Mathematics. — *New researches upon the centra of the integrals which satisfy differential equations of the first order and the first degree*". (First Part) by Prof. W. KAPTEYN.

1. In a very interesting memoir [1]) "Détermination et intégration d'une certaine classe d'équations différentielles ayant pour point singulier un centre" H. DULAC has investigated the conditions which must be fulfilled when the origin of coordinates is a centrum for the differential equation

$$\frac{dy}{dx} = -\frac{y + a x^2 + b xy + c y^2}{x + a' x^2 + b' xy + c' y^2}$$

that is, when the general integral may be written in this form

$$xy + F_3(xy) + F_4(xy) + \ldots = Const,$$

where $F_k(xy)$ represents a homogeneous polynomium of degree k. Supposing that the coefficients are real or complex he shows that the differential equation must be reducible to one of the following eleven forms, where μ and ν are arbitrary constants

$$(x + x^2 + \mu xy + \nu y^2)\, dy + (y + y^2 + \mu xy + \nu x^2)\, dx = 0 \quad . \quad . \quad . \quad (1)$$
$$(x + x^2 + 2xy + \mu y^2)\, dy + (y + y^2 + 2xy + \nu x^2)\, dy = 0 \quad . \quad . \quad . \quad (2)$$
$$(x + x^2 + \mu y^2)\, dy + (y + y^2 + \nu x^2)\, dx = 0 \quad . \quad . \quad . \quad (3)$$
$$(x + 2x^2 - xy + \mu y^2)\, dy + \left(y + 2y^2 - xy + \frac{1}{\mu} x^2\right) dx = 0 \quad . \quad . \quad (4)$$
$$(x + x^2)\, dy + (y + \mu x^2 + \nu xy)\, dx = 0 \quad . \quad . \quad . \quad (5)$$
$$(x + x^2 + y^2)\, dy + (y + \mu x^2 + 2xy)\, dx = 0 \quad . \quad . \quad . \quad (6)$$
$$(x + x^2 + y^2)\, dy + (y + \mu x^2)\, dx = 0 \quad . \quad . \quad . \quad (7)$$
$$(x + 2x^2 + y^2)\, dy + (y - xy)\, dx = 0 \quad . \quad . \quad . \quad (8)$$
$$(x + xy + \mu y^2)\, dy + (y + xy + \mu x^2)\, dx = 0 \quad . \quad . \quad . \quad (9)$$
$$x\, dy + (y + xy + \mu x^2)\, dx = 0 \quad . \quad . \quad . \quad (10)$$
$$(x + \mu y^2)\, dy + (y + \nu x^2)\, dx = 0 \quad . \quad . \quad . \quad (11)$$

The object of this memoir is the same as that which we have treated in our paper "On the centra of the integral curves which satisfy differential equations of the first order and the first degree". Our point of view however was narrower than that of DULAC, because we only considered real coefficients and real integrals. The object of the present paper now is to extend our former results by adopting also the wider supposition that the coefficients as well as the integrals may be either, real or complex. Our way of solving the

[1]) Bull. Sci. math. Paris (Sér. 2) 32, 1908.

[2]) These Proceedings 10 May 1911.

problem, being very different from that of DULAC, leads to a new classification of the results. Having obtained this we wish finally to compare the two solutions.

2. Let the differential equation be witten in POINCARÉ's form

$$\frac{dy}{dx}=\frac{-x+a'x^2+2b'xy+c'y^2}{y+ax^2+2bxy+cx^2}=\frac{-x+Y}{y+X} \quad . \quad . \quad . \quad . \quad (1)$$

which is different from the form adopted by DULAC, and let the coefficients be real or complex.

By substituting

$$\xi = hx + ky \qquad \eta = -kx + hy$$

the form of the equation is not changed, for we get

$$\frac{d\eta}{d\xi}=\frac{-\xi+\alpha'\xi^2+2\beta''\xi\eta+\gamma'\eta^2}{\eta+\alpha\xi^2+2\beta\xi\eta+\gamma\eta^2}$$

where

$$(h^2+k^2)^2\,\alpha = ah^3 + (a'+2b)\,h^2k + (2b'+c)\,hk^2+c'k^3$$
$$(h^2+k^2)^2\,\beta = bh^3 - (a-b'-c)\,h^2k - (a'+b-c')\,hk^2-b'k^3$$
$$(h^2+k^2)^2\,\gamma = ch^3 - (2b-c')\,h^2k + (a-2b')\,hk^2+a'k^3$$
$$(h^2+k^2)^2\,\alpha' = a'h^3 - (a-2b')\,h^2k - (2b-c')\,hk^2-ck^3$$
$$(h^2+k^2)^2\,\beta' = b'h^3 - (a'+b-c')\,h^2k + (a-b'-c)\,hk^2+bk^3$$
$$(h^2+k^2)^2\,\gamma' = c'h^3 - (2b'+c)\,h^2k + (a'+2b)\,hk^2-ak^3.$$

