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**Mathematics.** — *"On some relations between Bessel's functions"*. By Prof. W. KAPTEYN.

1. When $r$ and $s$ are real numbers and

$$\alpha = x \sin \omega \qquad \beta = x \cos \omega$$

the following expressions

$$S = \cos r\alpha \cos s\beta + \cos r\beta \cos s\alpha$$
$$T = \sin r\alpha \sin s\beta + \sin r\beta \sin s\alpha$$

may be developed in trigonometrical series.

For, writing

$$r\alpha + s\beta = rx \sin \omega + sx \cos \omega = \varrho \cos (\omega - \varphi)$$

from which

$$\varrho = x \sqrt{r^2+s^2} \qquad tg\, \varphi = \frac{r}{s}$$

the equations

$$2S = \cos (r\alpha + s\beta) + \cos (r\alpha - s\beta) + \cos (r\beta + s\alpha) + \cos (r\beta - s\alpha)$$
$$2T = \cos (r\alpha - s\beta) - \cos (r\alpha + s\beta) + \cos (r\beta - s\alpha) - \cos (r\beta + s\alpha)$$

may be reduced by means of the known series

$$\cos (r\alpha + s\beta) = \cos [\varrho \cos (\omega - \varphi)] = I_0 (\varrho) - 2I_2 (\varrho) \cos 2 (\omega - \varphi) + $$
$$+ 2I_4 \cos 4 (\omega - \varphi) - \dots$$
$$\cos (r\alpha - s\beta) = \cos [\varrho \cos (\omega + \varphi)] = I_0 (\varrho) - 2I_2 (\varrho) \cos 2 (\omega + \varphi) + $$
$$+ 2I_4 \cos 4 (\omega + \varphi) - \dots$$
$$\cos (r\beta + s\alpha) = \cos [\varrho \sin (\omega + \varphi)] = I_0 (\varrho) + 2I_2 (\varrho) \cos 2 (\omega + \varphi) + $$
$$+ 2I_4 \cos 4 (\omega + \varphi) + \dots$$
$$\cos (r\beta - s\alpha) = \cos [\varrho \sin (\omega - \varphi)] = I_0 (\varrho) + 2I_2 (\varrho) \cos 2 (\omega - \varphi) + $$
$$+ 2I_4 \cos 4 (\omega - \varphi) + \dots$$

Thus we obtain

$$2S = 4I_0 (\varrho) + 8I_4 (\varrho) \cos 4\varphi \cos 4\omega + 8I_8 \cos 8\varphi \cos 8\omega + \dots$$

and in the same way

$$2T = 8I_2 (\varrho) \sin 2\varphi \sin 2\omega + 8I_6 (\varrho) \sin 6\varphi \sin 6\omega + 8I_{10} (\varrho) \sin 10\varphi \sin 10\omega + \dots$$

2. We may also express the quantities $S$ and $T$ by multiplication of trigonometrical series.

For if

$$f(x) = \tfrac{1}{2} a_0 + a_1 \cos x + a_2 \cos 2x + \dots$$
$$= b_1 \sin x + b_2 \sin 2x + \dots$$
$$\varphi (x) = \tfrac{1}{2} a'_0 + a'_1 \cos x + a'_2 \cos 2x + \dots$$

their product may be represented in one of the following forms

$$f(x) \varphi (x) = \tfrac{1}{2} A_0 + A_1 \cos x + A_2 \cos 2x + \dots$$
$$f(x) \varphi (x) = B_1 \sin x + B_2 \sin 2x + \dots$$

where [1])

$$A_n = \tfrac{1}{2} \sum_0^n a'_m a_{n-m} + \tfrac{1}{2} \sum_1^\infty (a'_m a_{m+n} + a_m a'_{m+n})$$

$$B_n = \tfrac{1}{2} \sum_0^n a'_m b_{n-m} + \tfrac{1}{2} \sum_1^\infty (a'_m b_{m+n} - b_m a'_{m+n}).$$

These coefficients may be written more compactly by observing that $a_{-p} = a_p$, $a'_{-p} = a'_p$, $b_{-p} = -b_p$ and $b_0 = 0$.

