection with (17) follows from this that the spinodal line has the smallest radius of curvature and the curve of contact the largest.

From (18) we can furthermore deduce:

$$\frac{2}{R_r}=\frac{3}{R_b}-\frac{1}{R_s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18^a)$$

Out of this relation it is evident that R_r is *also* independent of the direction of the line OP; for R_b and R_s are quantities, which depend exclusively on the shape of the surface at point O.

[1]) D. J. Korteweg. l.c. 61 (1891).

If we introduce instead of the radii of curvature R the curvatures K we find

$$2K_r = 3K_b - K_s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18^b)$$

For the rest the curve of contact has nothing remarkable in the vicinity of the plaitpoint except that its course there is in a high degree independent of the situation of the vertex P of the cone, if but this vertex is not too close to the plaitpoint or not too close to the tangent to the spinodal line in the plaitpoint.

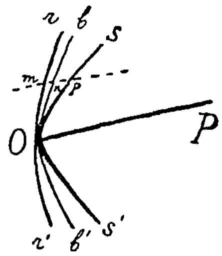

Fig. 2.

In fig. 2 sOs' represents the spinodal line, bOb' the binodal line, and rOr' the curve of contact of the cone P, or in other words the line of saturation of the solid substance P. As ensues out of the curvatures (18^a and 18^b) of these three lines, these must have a position with respect to each other as in fig. 2.

If we draw in this figure a line mnp parallel to and in the vicinity of OP, then np must be equal to $2mn$. If namely we calculate x_r, x_s, and x_b out of (13), (14) and (15) we find for a same value of y:

$$2\,(x_b - x_r) = x_s - x_b\,.$$

In so far as the binodal line has been drawn in fig. 2 the conjugated pairs of fluids represented by it are metastable; they all break up into the solid substance P and a solution of the line of saturation rOr'.

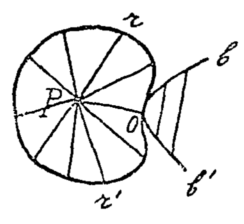

Fig. 3.

In fig. 3 the point P lies on the other side of the tangent in O as in fig. 2. Line rOr' is the line of saturation, bOb' the binodal line; the spinodal line has not been drawn.

In the vicinity of the plaitpoint the line of saturation must be curved, as has been represented in fig. 3, in the same direction as the binodal line. In its further course two or more points of inflection can of course appear. If e. g. P is a ternary solid substance, so that the line of saturation is a curve enclosing point P, then at least two points of inflection must appear, as has been assumed in fig. 3.

If now we change the temperature or the pressure, then the ζ-surface changes according to position and form; point P rises and falls. Now the binodal line and the line of saturation of course also change their form.

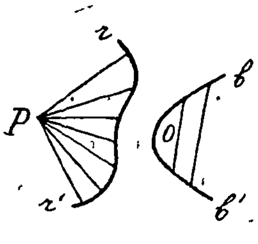

Fig. 4.

We now imagine temperature or pressure changed a little in such a direction that the two curves of fig. 3 move away from each other. We then obtain fig. 4, in which the line of saturation has been represented but partly. It is clear that it now likewise has to show two points of inflection. If both curves move still farther from each other, then of course both points of inflection can vanish.

If we change temperature or pressure in opposite direction, then we cause both curves of fig. 3 to overlap somewhat. We then find fig. 5, in which $baa'b'$ represents the binodal line and $raa'r'$ the line of saturation of P.

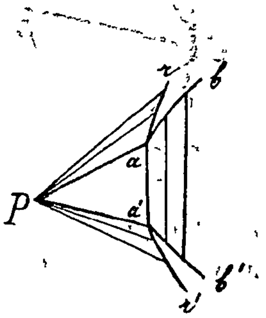

Fig. 5.

On the part aa' not represented of the binodal line lies the plaitpoint; the part aa' of the line of saturation lies between the part aa' of the binodal line and the straight line aa'. The line of saturation of P is only partly drawn.

As long as a and a' lie but close enough to each other, ar and $a'r'$ must lie as in fig. 5, they must run namely from a and a' to that side of line aa', where the stable part of the binodal line lies. In their further course the lines ar and $a'r'$ can of course intersect the line aa'.

