issuing from his lips when loudly sounding "a" amounted to 2.35 megaergs per second. According to this standard the effusion of sound from the other parts of the head may be determined. This is set forth in Table V.

TABLE V.

Acoustic energy emitted by the head per second whilst "a" was pronounced with chest-voice.

Regiones	Energy per cM^2 in megaergs	Extent of the surface in cM^2.	Total energy in megaergs per sec.	Remarks.
Nose	0.131	1.1	0.144	both nostrils together
Mouth	0,524	4.4	2.306	
Ears	0.000297	0.475	0.00014	both ears together
Bony parts	0.0000297	2233	0.066	
Soft parts	0.000297	559	0.166	
			2.68	

The total effusion of sound amounted therefore in the above experiment to 2.68 Megaergs per second. Of this the greater part viz. 2.45 Megaers left the head by the mouth and the ears, an extremely small part by the auditory passages and about $^1/_{10}$ by the hard and soft parts together. These data we offer uncorrected i. e. without an estimate of the efficiency of a well-regulated organ-pipe. Not all the energy imparted to the pipe is transformed into sound. Some of it is lost in the vortices of air. Hence our values are greater than the real acoustic values. Though the latter according to a recent publication by ZERNOV [1]) may be esteemed of about the same order, yet it seems to me that the importance of Dr. P. NIKIFOROWSKY's figures lies in the mutual relation of effusions which differ topographically.

Mathematics. — *"On partial differential equations of the first order".* By Prof. W. KAPTEYN.

1. When a partial differential equation of the first order

$$F(x,y\ z,p,q) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is transformed by a tangential transformation, the new equation will generally show the same form. Sometimes however the transformed equation will be linear. In this case the complete primitive of the

[1]) ZERNOV, Ueber absolute Messungen der Schallintensität. Die Rayleighsche Scheibe. Ann. d. Physik. (4). Bd. 26 p. 79. 1908.

non-linear equation and the integralsurface passing through a given curve (the problem of CAUCHY) may be obtained from the transformed linear equation.

The object of this paper is firstly to determine the necessary and sufficient conditions which must be fulfilled by the equation (1) when it may be reduced to the linear form by one of the two known tangential transformations of LEGENDRE and AMPÈRE; secondly to show how in these cases the problem of CAUCHY may be solved.

2. If the tangential transformation of LEGENDRE

$$x = P, \quad y = Q, \quad z = PX + QY - Z, \quad p = X, \quad q = Y \quad . \quad (2)$$

reduces (1) to the linear equation

$$A\,(X,Y,Z)\,P + B\,(X,Y,Z)\,Q = C\,(X,Y,Z)$$

where A, B and C are arbitrary functions, the former evidently must be equivalent with

$$xA\,(p,q,px+qy-z) + yB\,(p,q,px+qy-z) = C\,(p,q,px+qy- \;).$$

Therefore, writing

$$A\,(X,Y,Z)\,P + B\,(X,Y,Z)\,Q - C\,(X,Y,Z) = \psi\,(X,Y,Z,P,Q) = 0$$

we have

$$\frac{\partial\psi}{\partial P} = A\,(X,Y,Z)\;,\;\frac{\partial\psi}{\partial Q} = B\,(X,Y,Z)$$

and

$$\frac{\partial^2\psi}{\partial P^2} = \frac{\partial^2\psi}{\partial P\partial Q} = \frac{\partial^2\psi}{\partial Q^2} = 0.$$

Inversely, these conditions being fulfilled, ψ represents a linear form with regard to the variables P and Q.

These conditions may be transformed in the following way

$$F\,(x,y,z,p,q) = \psi\,(X,Y,Z,P,Q)$$

therefore

$$\frac{\partial\psi}{\partial P} = \frac{\partial F}{\partial x} + X\frac{\partial F}{\partial z}\;,\;\frac{\partial\psi}{\partial Q} = \frac{\partial F}{\partial y} + Y\frac{\partial F}{\partial z}$$

$$\frac{\partial^2\psi}{\partial P^2} = \frac{\partial^2 F}{\partial x^2} + 2X\frac{\partial^2 F}{\partial x\partial z} + X^2\frac{\partial^2 F}{\partial z^2}$$

$$\frac{\partial^2\psi}{\partial P\partial Q} = \frac{\partial^2 F}{\partial x\partial y} + Y\frac{\partial^2 F}{\partial x\partial z} + X\frac{\partial^2 F}{\partial y\partial z} + XY\frac{\partial^2 F}{\partial z^2}$$

