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marrow fibres that are found there and border the central grey substance. This strongly lateral situation is less pronounced in man though extant in principle, as appears from the figures subjoined to my previous communication (l.c.).

As to the nucleus trochlearis principalis, its situation corresponds to a high degree in all examined animals. In its entire length however it is imbedded in a more lateral part of the fasciculus longitudinalis than with rabbits. Slight local differences — apparently depending upon a stronger dorsal curvature of the posterior longitudinal bundle — occur however. Only in the cat the gradual transition into the nucleus oculomotorii is missing. The most distal cells of this nucleus are not situated strictly dorsal from the place where the nucleus trochlearis was found, but almost in the prolongation of the latter. Only a few preparations further frontal these cells pass into elements situated dorsomedially that have been developed in the mean time. In the dog the medial nucleus part however is found already in the same level where the nucleus trochlearis is still present. The transition of the latter into the lateral cells of the oculomotorian nucleus is gradual, as in all other examined animals (with the exception of the cat).

At last I still fix the attention to the great asymmetry of the split nuclei trochleares of the two rabbits which is distinctly expressed in the above lists. In rabbit 1 to the left a nucleus posterior "lagging far behind", to the right another lying only 180 μ farther caudal. In rabbit 8 to the left no splitting at all, to the right a very distinct one, which caused the formation of a nucleus posterior comparatively very rich in cells.

No certain information can be given about the significance of the phenomenon which I could only ascertain in man and rabbit. It seems only clear that by the distolateral direction of the trochlearic-root the situation of the nucleus trochlearis is at least partially determined.

Mathematics. — "*Calculus rationum*". By Dr. G. DE VRIES. (Communicated by Prof. JAN DE VRIES).

(Communicated in the meeting of February 24, 1912).

So far mathematicians have adhered to the opinion that an operation of the fourth rank would teach nothing new; this opinion was in part founded on the conviction that base number and exponent of a power cannot be submitted to the commutative, associative, and distributive properties. I have done away with this objection by introducing the notion "*mutual power of two numbers*".

Doing so I have at the same time indicated the means of introducing operations of an arbitrary rank for which the above mentioned properties hold.

I hope I have opened a new field of investigation. The future must show whether it is of importance. At all events the considerations have led me to investigate groups of transcendent curves, to a logical classification and to a new analysis of them.

I have the outcome of my investigations in manuscript, the contents of which I wish to sketch in some lines. Starting from the algebraical part I arrive in connection with the above mentioned analysis to the geometrical applications. Only the operation of the fourth rank will be under discussion and for the rest only considerations relating to two variables will be allowed.

§ 1. If a power is submitted to a new involution the exponents may be mutually interchanged. In this truth lies practically the validity of the commutative property. Only a symbol is wanting for the continual involution together with the settling of a base number to determine univalently the mutual power of numbers. If we choose for this e then the forms in their simplest shape appear. This supposition is made in the following, whilst the Nap. logarithm shall be indicated by L.

"*The mutual power of two numbers is the power of e having the product of the logarithms of those numbers as exponents.*"

If we put

$$x = e^p \quad ; \quad y = e^q,$$

we shall write:

$$x,y = e^{pq} = x^{Ly} = y^{Lx} = y,x.$$

That for the mutual power of more numbers also the associative and the distributive property holds, will need no reasoning.

In a form as

$$u = x,y,z$$

we shall call $x, y,$ and z *efficients.*

§ 2. A continued involution or evolution with equal exponent is called "*gradation*" and the upper exponent appearing here *gradation index.* The symbol used for it follows out of:

$$e^{p^n} = {}^n(e^p) \quad ; \quad (\sqrt[p]{\ })^n e = e^{p^{-n}} = {}^{-n}(e^p).$$

These can be summarized in the form

$$^n(x) = e^{L^n x}.$$

"A continued mutual power with equal efficients is called gradation". The inverse operation is called *"descension"*.

"The n^{th} descension of a number is the number which put to the n^{th} gradation furnishes the original number".

The descension will be indicated by an inverse rootsign; a distinction of power and root descension is superfluous. The forms

$$e^{\sqrt[n]{p}} = \underset{n}{\diagup\!\!1}\, e^{p} \quad ; \quad e^{\sqrt[n]{\frac{1}{p}}} = \sqrt[\sqrt[n]{p}]{e} = \underset{n}{\diagup\!\!1}(\sqrt[r]{e})$$

can therefore be written as

$$\underset{n}{\diagup\!\!1}\, x = e^{\sqrt[n]{Lx}}\,.$$

When introducing negative and broken indices of gradations everything can be summarized in ${}^{n}(x)$.

The gradation has the precedency of the involution, the evolution of the descension.

§ 3. For a first consideration it is desirable to allow only positive base numbers. By means of the operation, however, complex powers may appear. Thus the number

$$\underset{2}{\diagup\!\!1}\frac{1}{e} = e^{i}$$

will prove to act here the part of $\sqrt{-1}$ in common mathematics.

