Metereology. — *"On the Influence of the Earth's Rotation on pure Drift-Currents".* By Dr. D. F. TOLLENAAR. (Communicated by Dr. J. P. VAN DER STOK).

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The water of a laterally unbounded sea, initially at rest, is supposed to be suddenly subjected to the influence of a wind of constant magnitude and direction. Assuming a left-handed axial system of which the Z-axis points vertically downwards, the current-components u and v will have to satisfy the differential equations:

$$\frac{\partial u}{\partial t} = av + b\frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial v}{\partial t} = -au + b\frac{\partial^2 v}{\partial z^2}$$

in which

$a = 2n \sin \varphi$.

$n =$ angular velocity of the earth $= 7.3 \times 10^{-5}$.

$\varphi =$ geographical latitude.

$b = \frac{\mu}{\varrho}$.

$\mu =$ viscosity coefficient of water

$\varrho =$ density.

Starting with the assumption of a sea of infinite depth, we have the conditional equations: $u = v = 0$ for $t = 0$ and $z = \infty$. The assumption of a constant wind, whose direction we take along the Y-axis, may be expressed by the equations:

$$\left(\frac{\partial u}{\partial z}\right)_{z=0} = +\frac{k}{\mu}u_0$$

$$\left(\frac{\partial v}{\partial z}\right)_{z=0} = -\frac{k}{\mu}(V - v_0),$$

if V represents the magnitude of the wind and k the external viscosity coefficient.

The differential equations are solved in the simplest way by introducing a new variable $u + iv = w$, by which substitution they are transformed into:

$$\frac{\partial w}{\partial t} = -iaw + b\frac{\partial^2 w}{\partial z^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Putting likewise $iV = W$ the conditional equations become: $w = 0$ for $t = 0$ and for $z = \infty$;

$$\left(\frac{\partial w}{\partial z}\right)_{z=0} = -\frac{k}{\mu}\,(W - w_0)\,.$$

Putting $w' = we^{\iota a t}$, the equations change to

$$\frac{\partial w'}{\partial t} = b\,\frac{\partial^2 w'}{\partial t^2}, \qquad (2)$$

$$w' = 0 \text{ for } t = 0 \text{ and } z = \infty,$$

$$\left(\frac{\partial w'}{\partial z}\right)_{z=0} = -c\,(We^{\iota a t} - w'_0),$$

$\frac{k}{\mu}$ being $= c$.

If we now introduce a function φ, connected with w' by

$$\varphi = -w' + \frac{1}{c}\frac{\partial w'}{\partial z} \qquad (3)$$

φ will also have to satisfy the differential equation (2) and secondly we must have

$$\varphi_0 = -w'_0 - (We^{\iota a t} - w'_0) = -We^{\iota a t}.$$

Hence φ satisfies the differential equation $\frac{\partial \varphi}{\partial t} = b\frac{\partial^2 \varphi}{\partial t^2}$ and φ_0 is a function of t.

The solution of this equation is known from the theory of heat-conduction and is

$$\varphi = -\frac{2}{\sqrt{\pi}}\int_{\frac{z}{2\sqrt{bt}}}^{\infty} We^{-\beta^2 + \iota a\left(t - \frac{z^2}{4b\beta^2}\right)}\,d\beta\,, \qquad (4)^{1)}$$

Here W is constant by assumption; if W, the magnitude of the wind, were itself a function $W(t)$ of t, we should have under the sign of integration instead of W: $W\left(t - \frac{z^2}{4b\beta^2}\right)$. This remark will be useful later on.

(3) gives

$$w' = -ce^{cz}\int_z^{\infty} e^{-c\lambda}\,\varphi(\lambda)\,d\lambda$$

and finally

$$w = \frac{2ce^{cz}}{\sqrt{\pi}}\,W\int_z^{\infty} e^{-c\lambda}\,d\lambda \int_{\frac{\lambda}{2\sqrt{bt}}}^{\infty} e^{-\beta^2 - \frac{\iota a \lambda^2}{4b\beta^2}}\,d\beta \qquad (5)$$

1) RIEMANN—WEBER. Partielle Different gleich. II p. 106.