We now have besides a series of solutions saturated with P (ar and $a'r'$) and a series of conjugate solutions (ab and $a'b'$) also a conjugate pair of fluids $L_a + L_{a'}$ saturated with solid P.

As the pieces aa' left out of the binodal line and of the line of saturation lie inside the three-phase-triangle Paa', the fluids represented by them separate into $P + L_a + L_{a'}$.

Examples of lines of saturation with two points of inflection, between which a curvature in the same direction as the binodal line, we find e.g. in the system [1]: water-$AgNO_3$ ethylene cyanide.

At $\pm$ 11° the line of saturation of $2\,C_2H_4\,(CN)_2 \,.\, AgNO_3 \,.\, H_2O$ touches the binodal line in its plaitpoint and two points of inflection appear as in fig. 3.

With a rise of temperature both lines move away from each other; the lines of saturation determined experimentally at 12°, 20° and 25° show distinctly the type of fig. 4.

[1]) W. Middelberg. Z. f. Phys. Chem. **43**. 305 (1903).

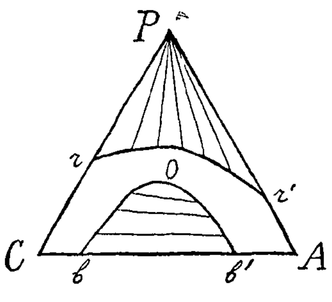

Fig. 6

If we lower the temperature below 11°, then the isotherms in the vicinity of the three-phase-triangle show a form as in fig. 5. Lines of saturation are also known which turn in their whole course their concave side to the plaitpoint of the binodal line; they have a form as rr' in fig. 6.

In the system[1]) water (C)—ether (A)—malonic acid (P) such a line of saturation and binodal line are determined at 15°.

In the system[2]): water (C) — ethylene cyanide (A) — benzoic acid (P) we find above 51° likewise isotherms as in fig. 6. When lowering the temperature the two curves of fig. 6 approach each other; at 51° the line of saturation rr' of the benzoic acid touches the binodal line in its plaitpoint O. At still lower temperatures a three-phase-triangle appears and the isotherms in the vicinity of that triangle show a form as in fig. 5.

Also in the systems[1]) water-phenol-alkali lines of saturation appear of the type as in fig. 4.

II_B. The line OP has the direction of the axis of the parabola.

We assume OP as Y-axis, then $p = 0$, $c_2 = 0$ and $c_3 = 0$. So the equation of the curve of contact becomes:

$$(d_2q - c_1)x^2 + 2d_3qxy + 3d_4qy^2 + \ldots = 0 \quad . \quad . \quad . \quad (19)$$

So the curve of contact has in point O a node, therefore it consists, as is drawn in fig. 8, of two intersecting branches rad and $r'ab$.

By a variation of parameter (on the ζ-surface temperature and pressure come into consideration for this) out of fig. 8 are formed fig. 7 and fig. 9. So fig. 8 is the transition form between fig. 7 and fig. 9.

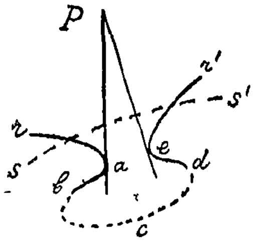

Fig. 7.

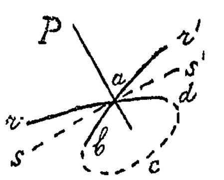

Fig. 8.

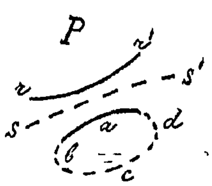

Fig. 9.

[1]) E. A. KLOBBIE. Z. f. Phys. Chem. **24**. 625.
[2]) F. A. H. SCHREINEMAKERS. Z f. Phys. Chem. **26**. 249 (1898).

In fig. 7 ss' represents the spinodal line; *rab* and *r'ed* are two branches of the curve of contact having in a and e a tangent passing through point P. In these points a and e we have the case considered sub I_B; a and e lie therefore both on the hyperbolically curved part of the surface.

