Mathematics. — *"On two linear congruences of quartic twisted curves of the first species"*. By Prof. JAN DE VRIES.

In a communication in the *Proceedings* of this Academy (Vol. XIV, p. 255) I have considered the congruence generated by the curve of intersection of two quadrics each of which belongs to a given pencil. This congruence is of the *first order* and of the *first class*.

In the following pages will be treated properties of two other congruences of quartic twisted curves also of the *first order* but of the *second* and *third* class.

1. We consider a pencil (φ^2) of quadrics φ^2 passing through the conics α^2, β^2, and a pencil (φ^3) of cubic surfaces φ^3 the base curve of which breaks up into α^2 and a twisted curve γ^7. By the intersection of each surface φ^2 and each surface φ^3 a congruence Γ of quartic twisted curves ϱ^4 of the first species is generated. Through an arbitrarily chosen point passes *one* surface of both pencils and therefore *one* ϱ^4; so Γ is linear or of the *first order.*

Through any point C of γ^7 passes *one* surface γ^2 of (φ^2) containing ∞^1 curves ϱ^4 passing through C; therefore γ^7 may be called a *singular curve, C a singular point of the second order.*

Also β^2 is *singular*; through any of its points B passes *one* surface β^3 containing all the curves ϱ^4 cutting B^2 in B; so B is a *singular point of the third order.*

Finally α^2 is *singular* too. For a φ^2 and a φ^3 touching each other in a point A of α^2 have a ϱ^4 passing through A in common. By making to correspond to each surface φ^2 the surface φ^3 touching it in A, the pencils, brought thereby in projective correspondence, generate a surface α^5 with α^2 as nodal curve and A as triple point, containing ∞^1 curves ϱ^4 cutting α^2 in A; so A and α^2 are *singular of order five.*

2. On an arbitrary straight line l the two pencils determine two involutions I^2, I^3; as these involutions admit two common couples, l is bisecant of two ϱ^4 and therefore Γ a congruence of the *second class.*

Any generator s of one of the φ^2 is cut by (φ^3) in an involution I^2 and therefore a *singular bisecant.* All these lines s form the congruence (2, 4) of the lines cutting α^2 and β^2 in two different points.

The planes α, β bearing α^2, β^2 form together a surface φ^2 intersected by any φ^3 in the combination of a line a with a cubic curve of β having with that line one point in common. Every line of β is a *singular trisecant*; for it has three points in common with each of the ∞^1 degenerated curves ϱ^4 determined by (φ^3) and (α, β).

A straight line t of any φ^3 not meeting α^2 is cut by (φ^2) in an I^2 and therefore a *singular bisecant*. It has three points in common with γ^7 and intersects α on the line of intersection a of φ^3 and α. This line a is met by ten lines of φ^3; all these lines are trisecants of γ^7. Moreover a is cut in its point of intersection with γ^7 in five other trisecants. So the singular bisecants t form a *ruled surface* of *order fifteen* with γ^7 as *fivefold curve*.

3. Let x be the order of the surface A formed by the curves ϱ^4 meeting the line l. Then the surfaces A and A' corresponding to l and l' have in the first place the x curves ϱ^4 in common, meeting l and l'. Through any other point of their intersection passes a ϱ^4 meeting l and an other ϱ^4 meeting l'; therefore the residual intersection can only be composed of singular curves. As l cuts the surfaces α^5, β^3, and γ^2, corresponding according art. 1 to points A, B, C of α^2, β^2, γ^7, into 5, 3, and 2 points respectively, α^2, β^2, γ^7 are respectively fivefold, threefold and double curves on A. So for the determination of x we find the relation [1] $x^2 = 4x + 5^2 \times 2 + 3^2 \times 2 + 2^2 \times 7$ giving $x = 12$.

We can verify this result as follows. If we make to correspond to each other any two surfaces φ^2 and φ^3 meeting one another on l we generate a correspondence (3,2) between them; thereby the points of an other line m are arranged in a (6,6)-correspondence, any coincidence of which is a point of intersection of two surfaces φ^2 and φ^3 having a point of l in common, and therefore a point of an ϱ^4 resting on l. Therefore A is of *order twelve*.

Besides the three multiple curves above mentioned A admits still as double curves the two ϱ^4 with l as bisecant.

4. On a plane φ the congruence Γ determines a quadruple involution. If the point L describes the line l of φ the three points joined to L by a quadruple will generate a curve λ^{11} determined by the surface A^{12} corresponding to l (art. 3).

Among the points of intersection of l and λ^{11} the two couples of points of the ϱ^4 for which l is a bisecant present themselves; the remaining seven are points of coincidence, i. e. points of contact of φ with curves ϱ^4.