The form of solution (5) gives rise to the following remarks: One would feel inclined, since (5) holds for a sea of infinite depth, to suppose that the solution for a sea of finite depth h could be given in the same form, the upper limit of integration for λ being h instead of ∞, since the differential equations and other conditional equations remain the same in this case. This conclusion would be wrong, however; the modified integral would be found no longer to satisfy the differential equation. The reason of this is found in the circumstance that although the function φ of eq. (3) must satisfy the same diff. eq. as w', this by no means involves that w' will satisfy this diff eq. together with φ. On closer investigation this appears to be the case only when the upper limit is infinite.

By introducing the new variable ζ and putting $\beta = \frac{\lambda}{2\sqrt{b\zeta}}$ the solution (5) may be transformed into

$$w = \frac{c e^{cz}}{2\sqrt{b\pi}} W \int_z^\infty e^{-c\lambda} d\lambda \int_0^t \lambda \frac{e^{-\frac{\lambda^2}{4b\zeta} - i a \zeta}}{\zeta^{3/2}} d\zeta$$

Now putting $\lambda = 2b\zeta\left(\frac{\lambda}{2b\zeta} + c\right) - 2cb\zeta$, we obtain:

$$w = cW\sqrt{\frac{b}{\pi}} \int_0^t \frac{e^{-\frac{z^2}{4b\zeta} - i a \zeta}}{\zeta^{1/2}} d\zeta - c^2 W \sqrt{\frac{b}{\pi}} \int_0^{\zeta = t} \int_0^{\lambda = \infty} \frac{e^{-\frac{(\lambda + z)^2}{4b\zeta} - c\lambda - i a \zeta}}{\zeta^{1/2}} d\lambda \, d\zeta \quad (6)$$

The solution for w was reduced to the form (6) in order to render comparison easier with the solution, given for the same problem by FREDHOLM [1]) and which for various reasons seemed to me to be theoretically inaccurate. For the assumption of a constant wind along the Y-axis is expressed by FREDHOLM in the conditional equations $\left(\frac{\partial u}{\partial z}\right)_{z=0} = 0$, $\left(\frac{\partial v}{\partial z}\right)_{z=0} = -\frac{T}{\mu}$, T being taken constant. To this form of the conditional equation he was probably led, because EKMAN had already found the solution of the stationary problem (in which the first members of the diff. eq. are put zero) and had used the same conditional equations as expressing a constant wind. Now these are right indeed in the stationary problem. For it is always possible to choose the axial system such that the X-axis has a direction along which the constant wind-velocity and the in this case constant surface-current velocity have equal components. The Y-axis then lies in the

[1]) EKMAN. On the influence of the earth's rotation on ocean currents. Archiv for Matematik, etc. p. 16. 1905.

direction of the relative motion of the wind with respect to the water of the surface. It will presently be shown that in this case also a similar choice of axes is not to be recommended. Still it is theoretically the correct expression for the existence of a wind of constant direction and magnitude. But in the non-stationary problem matters are different. Here also it is possible to choose at a certain moment the axes such that the components of wind and surface current along the X-axis are equal and hence the conditional equation $\left(\frac{\partial u}{\partial z}\right)_{z=0}=0$ holds, but this direction would have to rotate with the time, since now the current velocity is variable. Hence the conditional equations only express the condition of a constant wind with a variable system of axes. Since, however, the diff. equations only hold for a set of axes fixed in the earth, FREDHOLM's solution cannot be theoretically correct. FREDHOLM indeed finds a solution for which the u and v at the surface are, as we might expect, functions of t. Now if we bear in mind that his conditional equations have no other meaning than that the wind component in the X-direction is always equal to that of the surface current and that the difference of these components along the Y-axis remains constant, it follows at once that his conditional equations really presuppose a wind which is the same function of the time as the surface current.

Consequently, if FREDHOLM finds

$$w=\frac{iT}{\mu}\sqrt{\frac{b}{\pi}}\int_0^t\frac{e^{-\frac{z^2}{4b\zeta}-ia\zeta}}{\zeta^{1/2}}\,d\zeta \quad \ldots \quad (7)$$

this solution can only be correct for the assumption of a wind, the components of which must be given by

$$U=\frac{T}{\mu}\sqrt{\frac{b}{\pi}}\int_0^t\frac{\sin a\zeta}{\zeta^{1/2}}\,d\zeta$$

$$V=\frac{T}{k}+\frac{T}{\mu}\sqrt{\frac{b}{\pi}}\int_0^t\frac{\cos a\zeta}{\zeta^{1/2}}\,d\zeta.$$

Comparing FREDHOLM's solution (7) with the solution (6) found by us, and remembering that $W=iV$ and $c=\frac{k}{\mu}$. it will be seen that the first part of (6) corresponds with FREDHOLM's solution and the second part ought to contain the theoretically required correction.