The elementary operation is multiplication; when comparing two quantities we must therefore pay attention to their ratio. For the construction of figures the axes of coordinates are divided in such a manner, that the successive abscissae (and ordinates) form a geometrical series. The lines drawn through the dividing points parallel to the axes form a net of coordinates which shall be called *"field of ratio"* (in contrast to the wellknown "field of difference").

For constructions it is advisable to take as base a number differing but little from unity.

Just as the difference of two numbers is bivalent this proves likewise the case with their ratio. Referring to what HOÜEL[1]) says about operation-modulae we shall assign the same absolute value (in a rational sense) to

$$\frac{x}{y} \text{ and } \frac{y}{x}.$$

(Numbers smaller than unity have for the field of ratio the same

[1]) Cours de calcul infinitésimal I; § IV etc.

meaning as negative numbers for the field of difference). The following can serve as a confirmation of the assumed:

$$1 \times \frac{x}{y} = 1 : \frac{y}{x} \text{ and } {}^{2}\left(\frac{x}{y}\right) = {}^{2}\left(\frac{y}{x}\right).$$

§ 4. Out of the definition of mutual power follows that only one indirect operation can be deduced from this, viz.:

$$x = \sqrt[Ly]{z} \quad ; \quad y = \sqrt[Lx]{z}\,.$$

If we use the word *"mutual root"* there is still ambiguity. The absolute value of the two mutual roots (represented as follows):

$$x|y = \sqrt[Ly]{x}\,; \quad y|x = \sqrt[Lx]{y}$$

must, however, be regarded as an equal one for the field succeeding the field of ratio. We might call the former the root of x to y.

It is useful to give forms as

$$e|x = \sqrt[Lx]{e} = {}^{-1}(x)$$

the name of "reciprocal power".

For gradations and descensions we might mention a series of properties corresponding entirely to those for powers and roots. The further development of rational algebra is analogous to ordinary algebra. So we can speak here of a gradation binomium, of remarkable roots, continuous roots, etc. In the geometrical part of course the logarithmic proportion

$$a|b = c|d$$

comes to the foreground. And for different base numbers the form

$$^{a}Lu : {}^{a}Lv = {}^{b}Lu : {}^{b}Lv$$

is of importance. Likewise is of importance for geometry: *"the middle descent of two numbers"*. This is the number whose 2nd gradation is equal to the mutual power of the two numbers.

As was to be expected this is independent of the chosen base number. So if instead of e we take the base number a then, if we introduce this as index,

$$\underset{2}{\Delta}1(u, v)_a = \underset{2}{\Delta}1(u, v)_e\,.$$

This property corresponds thus to the property that the geometrical mean is independent of the chosen unity. Further on holds

$$\sqrt{uv} > \underset{2}{\Delta}1(u, v).$$

§ 5. The question arises whether in the equation

$$y = {}^{n}(x) = e^{p^n},$$

when y and p have an assumed value, there is always a value to be found for n. To show the possibility of this we have but to follow the reasoning of the algebra handbooks for the existence of logarithms. The immensurability of this number is, however, in general of a different order from that of the logarithms.

If n is called the second logarithm of y then this number is not definite until a definite value is indicated for p. The simplest assumption is $p = e$. There the e^{th} power of e shall be taken as *"base power."*

If now immensurable numbers are allowed as index of gradation the result is regarded as limit to which a descension (changing in a definite manner) of a gradation tends, and as definition of the 2nd logarithm follows:

"The 2nd logarithm of a number is the number indicating to which gradation the base power must be brought to furnish the given number."

The four principal properties are:

$$LL(u, v) = LLu + LLv; \quad LL^n(u) = n \,.\, LLu;$$

$$LL(u|v) = LLu - LLv; \quad LL\sqrt[n]{u} = \frac{1}{n} \,.\, LLu.$$

By the introduction of the notion *"mutual gradation"* of two numbers the difference between the two indirect operations which are deduced out of a gradation disappears. If we put

$$x = e^{e^p}; \quad y = e^{e^q},$$

then x and y can be interchanged in the following equation (their mutual gradation)

$$z = [x\,;y] = {}^{LLx}(y) = {}^{LLy}(x) = [y\,;x].$$

This form is equivalent to

$$LLz = LLx \,.\, LLy.$$

The mutual gradation as starting point leads to the investigation of the *"rootfield"*. For this the evolution is the elementary operation. The operation of the fifth rank following out of this shares the fundamental properties of the operations of lower rank.

Just as operations of an arbitrary higher rank may be introduced and may give rise to the investigation of definite groups of curves, operations of a rank lower than the first may also be introduced geometrical considerations may also be connected with these.

§ 6. The equation of the curves of the first gradation is found by elimination of n out of the following two

$$x = x_0 a^n \quad \text{and} \quad y = y_0 b^n.$$

For the construction of the points we must make use of different base numbers for abscissa and ordinate, unless the logarithms of the base numbers (a and b) used have a mensurable ratio.