Hence

$$\sum_{-n}^0 (a'_m a_{m+n} + a_m a'_{m+n}) = 2 \sum_0^n a'_m a_{n-m}$$

$$\sum_{-n}^0 (a'_m b_{m+n} - b_m a'_{m+n}) = 2 \sum_0^n a'_m b_{n-m}$$

and

$$4A_n = \sum_{-n}^\infty \varepsilon_m (a'_m a_{m+n} + a_m a'_{m+n})$$

$$4B_n = \sum_{-n}^\infty \varepsilon_m (a'_m b_{m+n} - b_m a'_{m+n})$$

where $\varepsilon_m$ represents the value 1 if $m = -n, -n+1, \ldots -2, -1, 0$ and the value 2 if $m = 1, 2, 3 \ldots$

3. Now let us apply these formulae in the first place to expand $S$ in a trigonometrical series.

Multiplying

$$\tfrac{1}{2} \cos r\alpha = \tfrac{1}{2} I_0(rx) + I_2(rx) \cos 2\omega + I_4(rx) \cos 4\omega + \ldots$$

$$\tfrac{1}{2} \cos s\beta = \tfrac{1}{2} I_0(sx) - I_2(sx) \cos 2\omega + I_4(sx) \cos 4\omega - \ldots$$

we have, $m$ being an even integer

$$a_m = I_m(rx) \qquad a'_m = (-1)^{\frac{m}{2}} I_m(sx)$$

therefore, writing the product

$$\frac{1}{4} \cos r\alpha \cos s\beta = \frac{1}{2} A_0 + A_2 \cos 2\omega + A_4 \cos 4\omega + \ldots$$

the coefficients $A$ are determined by

$$4A_n = \sum_{-n}^\infty (-1)^{\frac{m}{2}} \varepsilon_m \left[I_m(sx) I_{m+n}(rx) + (-1)^{\frac{n}{2}} I_m(rx) I_{m+n}(sx)\right].$$

In the same way supposing

$$\frac{1}{4} \cos r\beta \cos s\alpha = \frac{1}{2} A'_0 + A'_2 \cos 2\omega + A'_4 \cos 4\omega + \ldots$$

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[1]) Proceedings of the meeting of Febr. 29, 1908.

we have

$$4 A'_n = \sum_{-n}^{\infty}{}' (-1)^{\frac{m}{2}} \varepsilon_m \left[I_m(rx) I_{m+n}(sx) + (-1)^{\frac{n}{2}} I_m(sx) I_{m+n}(rx)\right],$$

and adding the two results

$$4(A_n + A'_n) = \{1 + (-1)^{\frac{n}{2}}\} \sum_{-n}^{\infty} (-1)^{\frac{m}{2}} \varepsilon_m \left[I_m(sx) I_{m+n}(rx) + \right.$$
$$\left. + I_m(rx) I_{m+n}(sx)\right].$$

Therefore, if $n$ has the values $2, 6, 10, \ldots$ this coefficient vanishes, and if $n = 0, 4, 8, \ldots$ we obtain

$$2(A_n + A'_n) = \sum_{-n}^{\infty} (-1)^{\frac{m}{2}} \varepsilon_m \left[I_m(sx) I_{m+n}(rx) + I_m(rx) I_{m+n}(sx)\right].$$

Hence, writing

$$\tfrac{1}{2} S = C_0 + 2\, C_4 \cos 4\omega + 2 C_8 \cos 8\omega + \ldots$$

the coefficients are determined by

$$2\, C_{4q} = \sum_{-2q}^{\infty} (-1)^p \varepsilon_{2p} \left[I_{2p}(sx) I_{2p+4q}(rx) + I_{2p}(rx) I_{2p+4q}(sx)\right].$$

Comparing this, with the first expansion

$$\tfrac{1}{2} S = I_0(\varrho) + 2 I_4(\varrho) \cos 4\varphi \cos 4\omega + 2 I_8(\varrho) \cos 8\varphi \cos 8\omega + \ldots$$

we obtain the remarkable relation

$$I_{4q}(x\sqrt{r^2+s^2}) \cos 4q\varphi = \tfrac{1}{2} \sum_{-2q}^{\infty} (-1)^p \varepsilon_{2p} \left[I_{2p}(sx) I_{2p+4q}(rx) + \right.$$
$$\left. + I_{2p}(rx) I_{2p+2q}(sx)\right] \quad . \quad (1)$$

where $tg\,\varphi = \frac{r}{s}$.

