Mathematics. — *"Homogeneous linear differential equations of order two with given relation between two particular integrals".* (4[th] communication). By Dr. M. J. VAN UVEN. (Communicated by Prof. W. KAPTEYN).

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In the preceding communications we have discussed the parity of function $I(\tau)$ in connection with the parity of the functions $x(\tau)$ and $y(\tau)$. It then became evident that the same function $I(\tau)$ determines two mutually semi-equivalent curves $F(x,y)=0$ when it is a univalent even function of τ.

Let us now suppose that $I(\tau)$ is determined as root of even power out of a certain function of τ, then to the same curve $F(x,y)=0$ belong two opposite functions I, from which ensues that $F(x,y)=0$ is then semi-equivalent to itself.

We shall occupy ourselves in the following, in connection with the remark made here, in particular with algebraical curves $F(x,y)=0$.

Just as in equation (31) and (32) (1[st] communication page 398) we have expressed I in the integrals x and y, we shall now also give $\dot{I}=\dfrac{dI}{d\tau}$ such a form.

We shall make use of the following abridgments:

$$\overline{\Phi}=\Phi_x F_y-\Phi_y F_x,$$
$$\overline{\overline{\Phi}}=(\overline{\Phi})_x F_y-(\overline{\Phi})_y F_x$$
etc.

With the aid of this we can write equation (31) in the form

$$G=F_z\overline{H}-3H\overline{F_z}.$$

We then find

$$\dot{I}=\frac{dI}{d\tau}=I_x\frac{dx}{d\tau}+I_y\frac{dy}{d\tau}=\frac{(n-1)F_z^{\frac{1}{2}}}{z^{\frac{1}{2}}H^{\frac{1}{2}}}(I_x F_y-I_y F_x)=\frac{(n-1)F_z^{\frac{1}{2}}\overline{I}}{z^{\frac{1}{2}}H^{\frac{1}{2}}}.$$

If we calculate I_x and I_y out of (32), we shall finally find:

$$I=\frac{dI}{d\tau}=\frac{(n-2)^2}{2zF_zH^3}(4F_zH\overline{F_z}\overline{H}+2F_z^2H\overline{\overline{H}}-6F_zH^2\overline{\overline{F_z}}-3F_z^2\overline{H}^2+3H^2\overline{F_z}^2). \quad (51)$$

For I^2 we can write

$$I^2=\frac{(n-1)^2}{zF_zH^3}(F_z\overline{H}-3H\overline{F_z})^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

In the supposition that $F(x,y,z)=0$ is an algebraical equation we shall arrive by eliminating the homogeneous variables x, y, and z out of (51), (52), and $F(x,y,z)=0$ at a rational equation between I^2 and $\dot{I}$;

$$\dot{I}=\Psi(I^2),$$

If the solution is

$$\Phi(I^2, I) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

we then find τ out of

$$\tau - \tau_0 = \int \frac{dI}{\Psi(I^2)} = \Omega(I) \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

and $I(\tau)$ by reversing the function $\Omega(I)$.

As $\Psi(I^2)$ is an algebraical function τ is an algebraical integral-function of I, and $I(\tau)$ is the reverse of it.

If we take $I^2 = X$ and $I = Y$ as rectangular coordinates then

$$\Phi(X, Y) = 0$$

will represent some algebraical curve.

We can conjugate the curve $\Phi(X, Y) = 0$ to the system of all curves $F(x, y) = 0$ which are mutually equivalent. A curve $F_1(x, y) = 0$, which is semi-equivalent to $F = 0$, determines an opposite I, hence an equal X and an opposite Y. The curve $F_1 = 0$ conjugate to $\Phi_1 = 0$ is thus the *image* of $\Phi = 0$ with respect to the X-axis.

The curve $\Phi = 0$, which is conjugate to a curve $F = 0$ *semi-equivalent to itself,* is therefore symmetrical with respect to the X-axis.

We shall now give a somewhat extensive treatment of the case in which $F = 0$ represents a conic.

By means of a homogeneous linear substitution (if necessary with complex coefficients) we can always make one of the points at infinity to be in the direction of the Y-axis. In this case the equation $F(x, y) = 0$ is linear in y and the equation $y = \varphi(x)$ is rational, so that operation with the equations (20),(21), and (22) (1st communication page 366) gives rise to few algebraic complications.

However as we have our formulae ready for I and $\dot{I}$ expressed by means of the implicit equation $F(x, y, z) = 0$ we shall, likewise with a view to greater symmetry, make use of the unsolved equation $F(x, y, z) = 0$.