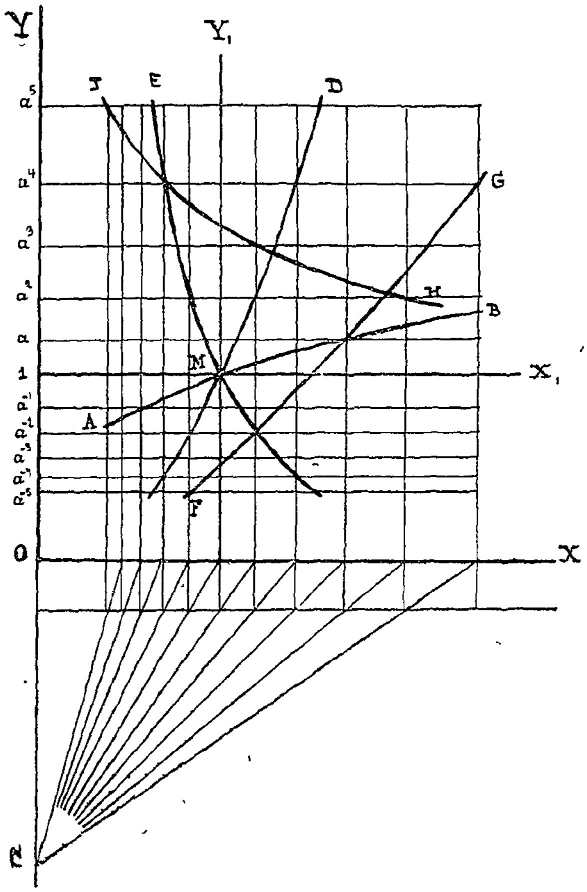

Fig. 1.

The lines of fig. 1 are drawn on this supposition; the above mentioned ratio is 3 for AB; $\frac{1}{2}$ for MD; $\frac{1}{2}$ for ME; 2 : 3 for FG and — 3 for HI. In the field of ratio the lines whose equation is:

$$\frac{y}{y_0} \Big| \, b = \frac{x}{x_0} \Big| \, a$$

are the simplest; they shall be called "*rationals*". They have the form of parabolae or hyperbolae according to their "director exponent" $\lambda = Lb : La$, being positive or negative. So

$$m = y_0 x_0^{-\frac{Lb}{La}}$$

represents the part cut off the line $x = 1$, counted to $M(1, 1)$. The lines $x = 1$ and $y = 1$ must be regarded as axes. They divide the positive field of ratio into four quadrants which are unequal, but which must be considered as equal in a rational sense (as will be evident later on); $x = 0$ and $y = 0$ correspond to $-\infty$ in the field of difference. Lines with equal director exponent are *"rational equidistant"*. A pencil of straight lines through O (rationals for which $\lambda = 1$) intersects two equidistant rationals in corresponding points. If the points of intersection of a selfsame rational are joined, then the corresponding figures are similar. So we might say that equidistant rationals are parallel in the smallest parts.

For $x_0 = 1$, $y_0 = 1$ the lines shall be called central rationals; they correspond to the right lines in the field of ratio drawn through the origin of coordinates. Rationals with mensurable director exponent find a continuation in one of the three other fields of ratio.

§ 7. The director exponent of the rationals has a simple geometrical meaning. Out of the equation

$$y = mx^{\lambda} \quad \text{or} \quad Ly = \lambda Lx + Lm$$

we find for it

$$\lambda = \frac{dLy}{dLx} = x : \frac{1}{y} \cdot \frac{dy}{dx},$$

that is *"the ratio of abscissa to subtangent."* For points of equidistant rationals with equal abscissa the tangents pass through one and the same point of OX (fig. 2); for given value of ordinates they intersect each other in one point of OY. Furthermore all rationals divide the rectangles of coordinates proportionally, so that the director exponent can be regarded as ratio of two integrals. Calling the parts J_x and J_y in which the line divides a rectangle (J) we find

$$\lambda = \int_0^y x dy : \int_0^x y dx = J_y : J_x .$$

As now the differences of the terms of a geometric series form again a geometric series that ratio also holds for area differences and differentials:

$$\lambda = \frac{\Delta J_y}{\Delta J_x} = \frac{dJ_y}{dJ_x}$$

For $\lambda > 0$ the strips lie outside each other; for $\lambda < 0$ overlapping takes place (fig. 2). Here

$$\lambda = tg\ AOB$$

represents the director exponent of the 3 equidistant rationals OP_1, OP_2, and OP_3.

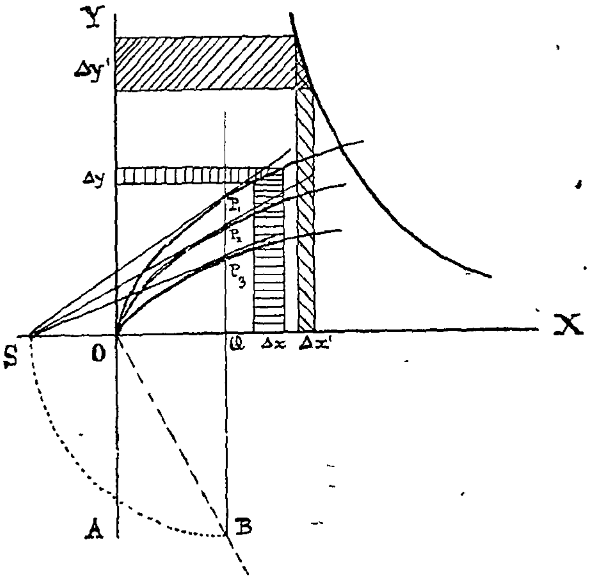

Fig. 2.