If we pursue the branches *ab* and *ed*, these can of course pass into each other[1]); in fig. 7 this continuation is represented by the dotted curve *bcd*.

In fig. 9 the curve of contact consists of the two branches rr' and *abcd*, separated from each other by the spinodal line ss' [2]).

The equation (19) can, however, also represent an isolated point; the curve of contact then consists of a single isolated point, lying on the spinodal line. For a small change in parameter this point then vanishes or a closed curve of contact is generated.

Inversely the closed curve of contact *abcd* of fig. 9 can thus contract so as to disappear in a point of the spinodal line.

To investigate whether the curve can possess other nodes or isolated points (in ordinary not conical points), we cause the Y-axis to coincide with OP. This is of course always possible and then we have $p=0$.

From (2) follows now as condition for a node $c_2 q=0$ and $2c_3 q=0$. So we find:

$$c_2=0 \quad c_3=0 \quad \text{and therefore also} \quad c_1 c_3 - \frac{1}{4} c^2_2 = 0.$$

This is just the condition for the generation of case II_B. So we find the nodes and isolated points only in case II_B, except of course in the points of osculation which can be regarded as a special case of it, where $c_1=0$.

So we can say:

"Nodes and isolated points of the curve of contact always lie on the spinodal line."

There would be an exception only if point P were on the surface itself; then of course there would always be in that place an isolated point or node; this however we do not discuss.

$II_{B\alpha}$. *Point O is a plaitpoint.*

We assume (fig. 10) OP as Y-axis so that besides $p=0$, $c_2=0$ and $c_3=0$, we find also $d_4=0$ [3]).

1) Comp. F. A. H. Schreinemakers. Z. f. Phys. Chem. **22**. 532 (1897).
2) Comp. F. A. H. Schreinemakers. Z. f. Phys. Chem. **22** 531 (1897).
3) D. J. Korteweg. Arch. Néerl. (I). **24** 61. (1891).

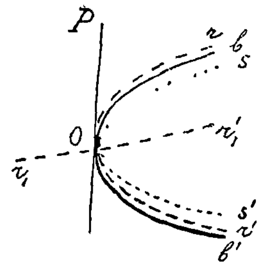

Fig. 10.

Out of (19) now follows that the plaitpoint is a node of the curve of contact and at the same time that the line OP itself is one of the tangents. To investigate this curve of contact further we write (2) (after having put there $p=0$, $c_2=0$, $c_3=0$ and $d_4=0$) in the form:

$$Ax^2+Bxy+Cx^3+Dx^2y+Exy^3+Fy^3+\ldots=0. \quad . \quad (20)$$

To satisfy this by

$$x=ky^2$$

we must have $Bk+F=0$. From this ensues now, as $B=2\,d_3q$ and $F=4\,qe_5$:

$$x=-\frac{2e_5}{d_3}y^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

thus reproducing the equation (12) of the binodal line in the vicinity of the plaitpoint.

Therefore the curve of contact coincides in the vicinity of the plaitpoint with the binodal line.

This coincidence does not hold for the higher terms, as is natural and as is shown still more clearly by the following.

We put namely:

$$x=ky^2+my^3$$

and we substitute this value in (20). As will immediately become evident, we must include in (20) still the term y^4. We write for it Gy^4.

So we find:

$$(Bk+F)y^3+(Ak^2+Bm+Ek+G)\,y^4+\ldots=0.$$

From this ensues:

$$m=-\frac{Ak^2+Ek+G}{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Now follows out of (2): $A=d_2q-c_1$, $B=2d_3q$, $E=3e_4q-2d_3$. If we calculate the coefficient G of y^4 in (2) we find:

$$G=(f_5\,p+5\,f_6q-3\,e_5),$$

so here, as $p=0$:

$$G=5\,f_6\,q-3\,e_5.$$

So we find, if we put for the curve of contact $m=m_1$:

$$m_1=\frac{(c_1-d_2q)\,\dfrac{4e_5^{\,2}}{d_3^{\,2}}+(-\,2d_3+3e_4q)\,\dfrac{2e_5}{d_3}+3e_5-5\,f_6q}{2\,d_3\,q} \quad . \quad (23)$$

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For the second term $m_b y^3$ of the binodal line we have:[1])

$$m_b = \frac{2\,(e_4 e_5 - d_3 f_6)}{d_3^{\,2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

so that the curve of contact and the binodal line differ in the term y^3.