Hence

$$(h^2+k^2)\,(\alpha+\gamma) = (a+c)\,h + (a'+c')\,k$$
$$(h^2+k^2)\,(\alpha'+\gamma') = (a'+c')\,h - (a+c)\,k$$

and

$$\frac{h}{h^2+k^2}=\frac{(a+c)\,(\alpha+\gamma)+(a'+c')\,(\alpha'+\gamma')}{(a+c)^2+(a'+c')^2}$$
$$\frac{k}{h^2+k^2}=\frac{(a'+c')\,(\alpha+\gamma)-(a+c)\,(\alpha'+\gamma')}{(a+c)^2+(a'+c')^2}.$$

Now generally h and k may be so chosen as to satisfy two conditions, for instance

$$\alpha+\gamma=0, \qquad \alpha'+\gamma'=\lambda$$

and thus the differential equation may be written in the simpler form

$$\frac{dy}{dx}=\frac{-x+a'x^2+2b'xy-a'y^2}{y+ax^2+2bxy+cy^2}.$$

This general case we examined in our former paper. If however

$$(a+c)^2+(a'+c')^2=0$$

this reduction is impossible. In addition to our former paper we therefore now must investigate the consequences of this supposition.

3. If $a + c = a' + c' = 0$
the corresponding differential equation

$$\frac{dy}{dx} = \frac{-x + a'x^2 + 2b'xy - a'y^2}{y + ax^2 + 2bxy - ay^2}$$

admits three particular integrals of the form

$$y = Ax + B$$

for, substituting this value and equalizing the coefficients of the different powers of x in the two members, we have

$$-aA^3 + (2b + a')A^2 + (a - 2b')A - a' = 0$$
$$A^2 + 1 - \{2aA^2 - 2(b + a')A + 2b'\}B = 0$$
$$B\{(aA - a')B - A\} = 0$$

which are satisfied by the roots of the cubic

$$aA^3 - (2b + a')A^2 - (a - 2b')A + a' = 0$$

and by

$$B = \frac{aA}{aA - a'}.$$

In this case, the general integral may be written

$$(y - y_1)^{\lambda_1}(y - y_2)^{\lambda_2}(y - y_3)^{\lambda_3} = const.$$

where $y_1\, y_2\, y_3$ stand for the three particular integrals and $\lambda_1\, \lambda_2\, \lambda_3$ are certain constants. To prove this, we may follow the way indicated by G. DARBOUX [1]) in his fundamental memoir "Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré".

Writing the differential equation in homogeneous coordinates we have

$$L(ydz - zdy) + M(zdx - xdz) + N(xdx - ydx) = 0$$

where

$$L = -2b'x^2 + a'xy - ay^2 + yz$$
$$M = a'x^2 - axy - xz - 2by^2$$
$$N = -(a + 2b')xz + (a' - 2b)yz.$$

Supposing

$$L\frac{\partial f}{\partial r} + M\frac{\partial f}{\partial x} + N\frac{\partial f}{\partial z} = Kf.$$

and replacing f by $A_i x - y - B_i z$, the function corresponding with this value of f is easily found, for

$$A_i L - M - B_i N = K_i(A_i x - y - B_i z)$$

gives immediately

$$K_i = -\left(2b' + \frac{a'}{A_i}\right)x - (2b - aA_i)y. \; (i = 1, 2, 3)$$

1) Bull Sci. math., Paris (Sér. 2) 2. 1878.