As we can also write

$$\frac{y_2}{y_1} \Big| \frac{x_2}{x_1} = e',$$

a rational is a line for which the mutual root of the ratio of coordinates of two points is constant.

§ 8. In the field of proportion coordinates and areas are quantities of the same exponential order. The area of the rectangle of coordinates determined by the point:

$$P_n(a^n, b^n) \quad \text{is} \quad J_n = (ab)^n.$$

The rational laid through this point and through P_m contains likewise P_{m+n}. The area of the rectangle of coordinates determined by this point is:

$$J_{m+n} = (ab)^{m+n} = J_m \times J_n,$$

so it is deduced out of a multiplication. Furthermore the coordinates can be regarded as areas namely of rectangles having the coordinates as base, the distance between OX_1 and OX (resp. OY_1 and OY) as height. A point is not yet determined by the area of a rectangle of coordinates only (equilateral hyperbola), an element for the direction must also be known. To a sum in the field of difference always corresponds a product here; likewise with the geometrical sum: $x + iy$ the geometrical product: $xy^i = \bar{J}$.

This determines the situation of the point entirely.

To the geometric sum:

$$\bar{r}_1 + \bar{r}_2 = x_1 + x_2) + i(y_1 + y_2)$$

corresponds here the geometric product·

$$\bar{J}_1 \times \bar{J}_2 = (x_1 x_2) \cdot (y_1 y_2)^i .$$

In fig. 3 this product has been constructed in two ways. The

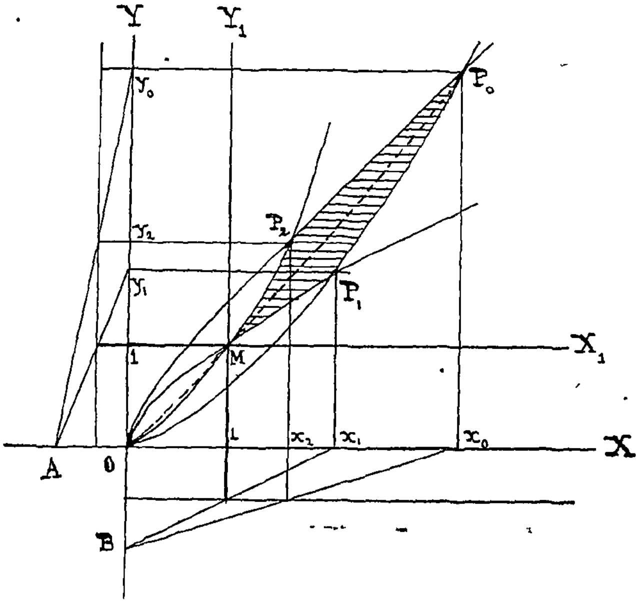

Fig. 3.

points $MP_1P_0P_2$ form a rational parallelogram whose rational through M and P_0 determines the diagonal. The rational distance between M and P_0 must therefóre be regarded as product of the distances between M and P_1 and M and P_2. So

$$xy^i \text{ and } xy^{-i}$$

are conjugate valu3s whose product and mutual power are real; viz.

$$x^2 \text{ and } {}^2(x) \,.\, {}^2(y).$$

When treating the analysis we shall define the "*rational angle*"; for a preliminary transition to polar coordinates we can put:

$$xy^i = r^{\cos\varphi + i \sin\varphi},$$

out of which follows for rational distance (radius) and *rat. tangent*:

$$r = \sqrt[2]{{}^2(x) \,.\, {}^2(y)} \quad ; \quad y | x = e^{tg\varphi}.$$

The first relation is the equation of the *rational* (or logarithmic) *circle*; the second equation represents the rational of the centre with director exponent $tg\,\varphi$. If we introduce for the rat. tangent $tg\,rat\,e$? and for the exponential quantity the letter u (meaning follows) we can then write (exchanging $tg\,rat$ by tr):

$$y|x = tr\,u.$$

According to definition the logarithmic circle is now the locus of the points possessing equal rational distance to a definite point (here M); this point is called centre. This has given an extension to the notion of ratio; this more general notion we shall call *"skew ratio"*. It is measured along the rational through P and M; the measure is the already mentioned radius r which is called the *"modulus"*, whilst to u the name of *"argument"* may be given.