Now the following reasoning suggests itself: If in formula (6) we suppose W not to be constant, but some function of t (which would modify this formula as explained under (4)) and if for this special function of t we take the above given expressions for U and V, which according to me contain FREDHOLM's equations of condition implicitly, we must obtain the solution of FREDHOLM. After some réductions of the integrals this appears to be indeed the case.

In order to judge of the practical value of the correction, we write (6) in this way:

$$w = cW\sqrt{\frac{b}{\pi}}\int_0^t \frac{e^{-\frac{z^2}{4b\zeta}-ia\zeta}}{\zeta^{1/2}}\left(1 - c\int_0^\infty e^{-\frac{\lambda^2+2z\lambda}{4b\zeta}-c\lambda}\,d\lambda\right)d\zeta.$$

Introducing the variable $\lambda' = \frac{\lambda}{2\sqrt{b\zeta}} + c\sqrt{b\zeta}$ we have for the surface current

$$w_0 = cW\sqrt{\frac{b}{\pi}}\int_0^t \frac{e^{-ia\zeta}}{\zeta^{1/2}}\left(1 - 2c\sqrt{b\zeta}\,e^{c^2b\zeta}\int_{c\sqrt{b\zeta}}^\infty e^{-\lambda'^2}\,d\lambda'\right)d\zeta.$$

Now $2xe^{x^2}\int_x^\infty e^{-\lambda'^2}d\lambda'$ is a function, the value of which is zero at $x = 0$, then rapidly rises and amounts already to 0.91 at $x = 2$ and then slowly approaches the value 1 for $x = \infty$. At $x = 0.06$, however, its value is only $^1/_{10}$. Now if t is such that the $c\sqrt{b\zeta}$ belonging to this limit does not exceed 0.06, the correction may practically be neglected. The data for a determination of $c\sqrt{b\zeta}$ are very scarce, but still it is possible to state something about the order of magnitude. This turns out, as will be explained presently, to be of the order 10^{-7}. Hence the correction may be neglected in practice if $c\sqrt{bt} < 0.06$ or $t < 4.10^4$ i.e. about eleven hours. Thus FREDHOLM's formula, although theoretically inaccurate, is practically serviceable for studying the development of the current. If in formula (6) without a correcting member we put W not constant but a function of t, the solution is, as we saw,

$$w = c\sqrt{\frac{b}{\pi}}\int_0^t W(t-\zeta)\,\frac{e^{-\frac{z^2}{4b\zeta}-ia\zeta}}{\zeta^{1/2}}\,d\zeta,$$

which formula would enable us to follow the development of the current if we assumed a time-function for the wind, which would then e.g. gradually increase or decrease.

Putting $t = \infty$ in formula (5) we must find the formulae for the stationary case. Introducing the quantity $a' = \sqrt{\frac{a}{2b}}$ we find:

$$w = \frac{cW}{c + (1 + i)\,a'}\, e^{-(1+i)a'z} \,,$$

from which follows for the current components

$$u = \frac{cV}{\sqrt{a'^2 + (a'+c)^2}}\, \sin\,(a'z + \xi)\, e^{-a'z},$$

where $tg\, \xi = \dfrac{a'}{a'+c}$

$$v = \frac{cV}{\sqrt{a'^2 + (a'+c)^2}}\, \cos\,(a'z + \xi)\, e^{-a'z}$$

Hence for the surface current

$$u_0 = \frac{cV}{\sqrt{a'^2 + (a'+c)^2}}\, \sin\, \xi = \frac{a'cV}{a'^2 + (a'+c)^2}\,, \qquad \frac{u_0}{v_0} = tg\xi$$

$$v_0 = \qquad\qquad = \frac{(a'+c)\,cV}{a'^2 + (a'+c)^2}\,,$$

$$s_0 = \frac{cV}{\sqrt{a'^2 + (a'+c)^2}}\,.$$

If in the accompanying figure OV represents the magnitude of the wind, OS_0 the surface current, $\angle\, VOS_0 = \xi$. It is easily found that

$$S_0V = \frac{a'\sqrt{2}}{\sqrt{a'^2 + (a'+c)^2}}\, V \quad \text{and} \quad \sin\, S_0VO = \frac{c}{\sqrt{a'^2 + (a'+c)^2}}\, {}^1/_2\sqrt{2}$$

This shows that the terminal point S_0 lies on a circular arc with chord OV and apical angle 135°. S_0V now represents the relative velocity of the wind which EKMAN took for Y-axis in his solution of the stationary case. Referring the result to these axes and putting $S_0V = V'$ we have.