1) The surfaces A have been used in a similar way by E. Veneroni, *Sopra alcuni sistemi di cubiche gobbe* (Rend. Palermo, XVI, 210).

The *curve of coincidences* φ^7, locus of these points, is obviously also the locus of the points of contact of the curves φ^2 and φ^3 of the pencils determined by the pencils (φ^2) and (φ^3). These pencils have two base points A_1, A_2 in common, whilst (φ^2) admits still the base points B_1, B_2 and (φ^3) the base points C_k $(k = 1, 2, .. 7)$. Through B_k passes one curve φ_k^3 containing ∞^1 quadruples with the common point B_k; one of these groups has a double point in B_k, from which ensues that B_k lies on φ^7. In an analogous way C_k belongs to ∞^1 quadruples lying on a φ_k^2 and is therefore likewise a point of φ^7. Finally A_k is a triple point of φ^7 and belongs to ∞^1 quadruples situated on the curve $\varphi^5{}_k$ in which α_k^5 (art. 1) meets φ. So the points of contact of the plane φ with curves ϱ^4 lie on a curve of *order seven* with *two threefold points.*

5. We will now consider the *branch curve,* i. e. the locus of the couples of points completing the coincidences to groups of the quadruple involution.

Any curve φ^2 is touched in the double points of the I^4 determined by (φ^3) on φ^2 by six curves φ^3. By a quadratic transformation with A_1, A_2, B_1 as fundamental points we find that φ^3 is touched by eight conics φ^2. So by considering as conjugate to each other two curves φ^2, φ^3 touching one another a correspondence (8, 6), is generated determining on any line m a correspondence (16, 18). The curve of order 34 generated by the two pencils breaks up into the curve of coincidences φ^7 counted twice and the branch curve φ^{20}.

By drawing m successively through A_k, B_k, C_k we find respectively an (8, 12), an (8,18), a (16, 12). Taking into account the known multiplicity of these points on φ^7 we come to the result that φ^{20} passes *eight* times through A_k, *six* times through B_k, *four* times through C_k.

6 By using once more a quadratic transformation with the fundamental points A_1, A_2, B_1 the curve of coincidences φ^7 is transformed into a curve π^7 having likewise triple points in A_1, A_2 and passing through B_1 and through the points B'_2, C'_k corresponding to B_2, C_k. The pencil (φ^2) passes into the pencil of lines with vertex B'_2, whilst the pencil (φ^3) is transformed into a pencil of quartic curves ψ^4, the base of which consists of double points in A_1, A_2 and the points B_1, C'_k. Obviously π^7 is the polar curve of B'_2 with respect to (ψ^4), i. e. the locus of the points of contact of the curves ψ^4 with tangents passing through B'_2. The class of π^7 is 30; to the 28 tangents concurring in B'_2 belong the lines through A_k, B_1, C'_k (in each of these points a curve ψ^4 is touched by the corresponding

line through B'_2). According to the definition of the polar curve the 18 other tangents are inflectional tangents of as many curves ψ^4 and therefore curves ψ^4 corresponding to conics φ^2 osculated by a φ^3. So the quadruple involution contains 18 groups with three coinciding points. In other words, *each plane is osculated by eighteen curves* ϱ^4.

The curves φ^7 and φ^{20} will touch each other in 18 points. In the base points A, B, C they have $2\times 3\times 8+2\times 6+7\times 4=88$ points in common. The remaining 16 points of intersection form 8 couples of coincidence and belong to eight quadruples. Otherwise, *each plane is bitangent plane for eight curves* ϱ^4.

7. Let us now consider the bisecants of the curves ϱ^4 through a point P and the surface Σ, locus of the couples of points S in which they meet these curves [1]). The curve ϱ^4_P through P is projected from P by a cubic cone σ^3, the edges of which touch Σ in P; so P is a threefold point of Σ. Each line through P contains moreover two couples S; so Σ is a surface of order seven.

On each edge of σ^3 still lies a second couple of points S; so these couples generate a curve σ^6 of order six. As σ^3 and Σ^7 with the common curves ϱ^4_P and σ^6 can have furthermore only straight lines in common, we find that *eleven singular bisecants* pass through P. To these belong the two common transversals s of α^2 and β^2 (art. 2); on the other ones (φ^2) and (ρ^3) must determine the same l^2, therefore they must meet γ^7. So there is still a congruence of *order nine* of *singular bisecants* with γ^7 as *director line.*

An analogous consideration furnishes with respect to a congruence of twisted curves ϱ^n of *order o* and *class c* the result that a point P bears in general $o\,(n-1)^2+(n-2)$ singular bisecants [2]). So this number is independent of the class.