Beforehand we remark that not all conics can be transformed into each other by means of homogeneous linear substitutions. For, only those conics can be transformed into each other by means of these substitutions where the anharmonic ratio σ between the points S_1 and S_2 at infinity and the points of contact R_1 and R_2 of the tangents out of the origin have the same value; in other words: equivalent conics have equal σ.

We shall now first express the value of σ in the coefficients of the equation $F = 0$.

The anharmonic ratio $\sigma = (S_1 S_2, R_1 R_2)$ is the anharmonic ratio of the four rays which join these points with a fifth point of the

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conic or of the four points which their tangents describe on a fifth tangent of the conic. Let us take as fifth tangent one of the tangents OR_1 out of O, then the point of contact R_1 must be considered as point of intersection of OR_1 with itself. The second tangent OR_2 out of O intersects OR_1 in O. Let us moreover call S_1', S_2' the points of intersection of OR_1 with the asymptotes of S_1^∞ and S_2^∞ then we can write

$$\delta = (S_1 S_2 . R_1 R_2) = (S_1' S_2', R_1 O) = \frac{S_1' R_1}{S_2' R_1} : \frac{S_1' O}{S_2' O},$$

or, because $S_1' R_1 = - S_2' R_1$,

$$\delta = - \frac{S_2' O}{S_1' O}.$$

The ratio $\frac{S_2' O}{S_1' O}$ can now be replaced by the ratio of the abscissae x_2 and x_1 of S_2' and S_1', so that we find as simplest expression

$$\delta = - \frac{x_2}{x_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

The conic may be represented by the equation

$$F(x,y,z) \equiv a_{11} x^2 + 2 a_{12} xy + a_{22} y^2 + 2 a_{13} xz + 2 a_{23} yz + a_{33} z^2 = 0. \quad (56)$$

We now put

$$\begin{vmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{12}, & a_{22}, & a_{23} \\ a_{13}, & a_{23}, & a_{33} \end{vmatrix} = \Delta$$

and we indicate the subdeterminants of $a_{11} \ldots a_{33}$ respectively by $A_{11} \ldots A_{33}$. The pair of asymptotes is then represented by the equation

$$a_{11} x^2 + 2 a_{12} xy + a_{22} y^2 + 2 a_{13} xz + 2 a_{23} yz + \left(a_{33} - \frac{\Delta}{A_{33}} \right) z^2 = 0.$$

The points of intersection of this pair of lines with the line

$$y = mx$$

through O are now determined by

$$(a_{22} m^2 + 2 a_{12} m + a_{11}) x^2 + 2 (a_{23} m + a_{13}) xz + \left(a_{33} - \frac{\Delta}{A_{33}} \right) z^2 = 0. \, (57)$$

If the indicated line through O is to touch the conic, then m must satisfy

$$A_{11} m^2 - 2 A_{12} m + A_{22} = 0,$$

or

$$(a_{22} a_{33} - a_{23}^2) m^2 - 2 (a_{13} a_{23} - a_{12} a_{33}) m + (a_{11} a_{33} - a_{13}^2) = 0,$$

or

$$(a_{23}m + a_{13})^2 = a_{33}(a_{22}m^2 + 2a_{12}m + a_{11}) \quad . \quad . \quad . \quad . \quad (58)$$

Out of (57) and (58) follows for both roots $x_1 . z_1$ and $x_2 : z_2$ or x_1 and x_2:

$$\frac{(x_1+x_2)^2}{4x_1x_2} = \frac{(a_{23}m+a_{13})^2}{(a_{22}m^2+2a_{12}m+a_{11})\left(a_{33}-\frac{\Delta}{A_{33}}\right)} = \frac{a_{33}}{a_{33}-\frac{\Delta}{A_{33}}} = \frac{a_{33}A_{33}}{a_{33}A_{33}-\Delta},$$

hence

$$\frac{(x_1+x_2)^2}{(x_1-x_2)^2} = \frac{a_{33}A_{33}}{\Delta},$$

and, on account of (55),

$$\left(\frac{1-\delta}{1+\delta}\right)^2 = \frac{a_{33}A_{33}}{\Delta} = \lambda^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

From this ensues

$$\frac{1-\delta_1}{1+\delta_1} = +\lambda \quad , \quad \frac{1-\delta_2}{1+\delta_2} = -\lambda,$$

so

$$\delta_1 = \frac{1-\lambda}{1+\lambda} \quad , \quad \delta_2 = \frac{1+\lambda}{1-\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

So we find as was to be expected two reciprocal values for δ. Equivalent conics have equal δ, therefore also equal λ. So for equivalent conics holds

$$\frac{a_{33}A_{33}}{\Delta} = constant.$$

To investigate how the value of δ depends on the form of the conic and on its situation with respect to the origin and of the line at infinity we can invert the relative situation of conic and origin. So we base our considerations on a fixed conic and we have then to investigate how the value of δ depends on the situation of the point O thought as variable with respect to the fixed conic.