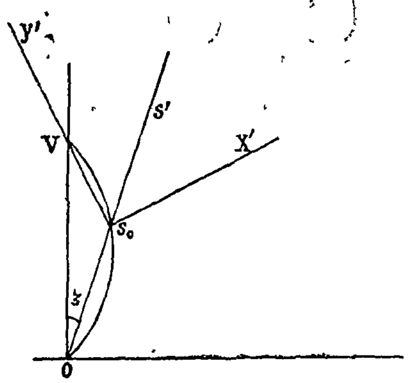
$$u' = \frac{cV'}{a'\sqrt{2}}\, e^{-a'z} \cos\,(45 - a'z) \qquad u'_0 = \frac{cV'}{2a'} \qquad \angle\, VS_0S' = 45°.$$

$$v' = \frac{cV'}{a'\sqrt{2}}\, e^{-a'z} \sin\,(45 - a'z) \qquad v'_0 = \frac{cV'}{2a'}$$

$$s'_0 = \frac{cV'}{a'\sqrt{2}}$$

These are indeed the formulae obtained by EKMAN, if $cV' = \dfrac{kV'}{\mu} = \dfrac{T}{\mu}$.

EKMAN's choice of axes and his formulae in the shape $u_0' = v_0' = \dfrac{T}{2\mu a'}$, the angle between surface current and relative wind velocity $= 45^\circ$, reckoned to the right for northern, to the left for southern latitude, led him to the wrong conclusion that, since a' becomes zero at the equator, the current there would become infinite and also that a sudden change in the direction of the current would take place there. It is not difficult to see now where the error in his conclusion lies It is only as long as one remains on the same spot that the quantity T, used by him, is constant, as soon as the results for different



latitudes are compared $T = kV' = \frac{ka'\sqrt{2}}{\sqrt{a'^2 + (a'+c)^2}}$ changes its value together with a'. And since T becomes zero at the equator together with a', the result is by no means that the current velocity becomes infinite, but simply that it becomes equal to V, as is obvious. At the same time we see from the value of ξ, that at the equator ξ approaches zero and that consequently the current, although rapidly, yet gradually approaches the absolute wind velocity and begins to deviate to the left for southern latitude.

If we put in the result for the stationary current velocity $s_0 = \frac{T}{\mu a'\sqrt{2}} = \frac{kV'\sqrt{b}}{\mu\sqrt{a}}$ $s_0\, a = n$ ($\varphi = 30^\circ$), it follows that $\frac{s_0^2}{V'^2} n = \frac{k^2 b}{\mu} = c^2 b$. This now enables us to make an estimate of the value $c^2 b$, which we wanted for the correction of FREDHOLM's formula. According to MOHN's observations $\frac{s_0}{V'}$ is of the order 4×10^{-2}, so that $c^2 b$ is $16 \times 10^{-4} \times 0.73 \times 10^{-4}$ i.e. of the order 10^{-7}, as we assumed.

The solution for a sea of finite depth was given by EKMAN only for the stationary case, neither does FREDHOLM deal with the non-stationary problem for this case. Since, as we saw, the theoretical correction of FREDHOLM's formula may be practically neglected for infinite depth, I felt justified in treating this case with FREDHOLM's equations of condition $\left(\frac{\partial u}{\partial z}\right)_{z=0} = 0$, $\left(\frac{\partial v}{\partial z}\right)_{z=0} = -\frac{T}{\mu}$; the formulae with the theoretically correct conditional equation become in this case unnecessarily complicated for practical application.