Remarking that $z=0$ is a fourth particular integral, and putting $f=z$, we obtain a fourth value for K

$$K_4 = \frac{N}{z} = -(a+2b')x + (a'-2b)y.$$

If now four numbers $\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4$ may be found such that

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$$

and

$$\alpha_1 K_1 + \alpha_2 K_2 + \alpha_3 K_3 + \alpha_4 K_4 = 0$$

the general integral may be written

$$(A_1\, x - y - B_1\, z)^{\alpha_1} (A_2\, x - y - B_2 z)^{\alpha_2} (A_3\, x - y - B_3\, z)^{\alpha_3} z^{\alpha_4} = const.$$

or in inhomogeneous form

$$(y-y_1)^{\lambda_1} (y-y_2)^{\lambda_2} (y-y_3)^{\lambda_3} = const.$$

where $\lambda_1 : \lambda_2 : \lambda_3 = \frac{\alpha_1}{\alpha_4} : \frac{\alpha_2}{\alpha_4} : \frac{\alpha_3}{\alpha_4}$.

The numbers $\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4$ must satisfy the equations

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$$

$$a'\left(\frac{\alpha_1}{A_1} + \frac{\alpha_2}{A_2} + \frac{\alpha_3}{A_3}\right) + a\,\alpha_4 = 0$$

$$a\,(\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3) + a'\,\alpha_4 = 0$$

hence

$$\frac{\alpha_1}{\alpha_4} = \frac{A_2 - A_3}{A_2 A_3}\left[-\frac{a'}{a} + A_2 + A_3 - \frac{a}{a'} A_2\, A_3\right] : D$$

$$\frac{\alpha_2}{\alpha_4} = \frac{A_3 - A_1}{A_1 A_3}\left[-\frac{a'}{a} + A_1 + A_3 - \frac{a}{a'} A_1\, A_3\right] : D$$

$$\frac{\alpha_3}{\alpha_4} = \frac{A_1 - A_2}{A_1 A_2}\left[-\frac{a'}{a} + A_1 + A_2 - \frac{a}{a'} A_1\, A_2\right] : D$$

D representing the determinant formed by the coefficients of the first members of the three preceding equations.

After a slight reduction we find

$$\lambda_1 = (A_2 - A_3)(aA_1^2 - 2bA_1 - a)$$

$$\lambda_2 = (A_3 - A_1)(aA_2^2 - 2bA_2 - a)$$

$$\lambda_3 = (A_1 - A_2)(aA_3^2 - 2bA_3 - a).$$

For small values of x and y the general integral may be expanded in the form

$$x^2 + y^2 + F_3 + F_4 + \ldots = const.$$

which proves that in this case the origin is a centrum.

4. Assuming in the second place $a' + c' = i\,(a + c)$, and referring to our former paper, the origin will be a centrum if it is possible to determine an infinite series of homogeneous polynomia P_k of degree

k such that the following conditions are satisfied:

$$x\frac{\partial P_1}{\partial y} - y\frac{\partial P_1}{\partial x} = \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) P_0$$

$$x\frac{\partial P_2}{\partial y} - y\frac{\partial P_2}{\partial x} = X\frac{\partial P_1}{\partial x} + Y\frac{\partial P_1}{\partial y} + \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) P_1$$

$$x\frac{\partial P_3}{\partial y} - y\frac{\partial P_3}{\partial x} = X\frac{\partial P_2}{\partial x} + Y\frac{\partial P_2}{\partial y} + \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) P_2$$

etc.
where $P_0 = 2$.

To expand the integral in the common form

$$x^2+y^2 + F_3+F_4 + \ldots = Const.$$

we must observe that between the functions F and P the following relations exist

$$(xY - yX)\, P_{n-3} - (x^2 + y^2)\, P_{n-2} + n\, F_n = 0.$$

Considering now the infinite series of conditions we easily see that it is always possible to determine P_1 by the first condition. Proceeding to the second it is evident that every constant factor of P_1 may be omitted.

Writing therefore

$$P_1 = p_0\, x + p_1\, y$$

$$p_0 = -(b + c') \qquad p_1 = a + b'$$

and putting

$$P_2 = q_0\, x^2 + q_1\, xy + q_2\, y^2$$

the second condition gives

$$q_1 = (3a + 2b')\, p_0 + a'\, p_1$$

$$2\, q_2 - 2\, q_0 = (4\, b + 2\, c')\, p_0 + (2\, a + 4\, b')\, p_1$$

$$- q_1 = c\, p_0 + (2\, b + 3\, c')\, p_1.$$

These equations can only be satisfied when

$$(3\, a + 2\, b' + c)\, p_0 + (a' + 2\, b + 3\, c')\, p_1 = 0$$

or introducing the values p_0 and p_1, when

$$aa' - cc' = b\,(a + c) - b'\,(a' + c').$$

From this condition and the given relation

$$a' + c' = i\,(a + c)$$

we deduce

$$a' = b - i\,(b' - c) \quad , \quad c' = -\, b + i\,(a + b')$$

so

$$p_0 = -\, i\,(a + b')$$

$$p_1 = a + b'.$$

If therefore

$$a + b' = 0$$

P_1 and all the following functions $P_2\, P_3 \ldots$ vanish, which proves that the origin is a centrum in this case.