We take for the conic of reference the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

If later on we wish to transfer our results to the ellipse, we have but to suppose b imaginary.

If x_0, y_0 are the coordinates of point O, and x_1, y_1 and x_2, y_2 those of the points S_1' and S_2', then holds

$$\delta = -\frac{OS_2'}{OS_1'} = -\frac{x_2-x_0}{x_1-x_0} = -\frac{y_2-y_0}{y_1-y_0},$$

so

$$(1+\delta)x_0 = \delta x_1 + x_2 \quad , \quad (1+\delta)y_0 = \delta y_1 + y_2 .$$

Because (x_1, y_1) and (x_2, y_2) lie on the asymptotes and their midpoint $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ on the conic, we find

$$\frac{x_1}{a} = \frac{y_1}{b} \quad , \quad \frac{x_2}{a} = -\frac{y_2}{b} \quad , \quad \frac{(x_1+x_2)^2}{a^2} - \frac{(y_1+y_2)^2}{b^2} = 4,$$

from which ensues

$$\frac{x_1 x_2}{a^2} - \frac{y_1 y_2}{b^2} = 2,$$

so that

$$(1+\delta)^2\left(\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2}\right) = \delta^2\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}\right) + 2\delta\left(\frac{x_1 x_2}{a^2} - \frac{y_1 y_2}{b^2}\right) + \left(\frac{x_2^2}{a^2} - \frac{y_2^2}{b^2}\right) = 4\delta,$$

or

$$\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = \frac{4\delta}{(1+\delta)^2} = 1 - \lambda^2 = k \quad . \quad . \quad . \quad . \quad . \quad (61)$$

So points O of equal δ lie on a conic similar and homothetic to the conic of reference.

In this way we can represent the values of δ in connection with those of k and λ in the following plan:

	I	II	III	VI	V	VI	VII=I
k	(+) $-\infty$	$> -\infty, < 0$	0	$> 0, < +1$	$+1$	$> +1, < +\infty$	$+\infty$ (−)
λ	$+\infty$	$< +\infty, > +1$	$+1$	$< +1, > 0$	0	$i\nu', 0 < \nu' < \infty$	$i\infty, +\infty$
δ_1	-1	$> -1, < 0$	0	$> 0, < +1$	$+1$	$e^{-i\psi}, 0 < \psi < \pi$	-1
δ_2	-1	$< -1, > -\infty$	$-\infty$ +	$< +\infty, > +1$	$+1$	$e^{+i\psi}, 0 < \psi < \pi$	-1

For the parabola $A_{33} = 0$, hence $\lambda = 0$, so always $\delta_1 = \delta_2 = +1$.

For the hyperbola we find the following state of affairs:

I. At infinity $\quad \delta_1 = -1 \quad , \delta_2 = -1$;

II. in the domain of the conjugate hyperbola $\quad -1 < \delta_1 < 0 \quad , -1 > \delta_2 > -\infty$;

III. on the asymptotes $\quad \delta_1 = 0 \quad , \delta_2 = \mp\infty$;

IV. between the asymptotes and the curve $\quad 0 < \delta_1 < +1 \quad , +\infty > \delta_2 > +1$;

V. on the curve $\quad \delta_1 = +1 \quad , \delta_2 = +1$;

VI. at the concave side of the curve $\delta_1 = e^{-i\psi}$, $0 < \psi < \pi$, $\delta_2 = e^{+i\psi}$, $0 < \psi < \pi$.

For the ellipse holds:

VII. At infinity	$\sigma_1 = -1$	, $\sigma_2 = -1$;
VI. outside the curve	$\sigma_1 = e^{-i\psi}, \pi > \psi > 0$,	$\sigma_2 = e^{+i\psi}, \pi > \psi > 0$;
V. on the curve	$\sigma_1 = +1$	, $\sigma_2 = +1$;
IV. inside the curve	$+1 > \sigma_1 > 0$	, $+1 < \sigma_2 < +\infty$;
III. in the centre	$\sigma_1 = 0$	, $\sigma_2 = +\infty$.

We shall now determine the form of the function $I(\tau)$.