The differential equations for the case of finite depth remain the same. The conditional equations now become

$$u = v = 0 \quad \text{at} \quad t = 0 \quad \text{and} \quad z = h, \quad \left(\frac{\partial u}{\partial z}\right)_{z=0} = 0, \left(\frac{\partial v}{\partial z}\right)_{z=0} = -\frac{T}{\mu}.$$

The solution is here found, as in the analogous problem in heat conduction, when the length of the bar is assumed finite, in the form of a FOURIER series. It runs:

$$w = f(z) - \frac{2}{h} \sum_{m=0}^{m=\infty} e^{-\frac{(2m+1)^2\pi^2}{4h^2}bt - ict} \cos\frac{(2m+1)\pi z}{2h} \int_0^h f(\lambda) \cos\frac{(2m+1)\pi\lambda}{2h} d\lambda \quad (8)$$

where

$$f(z) = \frac{T(1+i)}{2\mu a'} \frac{\sinh.(1+i)a'(h-z)}{\cosh.(1+i)a'h}$$

a' being $= \sqrt{\frac{a}{2b}}$. This $f(x)$ is the solution for $t = \infty$, i.e. for the stationary problem. It is easy to show that this solution satisfies the differential as well as the conditional equations.

Since

$$\int_0^h f(\lambda) \cos\frac{(2m+1)\pi\lambda}{2h} d\lambda = \frac{h}{2} \frac{T(1+i)}{2\mu a'} \frac{8(1+i)a'h}{(2m+1)^2\pi^2 + 4(1+i)^2 a'^2 h^2}$$

(8) may be written

$$w = f(z) - \frac{2iTb}{\mu h} \int_t^\infty e^{-ic\lambda} d\lambda \sum_0^\infty e^{-\frac{(2m+1)^2\pi^2 b}{4h^2}\lambda} \cos\frac{(2m+1)\pi z}{2h} . \quad . \quad (9)$$

Now the summation in (9) may be reduced to another form. We have namely:

$$\sqrt{\frac{p}{\pi}} \sum_{n=-\infty}^{\infty} e^{-p(2n+y)^2} = {}^1/_2 + \sum_{q=1}^{\infty} e^{\frac{-q^2\pi^2}{4p}} \cos q\pi y \quad ^{1)}$$

Putting likewise

$$\sqrt{\frac{p}{\pi}} \sum_{n=-\infty}^{\infty} e^{-p(2n+1-y)^2} = {}^1/_2 + \sum_{q=1}^{\infty} e^{\frac{-q^2\pi^2}{4p}} \cos q\pi(1-y)$$

and subtracting, we find

$$\sqrt{\frac{p}{\pi}} \left\{ \sum_{n=-\infty}^{\infty} \left(e^{-p(2n+y)^2} - e^{-p(2n+1-y)^2}\right) \right\} =$$

$$= \sum_{q=1}^{\infty} (1 - \cos q\pi) e^{\frac{-q^2\pi^2}{4p}} \cos q\pi y = 2 \sum_{m=0}^{\infty} e^{\frac{-(2m+1)^2\pi^2}{4p}} \cos(2m+1)\pi y.$$

1) RIEMANN—WEBER, l. c. II, p. 117.

Putting $y=\frac{z}{2h}$, $p=\frac{h^2}{b\lambda}$ we have

$$\sum_{m=0}^{\infty} e^{\frac{-(2m+1)^2\pi^2 b\lambda}{4h^2}} \cos\frac{(2m+1)\pi z}{2h} = \frac{1}{2}\frac{h}{\sqrt{b\pi}}\sum_{-\infty}^{\infty}\left(\frac{e^{\frac{-(4nh+z)^2}{4b\lambda}}}{\lambda^{1/2}} - \frac{e^{\frac{-(4nh+2h-z)^2}{4b\lambda}}}{\lambda^{1/2}}\right).$$

This transforms (9) to

$$w=f(z)-\frac{iT}{\mu}\sqrt{\frac{b}{\pi}}\int_t^{\infty}\sum_{-\infty}^{\infty}\left\{\frac{e^{\frac{-(4nh+z)^2}{4b\lambda}}}{\lambda^{1/2}}-\frac{e^{\frac{-(4nh+2h-z)^2}{4b\lambda}}}{\lambda^{1/2}}\right\}e^{-ia\lambda}\,d\lambda$$

Since at $t=0$ $w=0$, the final result is

$$w=\frac{iT}{\mu}\sqrt{\frac{b}{\pi}}\int_0^t (F-F')\,e^{-ia\lambda}\,d\lambda, \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

where

$$F=\sum_{n=-\infty}^{+\infty}\frac{e^{\frac{-(4nh+z)^2}{4b\lambda}}}{\lambda^{1/2}}$$

$$F'=\sum_{n=-\infty}^{+\infty}\frac{e^{-(4nh+2h-z)^2}}{\lambda^{1/2}}$$

As the way, followed in deriving (10) is rather long, it may not be superfluous to show that (10) satisfies the conditions of the problem. That the expression satisfies the diff. equat. needs no further proof, as this is the case with any form $\int_0^t \frac{e^{\frac{-(p+z)^2}{4b\lambda}-ia\lambda}}{\lambda^{1/2}}\,d\lambda$.