We now write:

$$x_r = ky^2 + m_r y^3 + \dots$$
$$x_b = ky^2 + m_b y^3 + \dots$$

from which ensues:

$$x_r - x_b = (m_r - m_b)\, y^3 + \dots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Out of (25) it is evident, that the binodal line bOb' and the curve of contact rOr' must have with respect to each other a position as in fig. 10. In this figure the part rO of the curve of contact has been drawn outside, the part $r'O$ inside the binodal line.

If we calculate with the help of (23) and (24) $m_r - m_b$ we then see that the sign of this difference depends on q, thus on the position of P. It is therefore also possible that for the same surface rO lies inside and $r'O$ outside the binodal line.

The curve of contact:

$$x = -\frac{2e^5}{d_3} y^2 + \dots$$

and the spinodal line (11):

$$x = -\frac{6\,c_1 e_5 - d_3^{\,2}}{c_1 d_3} y^2 + \dots$$

differ already in the coefficient of y^2. Hence for a plaitpoint of the first kind the curve of contact will always fall as in fig. 10, just like the binodal line, on the outer side of the spinodal line.

As we have seen above the curve of contact consists of two branches intersecting in the plaitpoint; one is the branch rOr', considered above, the other the branch $r_1 O r_1'$.

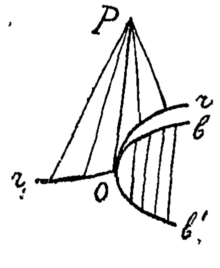

Fig. 11.

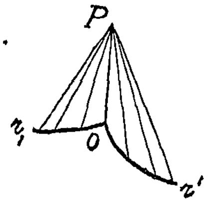

Fig. 12.

If in fig. 10 we restrict ourselves to that part of the lines representing stable conditions, we find fig. 11. Also the case represented

[1]) D. J. Korteweg, l.c. 69, 70.

in fig. 12 can of course appear, so that the binodal line vanishes because it falls inside the sector POr'.

Remarkable in both cases is that the stable part of the line of saturation of P, although it represents an unbroken series of solutions, yet shows a discontinuity. This makes its appearance in the critical solution saturated with solid P.

III. Point O is a point of osculation.

As in a point of osculation we have $c_1 = 0$, $c_2 = 0$ and $c_3 = 0$, we get from (2) for the equation of the curve of contact:

$$(3d_1 p + d_2 q) x^2 + (2d_2 p + 2d_3 q) xy + (d_2 p + 3d_4 q) y^2 + \ldots = 0$$

or if we make the X-axis to coincide with OP:

$$3d_1 x^2 + 2d_2 xy + d_3 y^2 + \ldots = 0 \quad . \quad . \quad . \quad . \quad . \quad (26)$$

So the curve of contact consists either of an isolated point or it shows in O a node. From (26) it is evident that the directions of the two tangents are independent of the distance from point P to point O; they depend only on the direction of the line OP.

The above-mentioned property that the curve of contact and the binodal line are curved in the same direction in the vicinity of the plaitpoint $(II_{A'})$ caused us to surmise that this also would be the case with a second branch of the binodal line, should such a one pass through the plaitpoint [1]).

This surmise can be affirmed in the following way and it can also be shown that the curvation of such a branch corresponds entirely to that of the curves of contact passing through the plaitpoint.

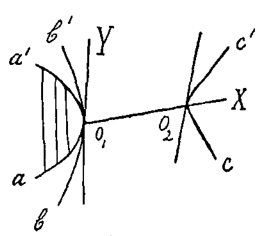

Fig. 13.

To that end we assume again as Y-axis the tangent to the spinodal and the binodal line of the plaitpoint O_1 (fig. 13); for the X-axis we choose the line of conjugation O_1O_2 and we put $O_1O_2 = p$.