From (53) follows:

$$F_x = 2(a_{11}x + a_{12}y + a_{13}z) \quad , \quad F_y = 2(a_{12}x + a_{22}y + a_{23}z).$$

$$F_z = 2(a_{13}x + a_{23}y + a_{33}z) = 2g.$$

$$H = \begin{vmatrix} 2a_{11}, & 2a_{12}, & 2a_{13} \\ 2a_{12}, & 2a_{22}, & 2a_{23} \\ 2a_{13}, & 2a_{23}, & 2a_{33} \end{vmatrix} = 8\Delta \ , \ \overline{H} = 0 \ , \ \overline{\overline{H}} = 0.$$

$$\overline{F}_z = F_{xz}F_y - F_{yz}F_x = 4\{a_{13}(a_{12}x + a_{22}y + a_{23}z) - a_{23}(a_{11}x + a_{12}y + a_{13}z)\} =$$
$$= 4(A_{23}x - A_{13}y).$$

$$\overline{\overline{F}}_z = (\overline{F}_z)_x F_y - (\overline{F}_z)_y F_x = 8\{A_{23}(a_{12}x + a_{22}y + a_{23}z) + A_{13}(a_{11}x + a_{12}y + a_{13}z)\} =$$
$$= 8\{(a_{11}A_{13} + a_{12}A_{23})x + (a_{12}A_{13} + a_{22}A_{23})y + (a_{13}A_{13} + a_{23}A_{23})z\} =$$
$$= 8\{-a_{13}A_{33}x - a_{23}A_{33}y + (\Delta - a_{33}A_{33})z\} = 8(\Delta z - A_{33}g).$$

$$I^2 = \frac{3^2 . 2^{10} . \Delta^2 (A_{23}x - A_{13}y)^2}{2^{10} zg\Delta^3} = 9 \frac{(A_{23}x - A_{13}y)^2}{zg\Delta},$$

$$\dot{I} = \frac{-3 . 2^{11} . \Delta^2 g(\Delta z - A_{33}g) + 3 . 2^{10} . \Delta^2 (A_{23}x - A_{13}x)^2}{2^{11} zg\Delta^3} =$$
$$= \frac{3}{2zg\Delta}\{(A_{23}x - A_{13}y)^2 - 2g(\Delta z - A_{33}g)\}.$$

We now find:

$$(A_{13}x - A_{13}y)^2 + A_{33}g^2 - 2\Delta gz =$$
$$= (A_{23}^2 + a_{13}^2 A_{33})x^2 + 2(-A_{13}A_{23} + a_{13}a_{23}A_{33})xy + (A_{13}^2 + a_{23}^2 A_{33})y^2 +$$
$$+ 2(a_{13}a_{33}A_{33} - \Delta a_{13})xz + 2(a_{23}a_{33}A_{33} - \Delta a_{23})yz + (A_{33}a_{33}^2 - 2\Delta a_{33})z^2 =$$
$$= (a_{33}A_{33} - \Delta)(a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2) - \Delta a_{33}z^2$$
$$= (a_{33}A_{33} - \Delta)F - \Delta a_{33}z^2,$$

or, because (x, y, z) satisfy $F = 0$,

$$(A_{23}x - A_{13}y)^2 + A_{33}g^2 - 2\Delta gz = -\Delta a_{33}z^2.$$

Hence we find

$$I^2 = \frac{9}{zg\Delta}(-A_{33}g^2 + 2\Delta gz - \Delta a_{33}z^2), \quad . \quad . \quad . \quad . \quad (62)$$

$$\dot{I} = \frac{3}{2zg\Delta}(A_{33}g^2 - \Delta a_{33}z^2) \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (63)$$

By elimination of g we find

$$36\,\dot{I}^2 = I^4 - 36 I^2 + 324\left(1 - \frac{a_{33}A_{33}}{\Delta}\right), \quad . \quad . \quad . \quad (64)$$

or, on account of (59),

$$36\,\dot{I}^2 = (I^2 - 18)^2 - 18^2\lambda^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

so

$$6\,\dot{I} = \pm \sqrt{\{I^2 - 18(1+\lambda)\}\{I^2 - 18(1-\lambda)\}},$$

or

$$\tau - \tau_0 = \pm \int \frac{6dI}{\sqrt{\{I^2 - 18(1+\lambda)\}\{I^2 - 18(1-\lambda)\}}}. \quad . \quad . \quad (66)$$

So I proves to be an *elliptic* function of τ.