To determine the integral we have

$$2\,(xY - yX) + 3F_3 = 0$$

which gives

$$x^2 + y^2 - \frac{2}{3}\,[a'\,x^3 + (2b' - a)\,x^2y + (c' - 2b)\,xy^2 - cy^3] = const.$$

with

$$a' = b + i\,(a+c) \quad , \quad b' = -a \quad , \quad c' = -b$$

5. When

$$a' + c' = i\,(a+c)$$
$$aa' - cc' = (b - ib')\,(a+c)$$

and

$$a + b' \neq 0$$

the factor $a' + b$ may be omitted; thus

$$P_1 = -ix + y.$$

Determining now P_2, the coëfficient q_0' may be chosen arbitrarily. Putting $q_0 = 0$, we get

$$P_2 = q_1\,xy + q_2\,y^2$$

where

$$q_1 = a' - i\,(3a+2b') \quad q_2 = 2a + 3b' - ib.$$

Assuming

$$P_3 = r_0\,x^3 + r_1\,x^2y + r_2\,xy^2 + r_3\,y^3$$
$$P_4 = s_0\,x^4 + s_1\,x^3y + s_2\,x^2\,y^2 + s_3\,xy^3 + s_4\,y^4$$

the following conditions give

$$r_1 = a'\,q_1$$
$$2r_2 - 3r_0 = (3a + 4b')\,q_1 + 2a'\,q_2$$
$$3r_3 - 2r_1 = (4b + 3c')\,q_1 + (2a + 6b')\,q_2$$
$$-r_2 = cq_1 + (2b + 4c')\,q_2$$

which always can be satisfied, and

$$s_1 = (5a + 2b')\,r_0 + a'r_1$$
$$2s_2 - 4s_0 = (8b + 2c')\,r_0 + (4a + 4b')\,r_1 + 2a'\,r_2$$
$$3s_3 - 3s_1 = 3cr_0 + (6b + 3c')\,r_1 + (3a + 6b')\,r_2 + 3a'\,r_3$$
$$4s_4 - 2s_2 = 2cr_1 + (4b + 4c')\,r_2 + (2a + 8b')\,r_3$$
$$-s_3 = cr_2 + (2b + 5c')\,r_3.$$

which are impossible unless

$$(5a + 2b' + c)\,r_0 + (a' + 2b + c')\,r_1 + (a + 2b' + c)\,r_2 + (a' + 2b + 5c')\,r_3 = 0.$$

This condition may be witten

$$Ar_1 + B(2r_2 - 3r_0) + C(3r_3 - 2r_1) + D(-r_2) = 0$$

where

$$A = 5a' + 10b + 13c'$$
$$B = -(5a + 2b' + c)$$
$$C = a' + 2b + 5c'$$
$$D = -(13a + 10b' + 5c).$$

Introducing q_1 and q_2 this takes the form

$$q_1 [a' A + (3a+4b') B + (4b+3c') C + cD]$$
$$+ q_2 [2a' B + (2a+6b') C + (2b+4c') D] = 0$$

or, eliminating a' and c'

$$q_1 [-(30a^2+40ab'+32ac+12b'^2+20b'c+10c^2) + i(12ab+8bc+4bb')]$$
$$+ q_2 [(12ab+8bc+4bb') - i(42a^2+44ab'+28ac+12b'^2+16b'c+2c^2)] = 0.$$

To simplify this condition we write

$$q_1 [-(3a+b'+2c)(10a+10b'+4c) - 2(b'-c)^2 + 4ib(3a+b'+2c)]$$
$$+ q_2 [4b(3a+b'+2c) - i(3a+b'+2c)(14a+10b') - 2i(b'-c)^2] = 0,$$

or

$$-2(b'-c)^2 (q_1+iq_2) +$$
$$+ (3a+b'+2c)[4b(iq_1+q_2) - 10b'(q_1+iq_2) - 10a(q_1+iq_2) - 4cq_1 - 4iaq_2] = 0.$$

Now

$$4b(iq_1+q_2) - 4cq_1 - 4iaq_2 = 4(2a+3b'-c)(q_1+iq_2)$$

so

$$-2(q_1+iq_2)(a+c)(3a-2b'+5c) = 0$$

or

$$(a+c)[2b - i(a-c)](2b' - 3a - 5c) = 0.$$

This breaks up into three conditions which will be considered separately.