The tangential plane in a point x_1, y_1, z_1 in the vicinity of O_1 is:

$$Z - z_1 = (X - x_1)\frac{\partial z_1}{\partial x_1} + (Y - y_1)\frac{\partial z_1}{\partial y_1},$$

the one in a point x_2, y_2, z_2 in the vicinity of O_2 is:

$$Z - z_2 = (X - x_2)\frac{\partial z_2}{\partial x_2} + (Y - y_2)\frac{\partial z_2}{\partial y_2}$$

1) Comp. the paper of Mr. KUENEN (Proceedings of Oct. 1911, p. 420).

The conditions that x_1, y_1, z_1 and x_2, y_2, z_2 are conjugated points thus become:

$$\frac{\partial z_1}{\partial x_1} = \frac{\partial z_2}{\partial x_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

$$\frac{\partial z_1}{\partial y_1} = \frac{\partial z_2}{\partial y_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$x_1 \frac{\partial z_1}{\partial x_1} + y_1 \frac{\partial z_1}{\partial y_1} - z_1 = x_2 \frac{\partial z_2}{\partial x_2} + y_2 \frac{\partial z_2}{\partial y_2} - z_2 \quad . \quad . \quad . \quad (29)$$

In consequence of the choice of the Y-axis we find:

$$z_1 = c_1 x_1^2 + d_1 x_1^3 + d_2 x_1^2 y_1 + d_3 x_1 y_1^2 + e_1 x_1^4 + \ldots$$

If we put $x_2 = p + \xi_2$, so that ξ_2 is a small quantity we have:

$$z_2 = c'_1 \xi_2^2 + c'_2 \xi_2 y_2 + c'_3 y_2^2 + \ldots\ldots$$

We now write the equations (27) (28) and (29) in full; we then directly leave out the terms which are certainly small with respect to those written down, setting aside of what order $x_1\, y_1\, \xi_2$ and y_2 will prove to be with respect to each other. We then find:

$$2c_1 x_1 + d_3 y_1^2 + \ldots = 2c'_1 \xi_2 + c'_2 y_2 + \ldots \quad . \quad . \quad . \quad (27)'$$

$$d_2 x_1^2 + 2d_3 x_1 y_1 + 4e_5 y_1^3 + \ldots = c'_2 \xi_2 + 2c'_3 y_2 + \ldots \quad . \quad (28)'$$

$$c_1 x_1^2 + 2d_3 x_1 y_1^2 + 3e_5 y_1^4 + \ldots = 2pc'_1 \xi_2 + pc'_2 y_2 + \ldots \quad . \quad (29)'$$

If we solve out of (27)' and (28)' ξ_2 and y_2 at first approximation we find:

$$\xi_2 = \alpha x_1 + \beta y_1^2 \text{ and } y_2 = \alpha' x_1 + \beta' y_1^2$$

where α, β, α' and β' have definite values.

From this ensues that ξ_2 and y_2 will be of the same order of magnitude as x_1 and y_1^2, when namely those two correspond in order. If on the contrary x_1 and y_1^2 are of different order, then ξ_2 and y_2 must be of the lowest order as one of them (namely x_1 or y_1^2).

From (29)' however ensues that $2pc'_1 \xi_2 + pc_2 y_2$ and so also $2c'_1 \xi_2 + c'_2 y_2$ are of higher order than x_1 or y_1^2 or both; hence out of (27)' may be concluded at first approximation:

$$2c_1 x_1 + d_3 y_1^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

The equation of the branch bO_1b' of the binodal line (fig. 13) is therefore represented at first approximation by (30). This equation (30), however, corresponds entirely to (10) representing a curve of contact which touches the binodal line aO_1a' (fig. 13) in the plaitpoint.

In an entirely similar way as in $II_{A.}$, we can now deduce:

"an accidental branch of a binodal line passing through a plaitpoint is in this point always curved in the same direction as the binodal line to which the plaitpoint belongs."

Between the radii of curvature R_b, R'_b and R_s exists of course also the relation

$$\frac{2}{R'_b}=\frac{3}{R_b}-\frac{1}{R_s}$$

in which R_b represents the radius of curvature of the binodal line to which the plaitpoint belongs and R'_b the radius of curvature of the accidental branch of the binodal line passing through the plaitpoint.