In the special case that $r = s$, we have $\varphi = \frac{\pi}{4}$ and $\cos 4q\varphi = (-1)^q$, so

$$(-1)^q I_{4q}(xr\sqrt{2}) = \sum_{-2q}^{\infty} (-1)^p \varepsilon_{2p} I_{2p}(rx) I_{2p+4q}(rx),$$

which gives for $q = 0$

$$I_0(xr\sqrt{2}) = \sum_{0}^{\infty} (-1)^p \varepsilon_{2p} I^2_{2p}(rx).$$

a result which may be verified by expanding $I_0(z\sqrt{2})$ by NEUMANN's method in a series of the form

$$I_0(z\sqrt{2}) = \alpha_0 I_0^2(z) + \alpha_1 I_1^2(z) + \alpha_2 I_2^2(z) + \ldots$$

4. To determine in the second place $T$ we multiply the series

$$\tfrac{1}{2} \sin r\alpha = I_1(rx) \sin \omega + I_3(rx) \sin 3\omega + I_5(rx) \sin 5\omega + \ldots$$
$$\tfrac{1}{2} \sin s\beta = I_1(sx) \cos \omega - I_3 (sx) \cos 3\omega + I_5 (sx) \cos 5\omega - \ldots$$

which, compared with the notations of Art. 2, give

$$a'_0 = 0 \quad a'_m = (-1)^{\frac{m-1}{2}} I_m (sx) \quad b_m = I_m (rx)$$

$m$ being odd in this case.

Therefore writing

$$\frac{1}{4} \sin r\alpha \sin s\beta = B_2 \sin 2\omega + B_4 \sin 4\omega + \ldots.$$

the coefficients are determined by

$$4B_n = \sum_{-n}^{\infty} (-1)^{\frac{m-1}{2}} \varepsilon_m \left[I_m (sx) I_{m+n} (rx) - (-1)^{\frac{n}{2}} I_m (rx) I_{m+n}(sx)\right].$$

and in the same way

$$\frac{1}{4} \sin s\alpha \sin r\beta = B'_2 \sin 2\omega + B'_4 \sin 4\omega + \ldots$$

where

$$4B'_n = \sum_{-n}^{\infty} (-1)^{\frac{m-1}{2}} \varepsilon_m \left[I_m (rx) I_{m+n} (sx) - (-1)^{\frac{n}{2}} I_m (sx) I_{m+n} (rx)\right].$$

Hence

$$4 (B_n + B_n') = \{1-(-1)^{\frac{n}{2}}\} \sum_{-n}^{\infty} (-1)^{\frac{m-1}{2}} \varepsilon_m \left[I_m(sx)I_{m+n}(rx)+I_m(rx)I_{m+n}(sx)\right].$$

vanishes for $n = 0, 4, 8 \ldots$ and reduces to

$$2(B_n+B'_n) = \sum_{-n}^{\infty} (-1)^{\frac{m-1}{2}} \varepsilon_m\left[I_m(sx)I_{m+n}(rx) + I_m(rx)I_{m+n}(sx)\right].$$

for the values $n = 2, 6, 10, \ldots$

From this we may infer that

$$\tfrac{1}{2} T = D_2 \sin 2\omega + D_6 \sin 6\omega + D_{10} \sin 10\omega + \ldots$$

where

$$D_{4q+2} = \sum_{-2q-1}^{\infty} (-1)^p \varepsilon_{2p+1}\left[I_{2p+1} (sx) I_{2p+4q+3}(rx) + I_{2p+1} (rx) I_{2p+4q+3} (sx)\right].$$