§ 9. The properties of the rational goniometric functions are founded on the consideration of the logarithmic circle with radius e (represented in fig. 4). In the geometry of differences we arrive, gradually moving along a circle, from the value $+1$ to -1 and back; the absolute value of the difference does not change then. Here the logarithmic circle leads gradually from the value e to e^{-1}, where the absolute value of the "ratio" remains constant. Let us yet mention as particularity that in M the differential coefficient of the rationals is equal to the director exponent.

If we now allow the rational radius to revolve around M, then this coincides in 4 positions (P_0, P_3, P_6, and P_9) with the axes;

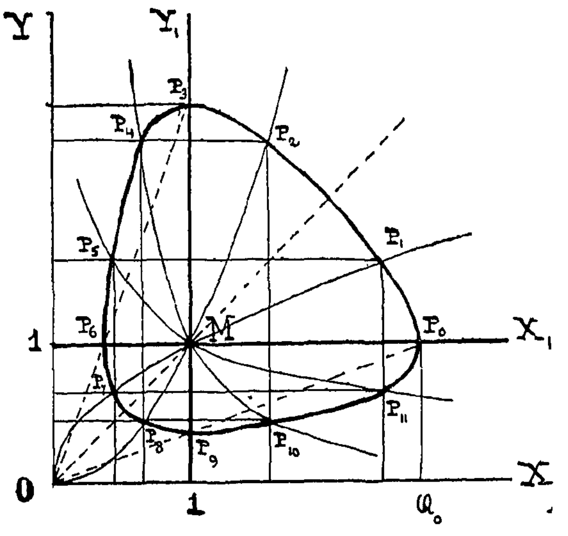

Fig. 4.

here the rational direction is identical to the direction of the line. Following GRASSMANN's notation for the field of ratio we might write:

$$P_0 : M = M : P_6 \text{ and } P_3 : M = M : P_9 .$$

To arrive now from the value

$$e = e^{(-1)^0} = {}^0\left(\frac{1}{e}\right)$$

at the value:

$$e^{-1} = e^{(-1)^1} = {}^1\left(\frac{1}{e}\right),$$

we must allow the index of gradation to vary continuously from 0 to 1. If by this change the index has reached the value $\frac{1}{2}$ (resp. $-\frac{1}{2}$), then the direction as well as the rational direction seen out of M is halved. Considered in this way the points P_3 and P_9 are therefore determined by

$$e^i \text{ and } e^{-i}.$$

In all intermediate positions the exponent is complex. For all points of a logarithmic circle the modulus is constant with variable argument. Here we have the widest definition of the notion "ratio". For two diametral points is

$$^{\frac{\varphi}{\pi}}\left(\frac{1}{e}\right) : 1 = 1 : {}^{\frac{\varphi}{\pi}+1}\left(\frac{1}{e}\right).$$

Also to the areas of the rectangle of coordinates (or the partial one separated by the rational) is applicable:

$$xy : 1 = 1 : x^{-1}y^{-1}.$$

If the radius of the log. circle is *one*, then by

$$^2(x)\ {}^2(y) = 1$$

is represented the point M, so that for all values of φ holds:

$$\overline{r} = 1^{(-1)^{\frac{\varphi}{\pi}}} = 1.$$

The directed area represented by $P_0 q_0 O1$ is therefore

$$J_0 = e \times 1^i ,$$

the one belonging to P_3 is

$$J_3 = e^i \times 1$$

§ 10. Although the curves of the field of ratio ought to be compared to the rationals, a good idea can be formed of their course by the comparison to straight lines. When tracing the inflectional points we must frequently strike into a particular path. As example the log.

circle will be investigated in its simplest position. Out of the equation :

$$^2(x) \,.\, ^2(y) = {}^2(r)$$

follows by differentiation

$$y' = -\frac{L\sqrt[x]{x}}{L\sqrt[y]{y}}; \quad y'' = \frac{y}{x^2 (Ly)^3}\{Lx \,.\, Ly \,.\, Lxy - (Lr)^2\}.$$

As y cannot become zero, x and Ly not ∞, the inflectional point condition is :

$$Lx \,.\, Ly \; Lxy = L^2 r.$$

If we now introduce the area of the rectangle of coordinates, then

$$xy = J,$$

furnishes, connected with the identity, where for convenience, sake we write $L^2 x$ for $(La)^2$, the condition :

$$L^3 J - L^2 r \,.\, LJ - 2L^2 r = 0.$$

There are six inflectional points when

$$r \geq e^{3\sqrt{3}}.$$

is satisfied.

The value of r, for which two more inflectional points exist, is deduced from the necessary condition for real values

$$L^2(xy) \geq 4Lx \,.\, Ly.$$

In connection with the inflectional point condition this becomes:

$$\frac{L^4 r}{L^2 x \,.\, L^2 y} \geq 4Lx \,.\, Ly,$$

from which ensues

$$Lx \,.\, Ly \leq \sqrt[3]{\frac{1}{4} L^4 r} \quad \text{and} \quad Lxy \geq \sqrt[3]{4L^2 r}..$$

For decreasing value of r the two inflectional points therefore coincide into a stationary point, if

$$L^2 r = L^2 xy - 2Lx \, Ly = \sqrt[3]{2L^4 r};$$

so that the radius of the *log.* circle is then:

$$r = e^{\sqrt{2}}.$$

§ 11. For the study of the general equation of the 2nd gradation comprising all logarithmic conics we should first have to treat the rational displacement and rational revolution round the axis. As this would lead us too far, I restrict myself to the logarithmic parabola and log. equilateral hyperbola.