However we must remark that this consideration does not hold for $o=1$, $c>1$ and $n=2$; for then the cone projecting the conic through P out of P becomes a plane and the curve of intersection lying in this plane will also pass through P. But then the singular bisecants are lines of the surface enveloped by the planes of the conics and in general they form no congruence.

The surface Σ^7_P contains the singular curve γ^7, for through each point C passes *one* ϱ^4 cutting CP a second time. However the singular

[1]) For bilinear congruences the surfaces Σ have already been used by VENERONI (l.c. p. 212).

[2]) For $o=1$, $c=1$ this number is n^2-n-1. It was found in an analogous way by VENERONI (l.c. p. 212).

curves α^2 and β^2 are nodal curves of Σ^7_P, each of their points A (B) bearing two curves ϱ^4 with AP(BP) as bisecant.

Any ϱ^4 has in common with Σ^7_P the two couples of points on its two bisecants through P. Every other point of intersection is evidently singular. As the 8 points of intersection with α^2 and β^2 have to be counted for 16, these conics being nodal curves of Σ^7_P, ϱ^4 must have eight points in common with γ^7. *So the curves ϱ^4 of Γ have eight points in common with γ^7, four points with α^2 and with β^2.*

8. The points of contact of the tangents through P to the surfaces φ^2 and φ^3 lie on the polar surfaces Π^2 and Π^5 of P with respect to these pencils. In a point A of α^2 a φ^3 is touched in such a manner by a φ^2 that the tangent plane passes through P; but in general the tangent in A to the common ϱ^4 does not pass through P. The surfaces Π^2 and Π^5, touching each other along α^2 have still a curve π^{11} in common, which passes through P and through the points of contact of tangents through P to curves ϱ^4. From this ensues that *the tangents to the curves ϱ^4 of Γ form a complex of order ten.*

Any plane through P cuts Σ^7_P in a curve σ^7 with a triple point in P and four double points on α^2 and β^2, and therefore of class 28. Of the 22 tangents through P ten belong to the complex of the tangents of Γ. The remaining 12 coincide in pairs to double tangents of σ^7, i.e. in lines on which the involutions determined by (φ^3) and (φ^2) have two coincided pairs in common, so they are cut by only *one* ϱ^4 in two points. Evidently the lines possessing this property form a *complex of order six.*

9. The surface Λ^{12} corresponding to a line l (§ 3) is cut by a plane according to a curve λ^{12} passing respectively 5 times, 3 times, 2 times through the base points A_k, B_k, C_k (art. 4). So it has in common with the curve of coincidences φ^7 in the base $2\times5\times3+2\times3+7\times2=50$ points; so there are 34 points of contact of φ with a curve ϱ^4 cutting l; in other words, the curves ϱ^4 touching a plane form a *surface of order* 34.[1]

The surface α^5 corresponding to a point A of α^2 intersects φ in a curve, passing twice through A_k, and once through B_k, C_k, and having therefore besides these points still $5\times7-2\times2\times3-2-7=14$ points in common with φ^7. So the surface φ^{31} mentioned just now meets 14 times the conic α^2; this agrees with the fact

[1]) This shows once more that the branch curve is a φ^{20}, for the surface under consideration has with φ in common the curve of coincidences counted twice.

that the curves φ^7 and φ^{20}, common to φ and φ^{31} pass together 14 times through A_k [1]).

In the same manner can be shown that β^2 is an eightfold curve and γ^7 a sixfold curve of φ^{34}.

The curve of coincidences ψ^7 of a plane ψ has in its points of intersection with $\alpha^2, \beta^2, \gamma^7$ evidently $2 \times 14 \times 3 + 2 \times 8 + 7 \times 6 = 142$ points in common with φ^{34}; in each of the remaining 96 points a ϱ^4 touches ψ. So *two* arbitrary *planes* are *touched* by 96 curves ϱ^4.

10. Let us now consider the projective nets of quadrics represented by

$$\lambda a^2_x + \mu b^2_x + \nu c^2_x = 0 \text{ and } \lambda f^2_x + \mu g^2_x + \nu h^2_x = 0.$$

The congruence Γ of the quartic twisted curves ϱ^4, forming intersections of corresponding surfaces, is *linear*. For through any point Y passes the curve determined by the relations

$$\lambda a^2_y + \mu b^2_y + \nu c^2_y = 0 \text{ and } \lambda f^2_y + \mu g^2_y + \nu h^2_y = 0.$$

But under the condition

$$M = \begin{vmatrix} a^2_y & b^2_y & c^2_y \\ f^2_y & g^2_y & h^2_y \end{vmatrix} = 0$$

Y is a singular point bearing ∞^1 curves ϱ^4.