6. When, in the first place

$$a' + c' = i(a+c)$$
$$aa' - cc' = (b - ib')(a+c)$$
$$a + c = 0$$

the second relation depends upon the first and the third.

This case has already been considered in Art. 2.

7. Supposing secondly

$$a + c' = i(a+c)$$
$$aa' - cc' = (b - ib')(a+c)$$
$$2b = i(a-c)$$

which may be written

$$2a' = i(a - 2b' + c)$$
$$2c' = i(a + 2b' + c)$$
$$2b = i(a - c)$$

then again the general integral of the corresponding differential equation may be constructed from a system of particular integrals.

Substituting

$$y = Ax + B$$

in the differential equation we find that it is satisfied when

$$B = \frac{A}{c' - cA}$$

and A is the common root of the two cubics

$$cA^3 + (2b - c')A^2 + (2b' - c)A + c' = 0$$
$$cA^3 + (2b - c')A^2 + (2b' + a)A - a' = 0.$$

Therefore

$$-(a + c)A + a' + c' = 0$$

which combined with the relation

$$a' + c' = i(a + c)$$

gives $A = i$ whence the corresponding particular integral may be written

$$y = ix + \frac{i}{c' - ic}$$

or

$$p \equiv x + iy - \frac{2i}{a + 2b' - c} = 0 .$$

This being the only possible integral of the first degree, now we will try to satisfy the differential equation by an equation of the second degree

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0.$$

Therefore

$$-\frac{Ax + Hy + G}{Hx + By + F} = \frac{-x + a'x^2 + 2b'xy + c'y^2}{y + ax^2 + 2bxy + cy^2}$$

must be equivalent with

$$(Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C)(mx + ny) = 0.$$

Thus the following relations must be satisfied

$$aA + a'H = mA$$
$$2bA + aH + 2b'H + a'B = 2mH + nA$$
$$cA + 2bH + c'H + 2b'B = mB + 2nH$$
$$cH + c'B = nB$$

$$aG - H + a'F = 2mG$$
$$A + 2bG - B + 2b'F = 2mF + 2nG$$
$$H + cG + c'F = 2nF$$
$$- F = mC$$
$$G = nC.$$

The last two equations give $mG + nF = 0$, therefore adding the 5th and 7th equations we get

$$(a+c)\,G + (a'+c')\,F = 0$$

which compared with the given relation

$$a' + c' = i(a+c)$$

shows that we may take

$$G = 1, \quad F = i, \quad C = -\frac{i}{m}, \quad n = im.$$

Considering now the 1th, 4th, and 5th equation we get

$$A = \frac{a'}{m-a} H \quad , \quad B = \frac{c}{im-c'} H, \quad H = a + ia' - 2m$$

and substituting these values in the 2th, 3th and 6th, we see that m must be such as to satisfy the following three relations

$$\frac{a'(2b-im)}{m-a} + \frac{a'c}{im-c'} + a + 2b' - 2m = 0$$

$$\frac{a'c}{m-a} + \frac{c(2b'-m)}{im-c'} + 2b + c' - 2im = 0$$

$$\left(\frac{a'}{m-a} - \frac{c}{im-c'}\right)(a+ia'-2m) + 2(b+ib'-2im) = 0$$

The roots of the last equation are easily found, for

$$H = a + ia' - 2m = \tfrac{1}{2}\,(a-c+2b'-4m) = -\,i\,(b+ib'-2im)$$

so a first root is

$$m_0 = \frac{a+ia'}{2} = \frac{a+2b'-c}{4}$$

and consequently the others

$$m_1 = \frac{3a+2b'+c}{4} \quad , \quad m_2 = \frac{a+2b'-c}{2} = 2m_0$$

Now m_0 does not satisfy the first and second equations, but m_1 and m_2 do. Corresponding with m_1 and m_2 we have therefore two solutions

$$u \equiv H_1\left(\frac{a'}{m_1-a}\,x^2 + 2xy + \frac{c}{im_1-c'}\,y^2\right) + 2x + 2iy - \frac{i}{m_1} = 0$$

and

$$H_2\left(\frac{a'}{m_2-a}x^2+2xy+\frac{c}{im_2-c'}y^2\right)+2x+2iy-\frac{i}{m_2}=0.$$

the last solution however may be easily reduced to

$$\left(x+iy-\frac{i}{2m_0}\right)^2=0$$

which is the square of the first particular integral, so that only one integral of the second degree is left.