The rational translation offers no great difficulties in this way; the logarithmic circle given by

$${}^{2}\left(\frac{x}{a}\right) \cdot {}^{2}\left(\frac{y}{b}\right) = {}^{2}(r)$$

will pass by means of the translation of the axes from $x = a, y = b$ into the above mentioned form The figure changes then according to the assumed notions, but as figure of the field of ratio it remains congruent to itself. The congruence in the field of ratio is equal to the similarity in the field of difference, when the translation takes place along straight lines through O; in general it takes place along arbitrary rationals, *so that the notion of similarity must be extended,* where the skew ratios come into consideration.

Whilst now in general the points of the logarithmic curves are constructed by means of a logarithmic line, they can be found in simple position by means of rectilinear constructions.

§ 12. If we draw in the field of ratio a series of lines whose equation is:

$$x_n = x_0 a^n ,$$

and then, with the aid of a pencil of rays through O the points $P_0, P_1, P_{-1}, P_2, P_{-2}, P_3, P_{-3} \ldots$ and likewise $P_0, Q_1, Q_{-1}, Q_2, Q_{-2} \ldots$ on the above mentioned lines, we then find ordinates given by means of:

$$y_n = y_0 a^{n^2}$$

By elimination of n out of these equations (regarding x_n and y_n as variable coordinates) we find the equation:

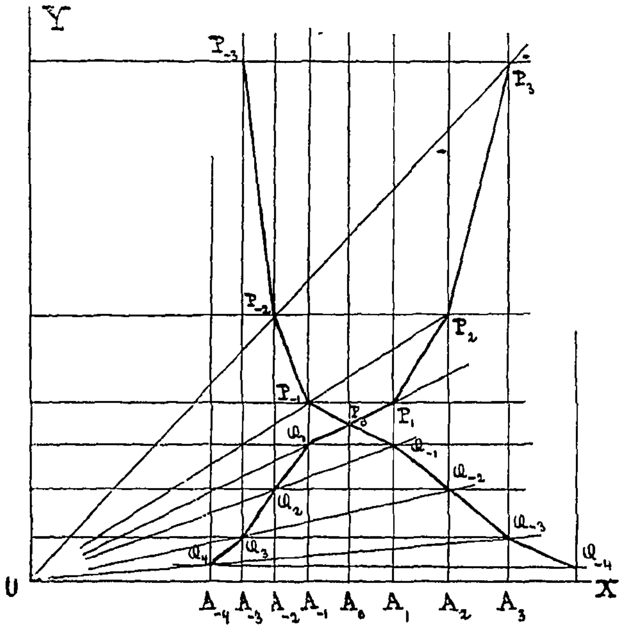

Fig. 5.

$$\frac{y}{y_0}, a = {}^2\left(\frac{x}{x_0}\right)$$

The points P form part of the logarithmic parabola: the points Q of the anti-parabola, having as parameter the reciprocal value of a.

For $x_0 = y_0 = 1$ we find the abscissa of the point of inflection out of

$$Lx = \frac{1}{4} La . \left(1 \pm \sqrt{1 - \frac{8}{La}}\right).$$

From this we can see that the anti-parabola has always two inflectional points; for the rest the condition for the existence of the inflectional points is for the former $r > e^8$. As locus of the inflectional points we find $y^2 = x : e$. At the interpolation the centre O displaces itself along the X-axis.