Comparing this result with the first expansion

$$\tfrac{1}{2} T = 2I_2(\varrho) \sin 2\varphi \sin 2\omega + 2I_6 (\varrho) \sin 6\varphi \sin 6\omega + \ldots$$

we find the identity

$$I_{4q+2} (x\sqrt{r^2+s^2}) \sin (4q+2) \varphi =$$
$$= \tfrac{1}{2} \sum_{-2q-1}^{\infty} (-1)^p \varepsilon_{2p+1} \left[I_{2p+1}(sx)I_{2p+4q+3}(rx) + I_{2p+1}(rx)I_{2p+4q+3}(sx)\right] \quad . (2)$$

which gives for $r = s$

$$I_{4q+2}(xr\sqrt{2}) = \sum_{-2q-1}^{\infty} (-1)^p \varepsilon_{2p+1} I_{2p+1}(sx) I_{2p+4q+3}(rx)$$

and reduces to

$$I_2(xr\sqrt{2}) = \sum_{-1}^{\infty} (-1)^p \varepsilon_{2p+1} I_{2p+1}(sx) I_{2p+3}(rx).$$

if $q = 0$.

5. It seems desirable to verify the two formulae (1) and (2). This may be done in the following way.

The first member of (1), considered as a function of $x$, satisfies the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(r^2 + s^2 + \frac{16q^2}{x^2}\right)y = 0$$

therefore the second member must also satisfy the same equation. To prove this we put

$$I_{2p}(rx) = u_{2p} \quad , \quad I_{2p}(sx) = v_{2p}$$

then it is evident that

$$\frac{d^2u_{2p}}{dx^3} + \frac{1}{x}\frac{du_{2p}}{dx} + \left(r^2 - \frac{4p^2}{x^2}\right)u_{2p} = 0$$

$$\frac{d^2v_{2p}}{dx^2} + \frac{1}{x}\frac{dv_{2p}}{dx} + \left(s^2 - \frac{4p^2}{x^2}\right)v_{2p} = 0.$$

Now, if $\frac{d}{dx}$ be represented by $D$, we will apply the operation

$$D^2 + \frac{1}{x}D + \left(r^2 + s^2 - \frac{16q^2}{x^2}\right)$$

on the general term

$$u_{2p}\, v_{2p+4q} + v_{2p}\, u_{2p+4q}$$

of the series in the second member of (1).

Determining in the first place

$$\left\{D^2 + \frac{1}{x}D + \left(r^2 + s^2 - \frac{16q^2}{x^2}\right)\right\}(u_{2p}v_{2p+4q}) = M_1$$

we have

$$D^2(u_{2p}v_{2p+4q}) = u_{2p}D^2v_{2p+4q} + 2Du_{2p}Dv_{2p+4q} + v_{2p+4q}D^2u_{2p}$$

$$\frac{1}{x}D(u_{2p}v_{2p+4q}) = \frac{1}{x}(u_{2p}Dv_{2p+4q} + v_{2p+4q}Du_{2p})$$

so by addition

$$u_{2p}\left(D^2v_{2p+4q} + \frac{1}{x}Dv_{2p+4q}\right) = \left(\frac{(2p+4q)^2}{x^2} - s^2\right)u_{2p}v_{2p+4q}$$

$$v_{2p+4q}\left(D^2 u_{2p}+\frac{1}{x}Du_{2p}\right)=\left(\frac{4p^2}{x^2}-r^2\right)u_{2p}v_{2p+4q}.$$

