Mathematics. — *"The theory of bilinear complexes of conics."* By Prof. D. MONTESANO. (Communicated by Prof. JAN DE VRIES).

1. The definitions given by me for the congruences and for the complexes of the conics of space can be transferred to the analogous systems of curves of any order.

We then find:

A doubly infinite algebraical system of curves of space of the same order n forms a congruence. The *order* of the congruence is the number of curves of the system passing through an arbitrary point; the *class,* for $n > 1$, is the number of curves of the system meeting in two points any straight line of space. A congruence of order 1 is *linear;* it is *bilinear* if order and class are equal to 1.

A triply infinite algebraical system of curves of space of the same order n forms a *complex.*

The *order* of a complex of plane curves of order $n > 1$ is the number of curves of the system belonging to any plane of space; the *class* is that of the cone enveloped by the planes of the curves of the system passing through an arbitrary point of space.

The complex is linear if it is of order 1; it is bilinear if order and class are equal to 1.

2. In 1892 I developed thoroughly [1]) the theory of the bilinear congruences of conics, showing in the first place that a congruence of this kind is formed by the intersections of the planes of a sheaf with the homologous quadrics of a homographic net [2]). I immediately completed the study of the bilinear complexes of conics; it had to form one of the chapters of my book: *"La Géométrie des coniques de l'espace"*, to appear shortly.

Mr. GODEAUX, who was already informed of my works and of the coming publication of my studies relating to the bilinear complexes of conics, occupied himself in 1907 with the same complexes of which I had provided him with the definition.

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1) Su di un sistema lineare di coniche nello spazio. Atti della R. Accademia delle Scienze di Torino, vol. XXVII, 1892.

2) Mr. GODEAUX says that this theorem is by M. VENERONI (Recherches sur les systèmes de coniques de l'espace, Mémoires de la Société royale des Sciences de Liège. 3me s. t. IX, 1911, p. 17). The demonstration of M. VENERONI (Sopra alcuni sistemi di cubiche gobbe, Rendiconti del Circolo matem. di Palermo, t. XVI, 1902, p. 215) is ten years posterior to mine!

The results of the researches which he has published [1]) can be gathered in the following theorems A), B), C).

A) *There are four types of bilinear complexes of conics according to the conics of the complex lying*

1. *on the quadrics of a linear system* ∞^3 Σ;
2. *on the quadrics of a net* P;
3. *on the quadrics of a pencil* Φ;
4. *on a single quadric.*

In the first case the correspondence (1,1) *between the quadrics of* Σ *and the planes of the conics situated on these quadrics is projective* [2]).

But this theorem:

is incomplete for the 2[rd] and the 3[rd] type, as the correspondences $(1, \infty)$, $(1, \infty^2)$ between the quadrics of the net P or of the pencil Φ and the planes of space are indeterminate; it is not true for the complex of the 4[th] type which, formed by the plane sections of a quadric, is of order 1 and of class 0; it has finally no foundation as the four types are reduced in all cases to a single one. In fact I have demonstrated that:

Every bilinear complex of conics (of space) is formed by the sections of the planes of space with corresponding quadrics of a triply infinite linear system, projective to the space as locus of the planes.

3. After this the two following theorems B and C of Mr. GODEAUX are no more of importance.

B). *Every bilinear complex of conics is generated by the intersection of the homologous elements of two varieties in birational correspondence; one of these varieties consists of the planes of space, the other of a homoloid triple infinity of quadrics belonging to a linear system* ∞^5.

C) *Every bilinear complex of conics is birationally equivalent to the complex generated by the intersection of the planes of space and the homologous quadrics of a linear system* ∞^3 *in birational correspondence* [3]).

But we can still observe that these two theorems do not express characteristic properties of the bilinear complex, for they hold for

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1) Notes 1[st] and 2[nd]: Bull. de l'Acad. roy. de Belgique, Classe des Sciences 1908, p. 597—601; 812—813; Note 3[rd]: Ibid. 1909, p. 499—500; Note 4[th]: Nouv. Ann. de Mathém. 4[e] série t. IX. p. 312—317.

2) Note 2[nd] p. 599 et 600.

3) Note 2[nd] p. 813.

any linear complex of conics. More precisely we can state instead of theorem C) this general proposition:

It is always possible to arrange a birational correspondence between the conics of any two bilinear complexes. This correspondence can be constructed by fixing a birational correspondence between the planes of space and by regarding as homologous two conics of the complexes which are lying in two homologous planes of this correspondence.

With respect to theorem C) we must still observe that:

A birational correspondence between the planes of space and the quadrics of a linear system ∞^3 *is necessarily projective, if the curves of intersection of the homologous elements form a bilinear complex.*

Indeed, it is evident that *the complex generated by the intersections of the homologous elements in a birational non-projective correspondence* $[m, n]$ *between the planes of space and the quadrics of an arbitrary*[1]) *linear system* ∞^3 *is of order* 1 *and of class* $m > 1$[2]).

Hence the correspondence of theorem C is *projective* besides *birational.*

4. The proposition on which Mr. GODEAUX depends for the demonstration of the theorems B), C) consists at bottom of the fact, that *two conics of a bilinear complex cannot have two points in common*[3]).