§ 13. Likewise the points of the logarithmic equilateral hyperbola

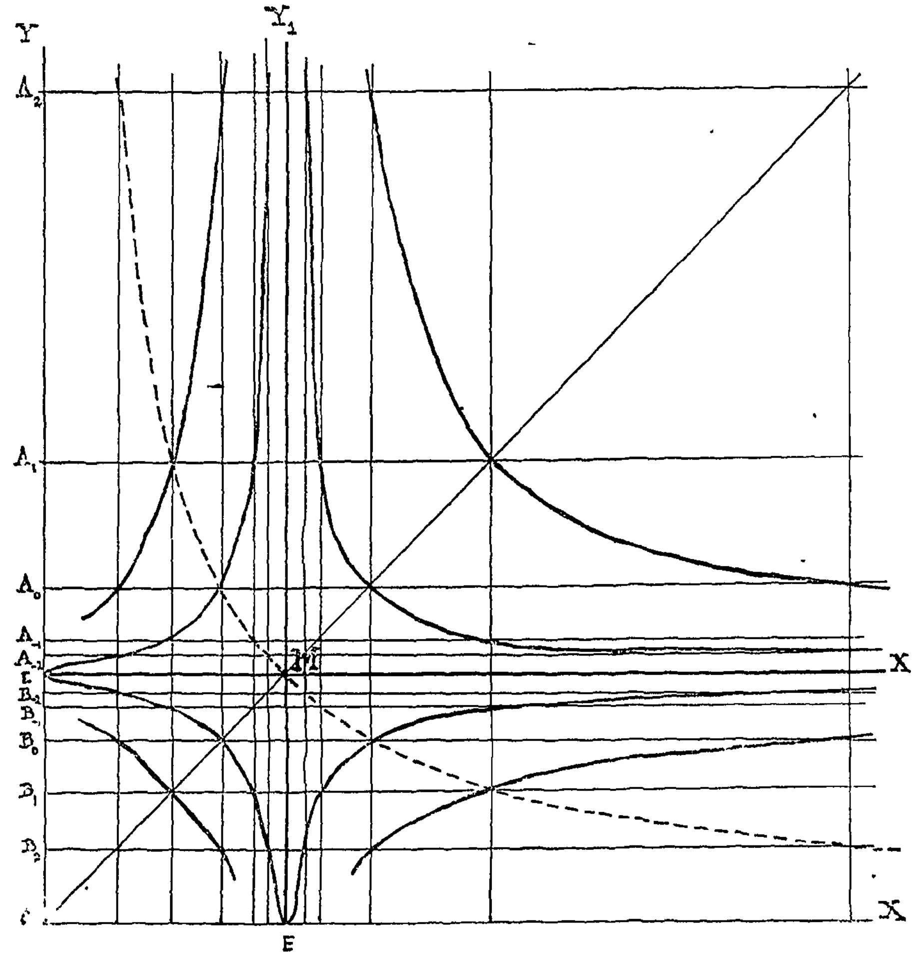

Fig. 6

and of the conjugate hyperbola can be immediately constructed, when OX_1 and OY_1 are the asymptotes; the equation is then

$$x\,,y = {}^2(a)^{\pm 1} \text{ or } Lx\,.\,Ly = \pm\, L^2\,a.$$

In this form it belongs to the curves of gradation; those are the simplest lines in the rootfield. In case the exponent be 2, the net of coordinates of the rootfield can be found by means of compasses and ruler. (The construction is not mentioned here). By OX_1 and OY_1 the positive field of ratio is divided into four rootfields; in these lie the branches of the logarithmic hyperbola and the conjugate one. In fig. 8 the above mentioned line is drawn, likewise the one with the base number a^2 (resp. a^{-2}). For the points A and B holds:

$$OA_n = a^{2^n} \quad ; \quad OB_n = a^{-2^n}.$$

The log. hyperbola is satisfied by: $x = a^{\pm 2^n}$; $y = a^{\pm 2^{-n}}$;

The conjugate one is satisfied by : $x = a^{\pm 2^n}$; $y = a^{\mp 2^{-n}}$.

Inflectional points appear in all root quadrants except the first. In the points E the line passes continually into the conjugate one. The locus of the inflectional points is $xy = e^{-2}$. Out of the fact that inflectional points are present follows already, that the curves touch the x- and y-axis in E; this is also to be seen algebraically in a roundabout way. So

$$y' = -\,(Ly^y : Lx^x)$$

seems to be indefinite in E. By the substitution :

$$x = p^z \quad , \quad y = \sqrt[z]{p},$$

in which p is a constant, the differential quotient becomes:

$$y' = -\,p^{\frac{1}{z} - z} : z^2.$$

For $z = -0$ that form is not any more decisive but taking

$$u = \frac{1}{z} - z$$

we find for $u = -\infty$

$$lim\, y' = lim\,[p^u : (\tfrac{1}{2}u^2 + 1 - \tfrac{1}{2}u\sqrt{(u^2+4)})] = 0.$$

§ 14. The investigation of the log. ellipse is again simplified by connecting p with two log. circles, whose radii are equal to the half axes. The simplest position is indicated by

$$^2(x|a)\,.\,^2(y|b) = e \quad \text{or} \quad (^aLx)^2 + (^bLy)^2 = 1.$$

The points of the line are points of intersection of the above mentioned circles with central rationals; the construction is based on the substitution :

ENGELMANN: "das Haar krümmt sich während der Vorwärtsbewegung stark concav, etwa wie ein Finger by starker Beugung".