To construct the general integral from the two solutions found, we write homogeneously

$$L(ydz-zdy)+M(zdx-xdz)+N(xdy-ydx)=0$$

where

$$L=-2b'x^2-c'xy+cy^2+yz$$
$$M=a'x^2-axy-xz-2by^2$$
$$N=-(a+2b')xz-(2b+c')yz.$$

Remarking that $z=0$ is also a particular integral and replacing f in

$$L\frac{\partial f}{\partial x}+M\frac{\partial f}{\partial y}+N\frac{\partial f}{\partial z}=Kf$$

by p, u, and z, we find the corresponding values of K to be

$$K_1=-\frac{1}{2}(a+2b'+c)x-iay$$

$$K_2=-\frac{1}{2}(a+6b'-c)x-\frac{i}{2}(3a+2b'-3c)y$$

$$K_3=\frac{N}{z}=-(a+2b')x-\frac{i}{2}(3a+2b'-c)y$$

Now three numbers α_1 α_2 α_3 may be found which satisfy the equations

$$\alpha_1K_1+\alpha_2K_2+\alpha_3K_3=0$$

or

$$(a+2b'+c)\alpha_1+(a+6b'-c)\alpha_2+2(a+2b')\alpha_3=0$$

$$2a\alpha_1+(3a+2b'-3c)\alpha_2+(3a+2b'-c)\alpha_3=0$$

and

$$\alpha_1+2\alpha_2+\alpha_3=0.$$

With these values

$$\frac{\alpha_1}{\alpha_3}=-\frac{3a+2b'+c}{a-2b'+3c}\qquad\frac{\alpha_2}{\alpha_3}=\frac{a+2b'-c}{a-2b'+3c}.$$

the general integral may be written

$$p^{\gamma_1}u^{\gamma_2}z^{\gamma_3}=const.$$

or returning to the inhomogeneous coordinates

$$\frac{\left[\frac{a+c}{a-2b'-c}\{i(a-2b'+c)x^2-(a-2b'-c)xy+2i y^2\}+x+iy-\frac{2i}{3a+2b'+c}\right]^{a+2b'-c}}{\left(x+iy-\frac{2i}{a+2b'-c}\right)^{3a+2b'+c}} = const.$$

For small values of x and y the first member of this equation may be expanded in the form

$$x^2 + y^2 + F_3 + F_4 + \ldots = const$$

which proves that in this case the origin is a centrum.

A remarkable case presents itself when

$$5a + 6b' - c = 0$$

for then

$$\alpha_1 = -\frac{4}{3}(a+c) \quad \alpha_2 = -\frac{2}{3}(a+c) \quad \alpha_3 = \frac{8}{3}(a+c)$$

$$p \equiv x + iy + \frac{3i}{a+c}$$

$$u \equiv (a+c)\left[i\frac{4a+c}{2a-c}x^2 - 2xy + i\frac{3c}{2a-c}y^2\right] + 2(x+iy) - \frac{3i}{a+c}$$

and the general integral

$$p^2 u = const.$$

Physics. — "*Preliminary account of some results obtained by the Netherlands Eclipse Expedition in observing the annular solar eclipse of April* 17*th*, 1912." By Prof. W. H. Julius.

The observation of the annular solar eclipse of April 17th, 1912 near Maastricht was favoured by an exceptionally clear sky.

The general plan included:

1. Visual observations on contacts and on positions of crescents.
2. Exposures with the photoheliograph.
3. Exposures with the objective prism spectrograph.
4. Determination of the minimum value of the total radiation at the instant of centrality.
5. Measurement of the entire process of radiation from the first until the fourth contact.
6. Photometric determination of the varying intensity of the sunlight from the first until the fourth contact for five spectral regions of 30 Å each.
7. Observation of various secondary phenomena.

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