This proposition has no foundation. Indeed I have demonstrated that:

On any conic of a bilinear complex exist 10 *pairs of points* $A_1A'_1, \dots A_{10}A'_{10}$, *in such a way that all conics of the complex passing through a point* A_i *pass also through* $A'_i\,(i=1, \dots 10)$.

On the cubic surface $\varphi_3\,(A_iA'_i)^2$, locus of these conics, the one passing through an arbitrary point of the line $d = A_iA'_i$ exists and it *is* determined, contrary to the statement of Mr. GODEAUX. This conic is situated in the tangential plane to the surface $\varphi_0^{(i)}$

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[1]) The characteristic numbers m, n of the correspondence are respectively the class of the surface enveloped by the planes corresponding to the quadrics of an arbitrary net of the system and the class of the developable enveloped by the planes corresponding to the quadrics of a pencil of the system.

[2]) We also find: *A birational correspondence being given between the quadrics of a net and the tangential planes of a homoloid surface of class* ν, *if to the quadrics of any pencil of the net the tangential planes of a developable of class* m *are homologous the congruence of the conics generated by the homologous elements of the correspondence is of order* m *and of class* ν.

If $\nu = m = 2$ we find the congruence considered by Mr. JAN DE VRIES: *A congruence of order two and class two formed by conics.* (Proceed. of the R. Acad. o. S., Amsterdam Nov. 23, 1904).

[3]) Note 2nd p. 813, N°. 4.

according to the line d, and it breaks up into this line and into the other line of intersection of the surface with the plane.

5. Amongst the numerous particular cases of the bilinear complex of conics, the one considered by Mr. HUMBERT is of special importance. This complex is determined by a complete skew pentagon in the sense that the conic of the complex situated in an arbitrary plane of space is the one with respect to which the intersection of the edge of the pentagon has as polar the intersection with the opposite face [1]).

This conic breaks up only if the plane passes through one of the vertices of the pentagon.

Mr. GODEAUX states that the complex of Mr. HUMBERT is identical to the complex generated by a projective correspondence between the planes of space and the quadrics having an autopolar tetrahedron [2]). Neither is this true. Indeed, in the complex of Mr. GODEAUX the planes of the conics breaking up into two lines envelop a single surface of class five, whilst in that of Mr. HUMBERT they form five pencils.

6. In a bilinear complex Γ the planes of the conics passing through an arbitrary point P of space form a pencil of which the axis p describes, if P varies, a *rational* complex in a perspective one-one correspondence with the system of the points P of space. [3])

The rays p passing through any point O correspond to the points P of the director-curve of the bilinear congruence of conics of the complex Γ contained in the planes of the sheaf (O); so they form a conic of order six projecting the director-curve of point O.

From this ensues that the complex of lines p is not a cubic complex, as Mr. GODEAUX states [4]); on the contrary it is of order six. And it is with the complex Γ in such a relation that every property of the former transforms itself into a property of the other. I have mentioned these properties in a paper read at the international congress of mathematicians at Rome (April 1908) [5]).

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1) *Sur un complexe remarquable de coniques*... (Journal de l'Ecole Polytechnique, Cahier LXIV. 1894).

2) Note 1st p. 600; Note 4th p. 317.

3) We say that a complex of lines is rational if it is possible to establish a birational correspondence between its rays and the points (or the planes) of space. This correspondence is perspective if every ray of the complex passes through the corresponding point (or if it is situated in the plane).

4) Note 4th p. 315.

5) Atti del IV Congresso internazionale dei Matematici, Roma 1909, p. 231—233.

7. I wish still to observe that *the methods followed in the study of the congruences and the bilinear complexes of conics can also serve in general for the congruences and the bilinear complexes of plane curves of any order.*

In particular: *An entirely arbitrary bilinear complex of plane curves Γ is in relation to a complex of lines K and with a birational and perspective representation H of the complex K on the system of points of space i.e. that to the rays of K contained in an arbitrary plane correspond in the representation H the points of the curve of Γ situated in this plane.*

Finally: *a rational complex of lines cannot always be brought into a birational and perspective correspondence with the system of points or with the planes of space; for it may happen that neither the one nor the other of these correspondences is possible, or that only the first can be possible, or that only the second is possible or that both of them are possible.*

Hence we find four different types of rational complexes of lines.

So e. g. in the most general case the quadratic complex of lines is of the first type; but if it contains the lines of a sheaf (or of a plane) without showing any other particularities, it belongs to the second (or the third) type; finally if it has linear congruences of lines with rectilinear directrices, it belongs to the 4th type [1]).

The linear [2]) complex and in general any complex containing a rational pencil of linear congruences with rectilinear directrices is also of the 4th type.

Capri, October 1911.

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1) Compare MONTESANO: *Su le* [illegible] *ivoche dello spazio che determinano complessi quadratici di* [illegible] del. R. Instituto lombardo. Serie II. Vol. XXV 1891; § 1).

2) Compare MONTESANO: *Su la curva gobba di 5⁰ ordine e di genere* 1 (Rendiconti della R. Accademia della Scienze di Napoli, Giugno 1888; § 12).