Sometimes, however, I succeeded, after long seeking, in finding in a preparation a cilium standing alone, which was mostly much longer than the other cilia. Such a cilium made then sometimes very regular whip-like movements. The entire period of such a

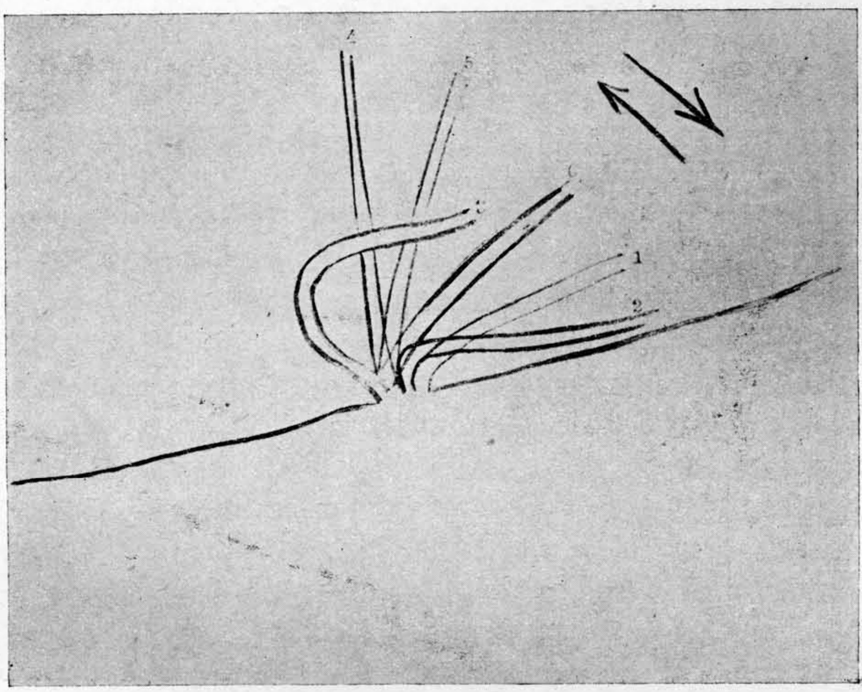

Fig. I.

movement is represented in Fig. I. The length of the whole period is $^{7}/_{28}$ sec., the characteristic difference between forward and backward movement is already visible. This was still more manifest in another film, made of another preparation. It shows the nature of the movement very distinctly. The number of pictures taken, amounted to 28 per sec.; the magnified pictures had been obtained with apochromatic objective 8 m.m. and projection-eye-piece No. 2, the film being about $^{3}/_{4}$ metres away from the eye-piece. The movements of the cilium had been made much slower by the refrigeration, and the periods took somewhat less than one second. By projecting a series of photos of this film at one place on paper, the drawings in Fig. II, III, IV, and V were obtained.

Let us now view Fig. II and IV. Although not alike they are of exactly the same type. The outstretched cilium, a small part of the basal part of which is visible, begins to bend, traversing whilst it is stretched out a circle segment, the basal part being approxi-

Naming the new coordinates (ξ, η) we find:

$$\xi = x^{\cos\varphi} \cdot y^{\sin\varphi} \quad ; \quad \eta = y^{\cos\varphi} : x^{\sin\varphi}$$

whilst for the rational distance to M holds:

$$\varrho = {}^2(\xi) \cdot {}^2(\eta) = {}^2(x) \cdot {}^2(y)$$

We must continually keep in view that rational distances are ratios; in the lack of a denominator we must assume 1 for it. If CP is an arbitrary rational through P, then — just as for rectilinear axes — the ratio of the parts into which $OB_1PA_{1\infty}$ is divided by this, is constant.

Let us call λ the director exponent of $M\Xi$, thus $-\frac{1}{\lambda}$ of MH; then PB_1 and PA_1 are given by:

$$y = px^{\lambda} \quad , \quad y = qx^{-\frac{1}{\lambda}}.$$

Let furthermore that exponent be μ for PC, then

$$J_\eta = xy\,(\mu - \lambda) : (1 + \mu)(1 + \lambda) \quad ; \quad J_\xi = xy\,(1 + \lambda\mu) : (1 + \mu)(1 - \lambda),$$

holds, when we put: $OB_1\,PCO = J_\eta$ and $OCPA_{1\infty}\ O = J_\xi$.

If λ and μ are moreover replaced by $tg\,\varphi$ and $tg\,\psi$, then follows from this:

$$J_\eta : J_\xi = tg\,(\psi - \varphi) \cdot tg\left(\frac{1}{4}\pi - \varphi\right),$$

which relation passes into the known one for $\varphi = 0$.

Finally we mention still that the connection between old coordinates and new ones can be given by *one* formula:

$$xy^i = \xi^{(-1)^{\frac{\varphi}{\pi}}} \cdot \eta^{(-1)^{\frac{\varphi}{\pi} + \frac{1}{2}}}.$$

With respect to the new axes the rational equation takes the form:

$$\eta = n\xi^{tg\,(\psi - \varphi)} \quad \text{(out of } y = mx^{\mu}\text{)}$$

in which we put for n:

$$m^{\cos\psi} \mid e^{\cos(\psi - \varphi)}.$$

Chemistry. — "*Some compounds of nitrates and sulphates*". By Prof F. A. H. Schreinemakers and A. Massink.

(Communicated in the meeting of February 24, 1912).

As has been known for a long time, several hydrated double salts can be obtained from solutions containing $NaNO_3$ and Na_2SO_4. It, therefore, was deemed important to investigate how the nitrates and sulphates of other metals, in the first place those of the alkali group would behave in this respect. The behaviour of NH_4NO_3 and $(NH_4)_2SO_4$