We now substitute in (28)′ and (29)′:

$$x_1=-\frac{d_3}{2c_1}y_1^2$$

and we find:

$$c'_2\,\xi_2+2\,c'_3\,y_2=\frac{4\,c_1\,e_5-d_3^{\,2}}{c_1}y_1^{\,3}+\ldots\ldots \quad (31)$$

$$2\,c'_1\,\xi_2+c'_2\,y_2=\frac{3\,(4\,c_1\,e_5-d_3^{\,2})}{4p\,c_1}y_1^{\,4}+\ldots\ldots \quad (32)$$

From this now follows:

$$(c'_2\,\xi_2+2\,c'_3\,y_2)^4=\alpha\,(2\,c'_1\,\xi_2+c'_2\,y_2)^3 \quad \ldots \quad (33)$$

in which:

$$\alpha=\frac{64}{27}\cdot\frac{4\,c_1\,e_5-d_3^{\,3}}{c_1}\cdot p^3.$$

The equation (33) represents at approximation the curve $c\,O_2\,c'$ (fig. 13); its tangent in point O is determined by

$$2\,c'_1\,\xi_2+c'_2\,y_2=0. \quad \ldots\ldots\ldots \quad (34)$$

The line determined by (34) is the diameter conjugated to the X-axis of the indicatrix in O_2; so we find that the tangent in O_2 and the conjugated line in $O_1\,O_2$ are conjugated diameters of the indicatrix in O_2. This property however has been known already for a long time[1]).

We now take that tangent in O_2 as new Y-axis, whilst we keep the line $O_1\,O_2$ as the X-axis.

Equation (33) now changes into:

$$(\lambda X+\mu Y)^4=\nu X^3$$

where λ, μ and ν have definite values. From this ensues as a first approximation of the binodal line in the vicinity of point O_2:

$$\mu^4 Y^4=\nu X^3\,,$$

or

$$Y^4=KX^3 \quad \ldots\ldots\ldots \quad (35)$$

If we calculate the radius of curvature in point O_2 we then find

[1]) D. J. Korteweg. l. c. p. 299.

that it is zero The branch cO_2c' of the binodal line has thus in point O_2 a somewhat angular shape, without however an angular point being really formed.

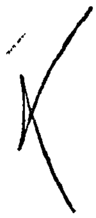

Fig. 14.

This shape is, indeed, the preparation to the wellknown form shown in fig. 14, generated in O_2 when the conjugated branch bO_1b' in fig. 13 begins to intersect the plait aO_1a'.

Moreover it is evident from the fact that ξ_2 and y_2 in the vicinity of the points O_1 and O_2 are of the same order of magnitude as y_1^3 and therefore much smaller than y_1, that the connode O_1 will displace itself there much quicker than the connode O_2.

Astronomy. — *"The Milky way and the star-streams."* By Prof. J. C. Kapteyn.

In a lecture, delivered before the Congress of Physicists and Physicians in the month of April, I arrived at the conclusion that "in passing from the stars of the spectral type B[1]) (Helium-stars) to those of the type A (Sirius-stars) and from these to those of the type G (solar-stars) there is a gradual change in the direction of the streams.

The stream-velocity was also found to be different. Owing to want of materials, however, the latter result was still even more uncertain than the former. Partly by the publication of Campbell's radial velocities of B stars[2]), partly by (not yet published) observations made on Mount Wilson, I have been able this summer materially to diminish this uncertainty.

It is true, that the increase of our data represents but a small fraction of what is urgently wanted. Still however, so much seems to have been gained already, that there is a pretty strong probability in favour of the conclusion that: not only the direction but also the velocity of the two great star-streams *gradually* changes in passing from type B to type A and thence to type G.

In these circumstances I feel justified in no longer suppressing a conclusion which was not yet communicated in my April lecture.

In what follows, stream-direction and stream-velocity will mean direction and velocity relative to the solar system, unless the contrary

1) In what follows the notations of Harvard college observatory have been adopted.

2) Lick Bulletin N°. 195.