As $\begin{vmatrix} a & b \\ f & g \end{vmatrix} = 0$ and $\begin{vmatrix} a & c \\ f & h \end{vmatrix} = 0$ determine two quartic surfaces having in common the curve $a = 0, f = 0$ not lying on $\begin{vmatrix} b & c \\ g & h \end{vmatrix} = 0$, *the locus of the singular points is a twisted curve* σ^{12} passing through the 16 base points of the two nets.

Evidently σ^{12} is the partial intersection of the quartic surfaces determined by

$$\begin{vmatrix} \alpha & \beta & \gamma \\ a^2_y & b^2_y & c^2_y \\ f^2_y & g^2_y & h^2_y \end{vmatrix} = 0, \quad \begin{vmatrix} \alpha' & \beta' & \gamma' \\ a^2_y & b^2_y & c^2_y \\ f^2_y & g^2_y & h^2_y \end{vmatrix} = 0.$$

For these surfaces have in common the curve of the congruence Γ determined by

$$\Delta \equiv \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ a^2_y & b^2_y & c^2_y \end{vmatrix} = 0 \text{ and } \Delta \equiv \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ f^2_y & g^2_y & h^2_y \end{vmatrix} = 0,$$

and the curve σ^{12} denoted by $M = 0$.

[1]) In connection with this it must be remarked that in art. 8 of my communication quoted above an error has slipped in. The surface A^{24} mentioned there passes six times through the curves β^4 and β'^4. So only 72 curves ρ^4 touch two planes.

Out of $\Delta = 0$ and $M = 0$ follows $\Delta' = 0$ or $a = b = c = 0$; out of $\Delta' = 0$ and $M = 0$ follows $\Delta = 0$ or $f = g = h = 0$. So $\Delta = 0$ and σ^{12} have, besides the 8 base points of the first net, 16 points in common lying at the same time on $\Delta' = 0$. Consequently each curve ϱ^4 of Γ cuts the *singular curve* σ^{12} in *sixteen* points.

11. On the line UV represented by $x_k = u_k + \varrho v_k$ the nets $\sum\limits_3 \lambda a^2_x = 0$ and $\sum\limits_3 \lambda f^2_x = 0$ determine the pairs of points

$$\sum_3 \lambda (a^2_u + 2\varrho a_u a_v + \varrho^2 a^2_v) = 0. \quad \sum_3 \lambda (f^2_u + 2\varrho f_u f_v + \varrho^2 f^2_v) = 0.$$

These two equations will determine the same pair, if the relations

$$\sum_3 \lambda a^2_u = \sigma \sum_3 \lambda f^2_u, \quad \sum_3 \lambda a_u a_v = \sigma \sum_3 \lambda f_u f_v, \quad \sum_3 \lambda a^2_v = \sigma \sum_3 \lambda f^2_v$$

hold. Eliminating λ, μ, ν we find the relation

$$\begin{vmatrix} a^2_u - \sigma f^2_u & b^2_u - \sigma g^2_u & c^2_u - \sigma h^2_u \\ a_u a_v - \sigma f_u f_v & b_u b_v - \sigma g_u g_v & c_u c_v - \sigma h_u h_v \\ a^2_v - \sigma f^2_v & b^2_v - \sigma g^2_v & c^2_v - \sigma h^2_v \end{vmatrix} = 0,$$

which proves that UV is bisecant of three curves ϱ^4; so *the congruence is of the third class.*

12. Through a singular point S pass ∞^1 curves ϱ^4 lying on a surface Σ^4, generated by two projective pencils of quadrics and having therefore a node in S.

The surface A formed by the ϱ^4 having a point in common with a line l passes four times through σ^{12}, for l cuts Σ^4 in four points. So the order x of A can therefore (§ 3) be deduced from the equation $x^2 = 4x + 192$ giving $x = 16$. So *two lines are met by sixteen* ϱ^4.

The ϱ^4 cutting l meet any plane φ through l in three points more lying on a curve λ^{15}; among the points common to l and λ^{15} occur the three couples of points which l has in common with the curves ϱ^4 of which it is a bisecant; in each of the remaining 9 points φ is touched by a curve ϱ^4. So *a plane is tangential plane for nine curves* ϱ^4.

The quadruple involution which the curves ϱ^4 determine in φ has therefore a curve of coincidences φ^9 of order nine. This curve is at the same time the locus of the double points of the net of quartic curves determined by

$$\begin{vmatrix} \alpha & \beta & \gamma \\ a_x^2 & b_x^2 & c_x^2 \\ f_x^2 & g_x^2 & h_x^2 \end{vmatrix} = 0.$$

As this net admits 12 base points (points of intersection of φ and σ^{12}) φ^9 has *twelve double points.*