$$\frac{\partial^2\psi}{\partial Q^2} = \frac{\partial^2 F}{\partial y^2} + 2Y\frac{\partial^2 F}{\partial y\partial z} + Y^2\frac{\partial^2 F}{\partial z^2}$$

so the necessary and sufficient conditions, in this case, may be written

$$\left.\begin{aligned}
&\frac{\partial^2 F}{\partial x^2} + 2p\frac{\partial^2 F}{\partial x \partial z} + p^2\frac{\partial^2 F}{\partial z^2} = 0 \\
&\frac{\partial^2 F}{\partial x \partial y} + q\frac{\partial^2 F}{\partial x \partial z} + p\frac{\partial^2 F}{\partial y \partial z} + pq\frac{\partial^2 F}{\partial z^2} = 0 \\
&\frac{\partial^2 F}{\partial y^2} + 2q\frac{\partial^2 F}{\partial y d z} + q^2\frac{\partial^2 F}{\partial z^2} = 0
\end{aligned}\right\} \quad . \; . \; . \; . \quad (3)$$

In the same way, considering the tangential transformation of AMPÈRE

$$x = X,\ y = -Q,\ z = Z - YQ,\ p = P,\ q = Y$$

we obtain the necessary and sufficient conditions

$$\left.\begin{aligned}
&\frac{\partial^2 F}{\partial p^2} = 0 \\
&\frac{\partial^2 F}{\partial y \partial p} + q\frac{\partial^2 F}{\partial z d p} = 0 \\
&\frac{\partial^2 F}{\partial y^2} + 2q\frac{\partial^2 F}{\partial y \partial z} + q^2\frac{\partial^2 F}{\partial z^2} = 0
\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (4)$$

3. Assuming now that the equation (1) has been transformed by the transformation of LEGENDRE in the linear form

$$A(X,Y,Z)P + B(X,Y,Z)Q = C(X,Y.Z)$$

we will proceed to examine how the integralsurface of (1) which passes through the curve

$$y = \varphi(x) \quad , \quad z = \psi(x)$$

may be obtained.

Let the integrals of the system of ordinary differential equations

$$\frac{dX}{A(X,Y,Z)} = \frac{dY}{B(X,Y,Z)} = \frac{dZ}{C(X,Y,Z)}$$

be

$$U(X,Y,Z) = a, \quad V(X,Y,Z) = b$$

where a and b are arbitrary constants, the difficulty of the problem consists solely in the determination of the relation or the relations which must exist between the constants a and b.

Designing a point x, y, z and a plane passing through this point with angular coefficients p and q by the name of element, the ∞^2 elements which are related by the three conditions

$$y = \varphi(x), \quad z = \psi(x), \quad dz = pdx + qdy$$

are transformed in the ∞^2 elements $(XYZPQ)$ which satisfy the three conditions

$$Q = \varphi(P), \quad PX + QY - Z = \psi(P), \quad \psi'(P) = X + Y\varphi'(P)$$

or

$$Q = \varphi(P), \quad XP + Y\varphi(P) - Z = \psi(P), \quad X + Y\varphi'(P) = \psi'(P).$$

These elements are precisely the elements of the developable surface generated by the plane

$$Xt + Y\varphi(t) - Z - \psi(t) = 0.$$

For this developable is obtained by eliminating t from

$$\left.\begin{array}{l} Xt + Y\varphi(t) - \psi(t) = Z \\ X + Y\varphi'(t) - \psi'(t) = 0 \end{array}\right\} \quad \ldots \ldots \quad (5)$$

and determining

$$\frac{\partial Z}{\partial X} = t + \{X + Y\varphi'(t) - \psi'(t)\}\frac{\partial t}{\partial X} = t$$

$$\frac{\partial Z}{\partial Y} = \varphi(t) + \{X + Y\varphi'(t) - \psi'(t)\}\frac{\partial t}{\partial Y} = \varphi(t)$$

it is evident that the angular coefficients of the tangent plane through the point X, Y, Z of the developable surface are related by

$$Q = \varphi(P).$$

Hence the constants a and b must be such that the curve

$$U(X,Y,Z) = a, \quad V(X,Y,Z) = b$$

touches the surface (5).

This condition leads to two or one relation between a and b.

In the first case we have

$$U(X,Y,Z) = m, \quad V(X,Y,Z) = n$$

where m and n represent the values found.

If now we transform again X, Y, Z in $x, y, z. p, q$ these relations give

$$U(p, q, xp + yq - z) = m, \quad V(p, q, xp + yq - z) = n$$

and by eliminating p and q from these and

$$F(x,y,z,p,q) = 0$$

we obtain the required integral surface.