Therefore

$$M_1=\frac{8p^2+16pq}{x^2}u_{2p}\,v_{2p+4q}+2Du_{2p}\,Dv_{2p+4q}$$

which is reducible to

$$M_1=rs\,(u_{2p-1}\,v_{2p+4q-1}+u_{2p+1}\,v_{2p+4q+1}).$$

by the known properties of the functions $u$ and $v$

$$2pu_{2p}=\frac{rx}{2}(u_{2p-1}+u_{2p+1})\qquad Du_{2p}=\frac{r}{2}(u_{2p-1}-u_{2p+1})$$

$$(2p+4q)\,v_{2p+4q}=\frac{sx}{2}(v_{2p+4q-1}+v_{2p+4q+1})$$

$$Dv_{2p+4q}=\frac{s}{2}(v_{2p+4q-1}-v_{2p+4q+1}).$$

In the same way, putting

$$M_2=\left\{D^2+\frac{1}{x}D+\left(r^2+s^2-\frac{16q^2}{x^2}\right)\right\}(u_{2p+4q}\,v_{2p})$$

we get

$$M_2=rs\,(u_{2p+4q-1}\,v_{2p-1}+u_{2p+4q+1}\,v_{2p+1})$$

Hence, if $P$ represents the second member of the equation (1) we have proved that

$$\frac{d^2P}{dx^2}+\frac{1}{x}\frac{dP}{dx}+\left\{r^2+s^2-\frac{16q^2}{x^2}\right\}P=$$

$$=\frac{rs}{2}\sum_{-2q}^{\infty}(-1)\,\varepsilon_{2l}\,[u_{2p-1}\,v_{2p+4q-1}+u_{2p+1}\,v_{2p+4q+1}+u_{2p+4q-1}\,v_{2p-1}+$$
$$+u_{2p+4q+1}\,v_{2p+1}\,]$$

Developing this series we obtain

$$\begin{array}{l}
u_{-4q-1}\,v_{-1}+u_{-4q+1}\,v_1+u_{-1}\,v_{-4q-1}+u_1\,v_{-4q+1}\\
-(u_{-4q+1}\,v_1+u_{-4q+3}\,v_3+u_1\,v_{-4q+1}+u_3\,v_{-4q+3})\\
+(u_{-4q+3}\,v_3+u_{-4q+5}\,v_5+u_3\,v_{-4q+3}+u_5\,v_{-4q+5})\\
\ldots\ldots\ldots\ldots\ldots\ldots\\
+(u_{-5}\,v_{4q-5}+u_{-3}\,v_{4q-3}+u_{4q-5}\,v_{-5}+u_{4q-3}\,v_{-3}\\
-(u_{-3}\,v_{4q-3}+u_{-1}\,v_{4q-1}+u_{4q-3}\,v_{-3}+u_{4q-1}\,v_{-1}\\
+(u_{-1}\,v_{4q-1}+u_1\,v_{4q+1}+u_{4q-1}\,v_{-1}+u_{4q+1}\,v_1)\\
-2\,(u_1\,v_{4q+1}+u_3\,v_{4q+3}+u_{4q+1}\,v_1+u_{4q+3}\,v_3)\\
+2\,(u_3\,v_{4q+3}+u_5\,v_{4q+5}+u_{4q+3}\,v_3+u_{4q+5}\,v_5)\\
-\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}$$

Observing that

$$u_{-n} = (-1)^n u_n \quad \text{and} \quad v_{-n} = (-1)^n v_n$$

it is easily seen, that the preceding series vanishes, which gives

$$\frac{d^2P}{dx^2} + \frac{1}{x}\frac{dP}{dx} + \left(r^2 + s^2 - \frac{16q^2}{x^2}\right) P = 0$$

and thus

$$P = C\, I_{4q}\left(x\sqrt{r^2+s^2}\right)$$

for the second integral is out of the question.

To determine the constant $C$, we must compare the coefficient of $\left(\frac{x}{2}\right)^{4q}$ in $P$ with the coefficient of $\left(\frac{x}{2}\right)^{4q}$ in $CI_{4q}\left(x\sqrt{r^2+s^2}\right)$, the latter being $\frac{(r^2+s^2)^{2q}}{(4q)!}C$.

To obtain the former we may observe that $\left(\frac{x}{2}\right)^{4q}$ only will be found in these terms of $P$

$$\frac{1}{2}\sum_{-2q}^{\infty}(-1)^p \varepsilon_{2p}\left[I_{2p}(sx)\, I_{2p+4q}(rx) + I_{2p}(rx)\, I_{2p+4q}(sx)\right]$$

where $\varepsilon_{2p}$ has the value 1 for all the terms existing.