If we introduce $I^2 = u$ as variable we find:

$$36\,I^2\dot{I}^2 = I^6 - 36\,I^4 + 324\,(1-\lambda^2)\,I^2,$$

or

$$9\left(\frac{du}{d\tau}\right)^2 = u^3 - 36u^2 + 324\,(1-\lambda^2)\,u$$
$$= u\,\{u - 18\,(1+\lambda)\}\,\{u - 18\,(1-\lambda)\},$$

thus

$$\tau - \tau_0 = \pm \int \frac{3du}{\sqrt{u\,\{u - 18\,(1+\lambda)\}\,\{u - 18\,(1-\lambda)\}}} \quad . \quad . \quad . \quad (67)$$

The singular points are now $u_1 = \infty$, $u_2 = 0$, $u_3 = 18\,(1+\lambda)$, $u_4 = 18\,(1-\lambda)$.

One of their six anharmonic ratios is therefore

$$\frac{u_4}{u_3} = \frac{1-\lambda}{1+\lambda} = \sigma_1\,.$$

The anharmonic ratio of the elliptic function $u = I^2 = Q(\tau)$ is therefore equal to the anharmonic ratio of the four characteristic points $S_1^{\infty}, S_2^{\infty}, R_1, R_2$ of the conics $F = 0$.

Evidently the invariant of this elliptic function is:

$$i = \frac{4\,(\sigma^2 - \sigma + 1)^3}{27\sigma^2\,(1-\sigma)^2} = \frac{(1+3\lambda^2)^3}{27\,(1-\lambda^2)^2\,\lambda^2} = \frac{(\Delta + 3a_{33}A_{33})^3}{27a_{33}A_{33}\,(\Delta - a_{33}A_{33})^2} \quad . \quad (68)$$

Before transforming the elliptic integral we shall first investigate in what case it degenerates. Degeneration takes place, when the equation $\frac{du}{d\tau} = 0$ has two coinciding roots. This occurs:

1. When $\lambda = 0$, thus $\sigma_1 = \sigma_2 = +1$; in this case either $a_{33} = 0$, holds or $A_{33} = 0$, i.e. either the conic passes through O, or it touches the line at infinity, in other words it is a parabola. These two types of curves are *not* equivalent, but they are semi-equivalent: so they have opposite functions I. This now coincides with the fact, that for

$\lambda = 0$ the form under the sign of the root in (66) is a perfect square, so that two separated functions I appear. To distinguish the types $a_{33} = 0$ and $A_{33} = 0$ properly we shall return to the equations (62) and (63). For $a_{33} = 0$ these take the following forms:

$$z\Delta I^2 = 9\,(-A_{33}g + 2\Delta z),$$

$$2z\Delta \dot{I} = 3A_{33}g.$$

By elimination of g we find

$$6\dot{I} = 18 - I^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69a)$$

so

$$\tau - \tau_0 = \int \frac{6dI}{18 - I^2} = +\sqrt{2}\,.\,tan\,h^{-1}\frac{I}{3\sqrt{2}},$$

or

$$I = +\,3\sqrt{2}\,.\,tan\,h\,\frac{\tau - \tau_0}{\sqrt{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (70a)$$

If on the other hand we put $A_{33} = 0$, then (62) and (63) pass into

$$gI^2 = 18g - 9a_{33}z,$$

$$2g\dot{I} = -\,3a_{33}z.$$

Elimination of g now leads to

$$6\dot{I} = I^2 - 18, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69b)$$

from which ensues

$$\tau - \tau_0 = -\int \frac{6dI}{18 - I^2} = -\sqrt{2}\,.\,tan\,h^{-1}\frac{I}{3\sqrt{2}},$$

or

$$I = -\,3\sqrt{2}\,.\,tan\,h\,\frac{\tau - \tau_0}{\sqrt{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (70b)$$

2. A second case of degeneration appears when $\lambda = +1$ (or $\lambda = -1$), so for $\sigma_1 = 0$, $\sigma_2 = \infty$ or $\sigma_1 = \infty$, $\sigma_2 - 0$; in this case we have $a_{33}A_{33} = \Delta$ or $a_{13}A_{13} + a_{23}A_{23} = 0$; the geometric meaning of this is that O lies on one of the asymptotes (for the ellipse in the centre). The equation (66) now runs:

$$\tau - \tau_0 = -\int \frac{dI}{I\sqrt{I^2 - 36}} = \pm\,sin^{-1}\frac{6}{I},$$

so that

$$I = \pm\,\frac{6}{sin\,(\tau - \tau_0)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

3. When at the same time $a_{33} = 0$ and $A_{33} = 0$ holds, i.e. when the conic is a parabola passing through O, then the equation (62) furnishes

$$I = \pm\,3\sqrt{2}.$$

This result has formerly been found (see 2[nd] communication page 590); we can regard it as the combination of (70a) and (70b) for $\tau_0 = \infty$.