In the second case, which is the general one, let

$$b = \theta(a)$$

be the only relation between the constants a and b. From these we deduce

$$V(X, Y, Z) = \theta\,[U(X, Y, Z)].$$

Differentiating with regard to X and Y, we have

$$\frac{\partial V}{\partial X}+P\frac{\partial V}{\partial Z}=\theta'\cdot\left(\frac{\partial U}{\partial X}+P\frac{\partial U}{\partial Z}\right)$$
$$\frac{\partial V}{\partial Y}+Q\frac{\partial V}{\partial Z}=\theta'\cdot\left(\frac{\partial U}{\partial Y}+Q\frac{\partial U}{\partial Z}\right).$$

Transforming now x, y, z, p, q and eliminating p and q from these, we get the integralsurface passing through the given curve.

4. The first case presents itself in the following problem.

Let

$$z=pq$$

be the given differential equation which satisfies the conditions (3) and let it be the question to determine the integralsurface passing through the curve

$$y=1,\quad z=x^2.$$

Transforming the differential equation, we get

$$XP+YQ=Z+XY$$

and

$$U(X,Y,Z)=\frac{Y}{X}=a,\quad V(X,Y,Z)=\frac{Z-XY}{X}=b.$$

The developable surface (5) being

$$Z=Y+\frac{X^2}{4}$$

the curve

$$\frac{Y}{X}=a,\ \frac{Z-XY}{X}=b$$

will touch this surface if

$$4a-1=0 \text{ and } b-a=0.$$

The solution of the linear equation is therefore

$$4Y-X=0,\quad 4Z-4XY-X=0.$$

which transformed to x,y,z,p,q gives

$$4q-p=0\quad 4(-z+px+qy)-4pq-p=0.$$

Joining to these

$$z=pq$$

and eliminating p and q we obtain the required solution

$$16z=(4x+y-1)^2$$

which satisfies the differential equation and passes through the curve

$$y=1,z=x^2.$$

The second case will be met with by taking the same differential equation with the condition that the integral passes through the line

51*

$$y = 2x \quad z = 2x.$$

Here U and V are the same as before, but now the ∞^2 elements $(XYZPQ)$ must satisfy the conditions

$$Q = 2P \;,\; (X+2Y-2)\,P - Z = 0 \;,\; X + 2Y - 2 = 0.$$

or

$$Q = 2\,P, \qquad Z=0, \qquad X + 2\,Y - 2 = 0$$

Here the developable surface reduces to a line and the ∞^2 elements consist of all points of this line with all planes passing through this line. For representing these planes by

$$Z = k\,(X + 2\,Y - 2)$$

it is evident that whatever k be, we have the relation

$$Q = 2P.$$

Expressing now that the curve

$$\frac{X}{Y} = a, \qquad \frac{Z - XY}{X} = b$$

meets the line

$$Z=0, \qquad X + 2\,Y - 2 = 0$$

we find but one relation between a and b, viz.

$$2a + b\,(1 + 2a) = 0.$$

This gives the solution of the linear equation

$$Z - XY = -\frac{2XY}{X+2Y},$$

and by differentiating with respect to X and Y

$$P - Y = -\frac{4Y^2}{(X+2Y)^2}$$

$$Q - X = -\frac{2X^2}{(X+2Y)^2}.$$

Transforming again and taking $p = \dfrac{z}{q}$ from the differential equation, the first of these equations and the quotient of the second and third give

$$2yq^4 - 4zq^3 + z\,(2x + y + 2)\,q^2 - 2z^2q + xz^2 = 0$$

$$2yq^4 - 2zq^3 + z^2q - xz^2 = 0.$$

Adding and subtracting these equations and writing

$$B = 2x + y + 2$$

we obtain

$$4yq^3 - 6zq^2 + Bzq - z^2 = 0$$
$$2q^3 - Bq^2 + 3zq - 2xz = 0$$

so finally

$$\begin{vmatrix} 3z-yB & 6y-B & z-4xy \\ z(6y-B) & B^2-16z-8xy & z(12x-B) \\ z-4xy & 12x-B & 3z-2xB \end{vmatrix} = 0.$$