Besides at $t=0$, $w=0$ and similarly $w=0$ for $z=h$, because the solution has the form $\varphi(z)-\varphi(2h-z)$. It remains to be shown that $\frac{\partial w}{\partial z}_{z=0}=-\frac{iT}{\mu}$. Now

$$\frac{\partial w}{\partial z}=\frac{iT}{\mu}\sqrt{\frac{b}{\pi}}\,(P-Q)$$

where

$$P=\int_0^t\sum_{-\infty}^{\infty}\left(-\frac{4nh+z}{2b\lambda}\,\frac{e^{\frac{-(4nh+z)^2}{4b\lambda}}}{\lambda^{1/2}}\right)d\lambda$$

$$Q=\int_0^t\sum_{-\infty}^{\infty}\left(-\frac{(4nh+2h-z)}{2b\lambda}\,e^{\frac{-(4nh+2h-z)^2}{4b\lambda}}\right)d\lambda.$$

Now it appears that for $z=0$ all the terms of P and Q annul each other excepting the term P for $n=0$, so that there finally remains:

$$\frac{\partial w}{\partial z}_{z=0}=\frac{iT}{\mu}\sqrt{\frac{b}{\pi}}\int_0^t-\frac{4nh}{2b\lambda}\,\frac{e^{-\frac{(4nh)^2}{4b\lambda}-ia\lambda}}{\lambda^{1/2}}\,d\lambda_{n=0} \quad . \quad . \qquad (11)$$

The value of this integral is found by introducing the variable $x^2 = \frac{(4nh)^2}{4b\lambda}$ equal to $-\sqrt{\frac{\pi}{b}}$, so that $\frac{\partial w}{\partial z}_{z=0} = -\frac{iT}{\mu}$.

The solution (10) admits of a remarkable interpretation. The analogon of the problem here dealt with in the theory of heat conduction is to find the temperature in a bar of small cross-section of length h, while at the end $z=0$ a definite temperature interval $\frac{\partial u}{\partial z} = -\frac{iT}{\mu}$ is maintained. Our result says that the temperature to be found can be conceived as due to a distribution of an infinitely large number of heat sources and sinks of equal strength [1]) The heat-sources lie at $z=0$ and $z = \pm[4nh]_{n=1}^{n=\infty}$, the sinks at $z = 2h(1 \pm 2n)_{n=0}^{n=\infty}$. This shows that with respect to the point $z=h$ the sources lie symmetrically with the sinks, so that to every source at distance p corresponds a sink at distance $-p$ and that consequently the temperature (and by analogy in our problem the current velocity) will remain zero here, if it is zero at $t=0$. With respect to the point $z=0$, however, the heat-sources lie symmetrically to each other and likewise the heat-sinks, while in addition there is one more heat-source at the point $z=0$. The symmetrical distribution of the sources makes at this point $\frac{\partial u}{\partial z} = 0$, and also the sinks do not contribute to this quantity, so that for $\frac{\partial u}{\partial z}$ at the point $z=0$ there results only the influence of the source there situated. Now this was calculated above and amounted exactly to $-\frac{iT}{\mu}$.

For the surface current we find thus:

$$w = \frac{iT\sqrt{b}}{\mu\sqrt{\pi}} \int_0^t \frac{e^{-ia\lambda}}{\lambda^{1/2}} \left[1 - 2e^{-\frac{4h^2}{4b\lambda}} + 2e^{-\frac{16h^2}{4b\lambda}} - 2e^{-\frac{36h^2}{4b\lambda}} + \text{etc.}\right] d\lambda.$$

Comparing this expression with that of Fredholm for infinite depth, we see that it is equal to the surface current for infinite depth, diminished by twice the current which in that case would exist at depth $z=2h$, increased by twice the current at depth $z=4h$, etc. As soon as z increases beyond a certain value, the current there is quite negligible against the surface current. With increasing h fewer terms will suffice and for $h=\infty$ only the first term remains, being Fredholm's formula for infinite depth.

[1]) W. Thomson. Math. and Phys. Papers 2, p. 41.