By changing $p$ in $-p$, this expression may be written

$$\frac{1}{2}\sum_{0}^{2q}(-1)^p\left[I_{-2p}(sx)\, I_{4q-2p}(rx) + I_{2p}(rx)\, I_{4q-2p}(sx)\right]$$

or

$$\frac{1}{2}\sum_{0}^{2q}(-1)^p\left[I_{2p}(sx)\, I_{4q-2p}(rx) + I_{2p}(rx)\, I_{4q-2p}(sx)\right].$$

Expanding the functions in this expression the coefficient of $\left(\frac{x}{2}\right)^{4q}$ is found to be

$$\frac{1}{2}\sum_{0}^{2q}(-1)^p \frac{s^{2p}r^{4q-2p}+r^{2p}s^{4q-2p}}{(2p)!\,(4q-2p)!}$$

or

$$\frac{s^{4q}}{(4q)!}\left[1 - \frac{4q\,(4q-1)}{2!}\lambda^2 + \frac{4q\,(4q-1)\,(4q-2)\,(4q-3)}{4!}\lambda^4 - \ldots + \lambda^{4q}\right]$$

where $\lambda = \frac{r}{s}$.

Now

$$(1+i\lambda)^{4q} + (1-i\lambda)^{4q} = 2\left[1 - \frac{4q(4q-1)}{2!}\lambda^2 + \frac{4q(4q-1)(4q-2)(4q-3)}{4!}\lambda^4 - \ldots + \lambda^{4q}\right]$$

or, supposing

$$1 + i\lambda = \sqrt{1+\lambda^2}\,(\cos\varphi + i \sin\varphi)$$

$$1 - i\lambda = \sqrt{1+\lambda^2}\,(\cos\varphi - i \sin\varphi)$$

$$(1+\lambda^2)^{2q} \cos 4q\varphi = 1 - \frac{4q(4q-1)}{2!}\lambda^2 + \frac{4q(4q-1)(4q-2)(4q-3)}{4!}\lambda^4 - \ldots + \lambda^{4q}$$

thus the required coefficient may be written

$$\frac{s^{4q}}{(4q)!}(1+\lambda^2)^{2q} \cos 4q\varphi = \frac{(r^2+s^2)^{2q}}{(4q)!} \cos 4q\varphi$$

where

$$tg\,\varphi = \lambda = \frac{r}{s}.$$

Comparing both coefficients we have

$$\frac{(r^2+s^2)^{2q}}{(4q)!} C = \frac{(r^2+s^2)^{2q}}{(4q)!} \cos 4q\varphi$$

or

$$C = \cos 4q\varphi$$

and finally

$$P = I_{4q}(x\sqrt{r^2+s^2}) \cos 4q\varphi$$

which proves the identity (1). In the same way the formula (2) may be verified.

6. From the formulae of Art. 1 we may at once deduce definite integrals for Bessel's functions of order $4q$ and $4q+2$.

For these give immediately by integration between the limits 0 and $\frac{\pi}{4}$

$$\frac{1}{2}\int_0^{\frac{\pi}{4}} S \cos 4q\omega d\omega = \frac{\pi}{4} I_{4q}(x\sqrt{r^2+s^2}) \cos 4q\varphi$$

$$\frac{1}{2}\int_0^{\frac{\pi}{4}} T \sin (4q+2)\,\omega d\omega = \frac{\pi}{4} I_{4q+2}(x\sqrt{r^2+s^2}) \sin (4q+2)\,\varphi$$

so when $r = s$

$$\int_0^{\frac{\pi}{4}} \cos (rx \sin\omega) \cos (rx \cos\omega) \cos 4q\omega = \frac{\pi}{4} \cos (q\pi)\, I_{4q}(rx\sqrt{2})$$

$$\int_0^{\frac{\pi}{4}} \sin (rx \sin\omega) \sin (rx \cos\omega) \sin (4q+2)\,\omega = \frac{\pi}{4} I_{4q+2}(rx\sqrt{2}).$$

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