5. Secondly we suppose that the given equation (1) satisfies the conditions (4). Then the transformation of AMPÈRE reduces it to the linear form

$$A(X, Y, Z)\,P + B(X, Y, Z)\,Q = C(X, Y, Z)$$

whose integrals may be written again

$$U(X, Y, Z) = a, \qquad V(X, Y, Z) = b.$$

The ∞^2 elements subjected to the conditions

$$y = \varphi(x). \qquad z = \psi(x) \qquad dz = pdx + qdy$$

will now be transformed in the ∞^2 elements $(X\,Y\,Z\,P\,Q)$ satisfying the conditions

$$-Q = \varphi(X), \qquad Z - YQ = \psi(X), \qquad \psi'(X) = P + Y\varphi'(X).$$

These elements consist evidently of every point of the surface

$$Z + Y\varphi(X) = \psi(X) \quad . \; . \; . \; . \; . \; . \; . \quad (6)$$

with the corresponding tangent plane. The curve

$$U(X, Y, Z) = a, \quad V(X, Y, Z) = b$$

touching this surface, it is evident that if we eliminate Y and Z, the resulting equation must have equal roots X.

This gives sometimes two, but generally one relation between a and b. Both cases may be treated in the same way as before, the only difference being the transformation, which is now

$$X = x, \quad Y = q, \quad Z = z - qy, \quad P = p, \quad Q = -y.$$

6. As the differential equation

$$z = pq$$

satisfies also the conditions (4), the transformation of AMPÈRE may also be applied if we wish to determine the integralsurface passing through the curve

$$y = x \quad z = (x+1)(x+2).$$

This transformation gives

$$YP + YQ = Z$$

whose integrals

$$Y - X = a, \quad \frac{Z}{Y} = b$$

are easily obtained.

Joining to these the equation (6)

$$Z + XY = (X+1)(X+2)$$

and eliminating Y and Z, we get

$$(a+b-3)X + ab - 2 = 0.$$

This equation does not admit equal roots unless

$$a + b - 3 = 0 \text{ and } ab - 2 = 0.$$

Hence

$$a = 1, \quad b = 2$$

and

$$Y - X = 1, \quad Z = 2Y.$$

The transformation applied to these equations gives

$$q - x = 1 \quad z - qy = 2q$$

and after elimination of q

$$z = (x+1)(y+2)$$

which is the required solution.

If, in the second place, the integralsurface through

$$y = 2x, \quad z = 2x$$

is required, the constants a and b of the integrals

$$Y - X = a, \quad \frac{Z}{Y} = b$$

must be such that this curve touches the surface (6)

$$Z + 2XY = 2X.$$

This condition gives

$$(b - 2a - 2)^2 = 16a$$

so the solution of the linear equation may be written

$$Z - 2Y(Y - X + 1) = 4Y\sqrt{Y - X}$$

from which by differentiation we obtain

$$P + 2Y = \frac{-2Y}{\sqrt{Y-X}}$$

$$Q - 4Y + 2X - 2 = \frac{6Y - 4X}{\sqrt{Y-X}}.$$

Hence, after transformation

$$z - qy - 2q(q - x + 1) = 4q\sqrt{q-x}$$

$$p + 2q = -\frac{2q}{\sqrt{q-x}}$$

$$4q + y - 2x + 2 = \frac{4x-6q}{\sqrt{q-x}}.$$

Introducing $z = pq$ and putting $\sqrt{q-x} = t$ the first and the third equation give

$$2t^4 + 4t^3 + Bt^2 + 4xt + xy + 2x - z = 0$$
$$4t^3 + 6t^2 + Bt + 2x = 0$$

therefore the discriminant of the first member of the first equation must be zero.

If we assume

$$24\,(xy - z) = A$$

this may be written

$$(A + B^2)^3 - \{B(3A - B^2) + 216z\}^2 = 0$$

or, after a slight reduction

$$A(A-3B^2)^2 - 432zB\,(3A - B^2) - 46656z^2 = 0.$$

This solution, though different in form from the former result, represents the same surface; that it passes through the line

$$y = z = 2x$$

may be easily verified.

Physics. — "*On some relations holding for the critical point*". By J. J. VAN LAAR. (Communicated by Prof. H. A. LORENTZ).

1. In this paper we will derive some important relations which exist between some critical quantities.

If it may be accepted that in the association to multiple molecules no generation of heat (change of energy) takes place, so that $q = 0$ may be put, we saw already in I, p. 291 that the relation

$$f = \left(\frac{T}{p}\frac{dp}{dT}\right)_k = 1 + \frac{a}{p_k v_k^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

holds.

If we now put $v_k : b_k = r$, and substitute for p_k its value, viz. (see I, p. 289)

$$p_k = \frac{1}{27} f_2 \frac{a}{b_k^2},$$

we find:

$$(f-1)\,r^2 = \frac{27}{f_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If instead of (1) we write: