The E-W component of the semi-diurnal lunar tide is then represented by the formula

$$0.''01144 \cos (2t - 251°53').$$

4. The amplitude of the theoretical tide, on the assumption that the earth is perfectly rigid, is

$$\frac{3m}{2M}\left(\frac{a}{r}\right)^3 \cos \Phi \cos^4 \frac{I}{2}\left(1 - \frac{5}{2} e^2\right)$$

m and M denoting the mass of moon and earth, a and r the radii of the earth and the moon's orbit, Φ the latitude, I the obliquity of the moon's orbit to the equator and e the excentricity of the moon's orbit. The assumed values are:

$$\frac{m}{M} = \frac{1}{81.4}, \quad \frac{a}{r} = \frac{1}{60.27} \cdot \Phi = 6°11', \quad I = 25°35'$$

and $e = 0.055$.

The lunar hour 0 corresponds with the time of the moon's upper transit.

Finally we find for the theoretical tide:

$$0.''0155 \cos (2t - 270°)$$

and for the real tide:

$$0.''0114 \cos (2t - 251°53').$$

Mathematics. — *"Infinitesimal iteration of reciprocal functions."* By M. J. VAN UVEN. (Communicated by Prof. JAN DE VRIES).

(Communicated in the meeting of April 29, 1910).

§ 1. A function $\varphi(x)$ will be called a *reciprocal function of order* n, when it satisfies the functional equation

$$\varphi_n(x) = \underbrace{\varphi\,[\varphi\,\{\ldots\ \varphi}_{n}(x)\,.\,.\}] = x.$$

The solution of this equation is known by the name of "the problem of BABBAGE" [1]).

In what follows we shall occupy ourselves exclusively with the reciprocal functions of order 2 which therefore satisfy

$$\varphi_2(x) = \varphi\,\{\varphi(x)\} = x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and which for short we shall call reciprocal functions.

The solution of the problem of BABBAGE shows us that the functional equation (1) must be satisfied by all the functions $y = \varphi(x)$

1) See inter alia LAURENT: Traité d'analyse t. VI, Paris 1890, p. 243.

connected to x by a symmetrical equation

$$S(x,y) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We now make it our task to build up these functions by infinitesimal iteration.

Let us call the index of iteration n, we have then to find a function f in such a way that

$$f(y) = f(x) + 1 \quad , \quad f(y_n) = f(x) + n,$$

where y_n is put equal to $\varphi_n(x)$.

If we still put $f(x) = v$, we find

$$x = f_{-1}(v) = g(v) \quad , \quad y = g(v+1) \quad , \quad y_n = g(v+n).$$

From (1) and (2) follows that y_n and y_{n+1} are connected by the relation

$$S(y_n, y_{n+1}) = 0.$$

As $y_2 = g(v+2) = x = g(v)$, then $g(v)$ must depend exclusively on a periodical function with period 2 for which function we shall choose

$$\sigma = e^{i\pi v}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3$$

The function $g(v)$ can therefore be written as a function of σ, in other words:

$$g(v) = h(\sigma).$$

Consequently we have

$$g(v+1) = h(-\sigma),$$

so that the function h is determined by the equation

$$S\{h(\sigma), h(-\sigma)\} = 0.$$

§ 2. A reciprocal function $y = \varphi(x)$ is evidently determined by the equation $S(x,y) = 0$. We have therefore to examine the various symmetrical equations $S(x,y) = 0$. We begin with the equation

$$S(x,y) \equiv x + y - 2k = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This equation passes on account of the substitutions

$$x = h(\sigma) \ , \ y = h(-\sigma)$$

into

$$h(\sigma) + h(-\sigma) = 2k$$

or

$$h(\sigma) - k = -\{h(-\sigma) - k\},$$

which is satisfied by choosing for $h(\sigma) - k$ an arbitrary odd function $\sigma \cdot \omega(\sigma^2)$. So we put

$$h(\sigma) - k = \sigma \cdot \omega(\sigma^2) \qquad (\omega \text{ arbitrary, but univalent}).$$

In this way we arrive at

$$\left.\begin{aligned} x &= h(\sigma) = k + \sigma \cdot \omega(\sigma^2) = k + e^{i\pi v}\,\omega(e^{2i\pi v}), \\ y &= h(-\sigma) = k - \sigma \cdot \omega(\sigma^2) = k - e^{i\pi v}\,\omega(e^{2i\pi v}), \\ y_n &= \qquad\qquad\qquad\qquad\ \ = k + e^{i\pi(v+n)}\,\omega(e^{2i\pi(v+n)}). \end{aligned}\right\} \quad . \quad (5)$$

In order to build up the function $y = \varphi(x) = 2k - x$ by infinitesimal iteration we have only to let n increase gradually. It is as easy to interpolate between x and y a certain number of functions.

The indefinite elements in the solutions are 1 the quantity ν, 2 the function ω.

If we have once chosen a function ω, then by the choice of ν we can assign to the variable x a given value. If we start e. g. from an initial value x_0 then we find ν out of the first equation (5). It goes without saying that this initial value ν_0 of ν can turn out complex. If e. g. $\nu_0 = \lambda + i\mu$, then by iteration the real part will increase, the imaginary one will remain constant.

If to give an example, we wish to interpolate one function between x and y and if we choose for ω

$$\omega(\sigma^2) = 1,$$

we find

$$x = y_0 = k + e^{i\pi\nu}, y_{\frac{1}{2}} = k + ie^{i\pi\nu}, y_1 = y = k - e^{i\pi\nu}, y_{1\frac{1}{2}} = k - ie^{i\pi\nu}.$$

If x is to have the initial value x_0 then ν_0 is determined out of

$$x_0 = k + e^{i\pi\nu_0}$$

or

$$\nu_0 = \frac{1}{i\pi} \log (x_0 - k).$$

For the relation existing between $y_{\frac{1}{2}}$ and x we find

$$y_{\frac{1}{2}} = k + i(x - k) = (1-i)k + ix,$$

and in general

$$y_{n+\frac{1}{2}} = (1-i)k + iy_n$$

§ 3. It is easy to see that all symmetrical equations of the form

$$S(x, y) \equiv \psi(x) + \psi(y) - 2k = 0 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

can be treated in the way followed in § 2.

We have but to put

$$\psi(x) = k + \sigma . \omega(\sigma^2) \quad , \quad \psi(y) = k - \sigma . \omega(\sigma^2),$$

hence

$$x = \psi_{-1}\{k + \sigma . \omega(\sigma^2)\} \quad , \quad y = \psi_{-1}\{k - \sigma . \omega(\sigma^2)\},$$

or

$$x = \psi_{-1}\{k + e^{i\pi\nu}\omega(e^{2i\pi\nu})\} \quad , \quad y = \psi_{-1}\{k - e^{i\pi\nu}\omega(e^{2i\pi\nu})\},$$

$$y_n = \psi_{-1}\{k + e^{i\pi(\nu+n)}\omega(e^{2i\pi(\nu+n)})\} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If we write the symmetrical equation in the form

$$S(x,y) = K,$$

then it is perhaps possible to regard $S(x,y)$ as a function of the expression $\psi(x)+\psi(y)$, so that

$$S(x,y)=F\{\psi(x)+\psi(y)\}=K, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

from which ensues

$$\psi(x)+\psi(y)=F_{-1}(K)=2k.$$

And with this we have returned to the preceding case.

If $S(x,y)$ is to be regarded as a function of the expression

$$\psi(x)+\psi(y)=T(x,y)$$

it must satisfy a certain differential equation. Let us now trace this equation.

It is clear that $T(x,y)$ satisfies

$$\frac{\partial^2 T}{\partial x \partial y}=0.$$

Let us put

$$\frac{\partial S}{\partial x}=S_x\,,\frac{\partial S}{\partial y}=S_y\,,\frac{\partial^2 S}{\partial x^2}=S_{xx}\,,\frac{\partial^2 S}{\partial x\partial y}=S_{xy}\,,\ \text{etc.},\ \frac{dF}{dT}=F'\,,\frac{d^2F}{dT^2}=F'',$$

we then find in the first place

$$S=F(T),$$

$$S_x=F'T_x\quad,\quad S_y=F'T_y\quad,\quad S_{xy}=F''T_xT_y+F'T_{xy}=F''T_xT_y\,;$$

hence

$$\frac{S_{xy}}{S_xS_y}=\frac{F''}{F'^2}=H(T)=G(S),$$

or

$$S_{xy}=GS_xS_y,$$

and therefore also

$$S_{xxy}=G'S_x^2S_y+GS_{xx}S_y+GS_xS_{xy}\quad,\quad S_{xyy}=G'S_xS_y^2+GS_{xy}S_y+GS_xS_{yy},$$

from which ensues by elimination of G and G'

$$S_xS_y\,(S_yS_{xxy}-S_xS_{xyy})=S_{xy}(S_{xx}S_y^2-S_{yy}S_x^2). \quad . \quad . \quad . \quad (9)$$

Let us still put

$$S_x=p,\quad S_y=q,\quad S_{xx}=r,\quad S_{xy}=s,\quad S_{yy}=t,\quad S_{xxy}=u,\quad S_{xyy}=v,$$

we then find

$$pq\,(qu-pv)=s\,(q^2r-p^2t). \quad . \quad . \quad . \quad . \quad . \quad . \quad (9a)$$

So each integral $S(x,y)$ of this differential equation can be regarded as a function of $T=\psi(x)+\psi(y)$.

The function F is determined as follows:

$$\frac{F''(T)}{F'^2(T)}=G(S)=G(F)$$

or

$$-\frac{d}{dT}\left(\frac{1}{F'}\right)=-\frac{dF}{dT}\cdot\frac{d}{dF}\left(\frac{dT}{dF}\right)=-\frac{\frac{d^2T}{dF^2}}{\frac{dT}{dF}}=G(F).$$

The solution of this is

$$\left.\begin{aligned} T &= C\int e^{-\int G(F,dF}dF+C'=\Phi(F)\\ F &= \Phi_{-1}. \end{aligned}\right\} \quad \ldots\ldots \quad (10)$$

so

As example we choose

$$S(x,y)=xy=K=k^2$$

or

$$S(x,y)=e^{\log x+\log y}=e^{2\log k},$$

consequently

$$\log x+\log y=2\log k,$$

from which ensues

$$\log x=\log k+\sigma\,.\,\omega(\sigma^2),\qquad \log y=\log k-\sigma\ \omega(\sigma^2)$$

or

$$x=k\,e^{\sigma\,.\,\omega(\sigma^2)},\qquad y=k\,e^{-\sigma\,.\,\omega(\sigma^2)},$$

or

$$x=k\,e^{e^{i\pi\nu}\omega(e^{2i\pi\nu})},\quad y=k\,e^{-e^{i\pi\nu}\omega(e^{2i\pi\nu})},\quad y_n=k\,e^{e^{i\pi(\nu+n)}\omega(e^{2i\pi(\nu+n)})}.$$

This result we can express somewhat differently. We put

$$\sigma\,.\,\omega(\sigma^2)=\chi(\sigma)-\chi(-\sigma)$$

and we then arrive at

$$x=k\,e^{\chi(\sigma)-\chi(-\sigma)}=k\,\frac{e^{\chi(\sigma)}}{e^{\chi(-\sigma)}}=k\,\frac{\Omega(\sigma)}{\Omega(-\sigma)}=k\,\frac{\Omega(e^{i\pi\nu})}{\Omega(-e^{i\pi\nu})}$$

therefore

$$y_n=k\,\frac{\Omega(e^{i\pi(\nu+n)})}{\Omega(-e^{i\pi(\nu+n)})}.\quad \ldots\ldots\ldots \quad (11)$$

We now put $k=1$ and $\Omega(\sigma)=1-\sigma$ and we find in that way

$$x=\frac{1-\sigma}{1+\sigma}=\frac{1-e^{i\pi\nu}}{1+e^{i\pi\nu}},\qquad y_n=\frac{1-e^{i\pi(\nu+n)}}{1+e^{i\pi(\nu+n)}},$$

consequently

$$i\pi(\nu+n)=\log\frac{y_n-1}{y_n+1}+i\pi,$$

$$i\pi\nu=\log\frac{x-1}{x+1}+i\pi,$$

and

$$\log\frac{y_n-1}{y_n+1}=\log\frac{x-1}{x+1}+i\pi n \quad \ldots\ldots \quad (12)$$

If on the contrary we put $k=1$ and $\Omega(\sigma)=e^{\frac{\sigma}{2}}$, we find

$$x=\frac{e^{\frac{\sigma}{2}}}{e^{-\frac{\sigma}{2}}}=e^{\sigma}=e^{e^{i\pi\nu}}\ ,\quad y_n=e^{e^{i\pi(\nu+n)}}\ ,$$

therefore

$$i\pi\,(\nu+n)=\log\log y_n\ ,$$
$$i\pi\nu=\log\log x$$

and

$$\log\log y_n=\log\log x+i\pi n\ \ldots\ldots\ (13)$$

Now we have formerly [1]) shown that the equation (12) determines the iteration of $y=\frac{1}{x}$, when $\frac{1}{x}$ is taken as a linear-broken function of x, whilst (13) indicates how $y=x^{-1}$ is iterated when x^{-1} is regarded as exponential function. From the above-mentioned it is evident that these two solutions of the iteration problem of $y=\frac{1}{x}$ are but two of an infinite number.

§ 4. If a certain symmetrical relation is given between x and y, e·g.

$$S(x,y)=0,$$

it may happen that by a symmetrical transformation

$$x=\Psi(\xi,\eta)\quad ,\quad y=\Psi(\eta,\xi)\ \ldots\ldots\ (14)$$

of the equation $S(x,y)=0$ we can arrive at a likewise symmetrical equation $\Sigma(\xi,\eta)=0$ of the form

$$\Sigma(\xi,\eta)\equiv\psi(\xi)+\psi(\eta)-2k=0.$$

In this case we have

$$\xi=\psi_{-1}\{k+\sigma\,.\,\omega(\sigma^2)\}\quad ,\quad \eta=\psi_{-1}\{k-\sigma\,.\,\omega(\sigma^2)\}\ ,$$
$$x=\Psi[\{k+e^{i\pi\nu}\,\omega(e^{2i\pi\nu})\}\,,\{k-e^{i\pi\nu}\,\omega(e^{2i\pi\nu})\}],$$
$$y=\Psi[\{k-e^{i\pi\nu}\,\omega(e^{2i\pi\nu})\}\,,\{k+e^{i\pi\nu}\,\omega(e^{2i\pi\nu})\}],$$
$$y_n=\Psi[\{k+e^{i\pi(\nu+n)}\,\omega(e^{2i\pi(\nu+n)})\}\,,\{k-e^{i\pi(\nu+n)}\,\omega(e^{2i\pi(\nu+n)})\}].$$

We shall dwell particularly on the *projective transformation*

$$x=\frac{\alpha\xi+\beta\eta+\gamma}{\delta(\xi+\eta)+\varepsilon}\quad ,\quad y=\frac{\beta\xi+\alpha\eta+\gamma}{\delta(\xi+\eta)+\varepsilon},\ \ldots\ (15)$$

where for abbreviation we shall put

$$\delta(\xi+\eta)+\varepsilon=\lambda.\ \ldots\ldots\ldots\ (16)$$

If $S(x,y)$ is a symmetrical *algebraical* function of order m, then

[1]) M. J. van Uven: "On the orbits of a function obtained by infinitesimal iteration n its complex plane. Proceedings of the Kon. Akad. Vol. XII, pages 503—512.

$S(x, y)$ will pass after the substitution (15) into an expression $S[\xi,\eta]$ of the form

$$S[\xi,\eta] = \frac{\Sigma(\xi.\eta)}{\lambda^m}.$$

The *equation* $S(x, y) = 0$ is then transformed into the equation $\Sigma(\xi, \eta) = 0$. The function $\Sigma(\xi, \eta)$ must now satisfy

$$\frac{\partial^2 \Sigma}{\partial\xi\partial\eta} = \Sigma_{\xi\eta} = 0.$$

So the differential condition becomes

$$\Sigma_{\xi\eta} = \frac{\partial^2}{\partial\xi\partial\eta}\{\lambda^m S[\xi, \eta]\} = 0,$$

or

$$\lambda^2 S_{\xi\eta} + m\delta\lambda\,(S_\xi + S_\eta) + m\,(m-1)\,\delta^2 S = 0 \quad . \quad . \quad . \quad (17)$$

We now have

$$S_\xi = S_x x_\xi + S_y y_\xi \quad , \quad S_\eta = S_x x_\eta + S_y y_\eta\,,$$

$$S_{\xi\eta} = S_{xx} x_\xi x_\eta + S_{xy}(x_\xi y_\eta + x_\eta y_\xi) + S_{yy} y_\xi y_\eta + S_x x_{\xi\eta} + S_y y_{\xi\eta}\,;$$

$$x_\xi = \frac{(\alpha-\beta)\,\delta\eta + (\alpha\varepsilon-\delta\gamma)}{\lambda^2} \quad , \quad y_\xi = \frac{-(\alpha-\beta)\,\delta\eta + (\beta\varepsilon-\delta\gamma)}{\lambda^2}$$

$$x_\eta = \frac{-(\alpha-\beta)\,\delta\xi + (\beta\varepsilon-\delta\gamma)}{\lambda^2}, \quad y_\eta = \frac{(\alpha-\beta)\,\delta\xi + (\alpha\varepsilon-\delta\gamma)}{\lambda^2},$$

$$x_{\xi\eta} = \delta\,\frac{(\alpha-\beta)\,\delta\,(\xi-\eta) - \{(\alpha+\beta)\,\varepsilon - 2\delta\gamma\}}{\lambda^3},$$

$$y_{\xi\eta} = \delta\,\frac{-(\alpha-\beta)\,\delta\,(\xi-\eta) - \{(\alpha+\beta)\,\varepsilon - 2\delta\gamma\}}{\lambda^3}.$$

From (15) ensues

$$\lambda = \delta\,(\xi+\eta) + \varepsilon = \frac{-\{(\alpha+\beta)\,\varepsilon - 2\delta\gamma\}}{\delta\,(x+y) - (\alpha+\beta)},$$

$$\xi = \frac{-(\alpha\varepsilon-\delta\gamma)\,x + (\beta\varepsilon-\delta\gamma)\,y + (\alpha-\beta)\,\gamma}{(\alpha-\beta)\,\{\delta\,(x+y) - (\alpha+\beta)\}},$$

$$\eta = \frac{(\beta\varepsilon-\delta\gamma)\,x - (\alpha\varepsilon-\delta\gamma)\,y + (\alpha-\beta)\,\gamma}{(\alpha-\beta)\,\{\delta\,(x+y) - (\alpha+\beta)\}}.$$

If we now put

$$\delta\,(x+y) - (\alpha+\beta) = l,$$
$$(\alpha+\beta)\,\varepsilon - 2\delta\gamma = c,$$

we finally find after reduction

$$x_\xi = \frac{l}{c}\,(\delta x-\alpha), \quad y_\xi = \frac{l}{c}\,(\delta y-\beta), \quad x_\eta = \frac{l}{c}\,(\delta x-\beta), \quad y_\eta = \frac{l}{c}\,(\delta y-\alpha)$$

$$x_{\xi\eta} = \frac{l^2}{c^2}\,\delta\,\{2\delta x - (\alpha+\beta)\}, \quad y_{\xi\eta} = \frac{l^2}{c^2}\,\delta\,\{2\delta y - (\alpha+\beta)\},$$

whilst at the same time holds

$$\lambda = -\frac{c}{l}.$$

The equation (17) now passes into

$$S_{xx}(\delta x-\alpha)(\delta x-\beta) + S_{xy}\{(\delta x-\alpha)(\delta y-\alpha) + (\delta x-\beta)(\delta y-\beta)\} +$$
$$+ S_{yy}(\delta y-\alpha)(\delta y-\beta) + \delta S_x\{2\delta x - (\alpha+\beta)\} + \delta S_y\{2\delta y - (\alpha+\beta)\} -$$
$$- m\delta S_x\{2\delta x - (\alpha+\beta)\} - m\delta S_y\{2\delta y - (\alpha+\beta)\} + m(m-1)\delta^2 S = 0,$$

or

$$\delta^2[x^2S_{xx} + 2xyS_{xy} + y^2S_{yy} - 2(m-1)(xS_x+yS_y) + m(m-1)S] -$$
$$-(\alpha+\beta)\delta[xS_{xx} + (x+y)S_{xy} + yS_{yy} - (m-1)(S_x+S_y)] +$$
$$+ [\alpha\beta S_{xx} + (\alpha^2+\beta^2)S_{xy} + \alpha\beta S_{yy}] = 0.$$

In order to give to this equation a more concise form we shall make the equation S homogeneous by introduction of a third variable, z.

We then have

$$m(m-1)S = x^2S_{xx} + 2xyS_{xy} + y^2S_{yy} + 2xzS_{xz} + 2yxS_{yz} + z^2S_{zz},$$
$$(m-1)S_x = xS_{xx} + yS_{xy} + zS_{xz},$$
$$(m-1)S_y = xS_{xy} + yS_{yy} + zS_{yz};$$

so

$$x^2S_{xx} + 2xyS_{xy} + y^2S_{yy} - 2(m-1)(xS_x + yS_y) + m(m-1)S = z^2S_{zz},$$
$$xS_{xx} + (x+y)S_{xy} + yS_{yy} - (m-1)(S_x + S_y) = -z(S_{xz} + S_{yz}).$$

If we now put $z=1$ we find for the differential condition

$$\delta^2S_{zz} + (\alpha+\beta)\delta(S_{xz} + S_{yz}) + [\alpha\beta S_{xx} + (\alpha^2+\beta^2)S_{xy} + \alpha\beta S_{yy}] = 0. \quad (18)$$

If we exclude for the present the case $\delta = 0$, corresponding to the *affine* transformation, we may put into the equation (18) without any objection $\delta = 1$; by this (18) takes the form

$$S_{zz} + (\alpha+\beta)(S_{xz} + S_{yz}) + [\alpha\beta S_{xx} + (\alpha^2+\beta^2)S_{xy} + \alpha\beta S_{yy}] = 0. \quad (18a)$$

We can now dispose arbitrarily of the quantities α and β.

If $S(x,y)$ is of order *two*, then all second derivatives are constant, so that the equation (18*a*) forms a connection between the constants of the equation and the constants of the transformation. So we can say:

The general symmetrical quadratic equation can be brought by an *infinite number of projective transformations* into the form $\psi(x) + \psi(y) = 2k$.

If e.g. is given

$$S(x,y) \equiv a_2(x+y)^2 + 2b_2xy + 2a_1(x+y) + a_0 = 0,$$

then we have

$$S_{xx} = 2a_2,\ S_{xy} = 2(a_2+b_2),\ S_{yy} = 2a_2,\ S_{xz} = 2a_1,\ S_{yz} = 2a_1,\ S_{zz} = 2a_0.$$

The condition (18a) now runs

$$a_0 + 2a_1(\alpha + \beta) + (a_2 + b_2)(\alpha + \beta)^2 - 2b_2\alpha\beta = 0 \quad . \quad . \quad (19)$$

Consequently if we choose α and β in such a way that (19) is satisfied, then S is brought to the form

$$(A\xi^2 + B\xi) + (A\eta^2 + B\eta) = 2C,$$

or

$$(\xi^2 - 2B'\xi + C') + (\eta^2 - 2B'\eta + C') = 2k,$$

or if we choose $C' = B'^2$

$$(\xi - B')^2 + (\eta - B')^2 = 2k,$$

so that

$$\psi(\xi) = (\xi - B')^2 = k + \sigma \, . \, \omega(\sigma^2) = k + e^{i\pi\nu}\omega(e^{2i\pi\nu}),$$

or

$$\xi = B' + \sqrt{k + e^{i\pi\nu}\omega(e^{2i\pi\nu})}.$$
$$\eta = B' + \sqrt{k - e^{i\pi\nu}\omega(e^{2i\pi\nu})},$$

whilst

$$\left.\begin{aligned} x &= \frac{\alpha\{B' + \sqrt{k + e^{i\pi\nu}\omega(e^{2i\pi\nu})}\} + \beta\{B' + \sqrt{k - e^{i\pi\nu}\omega(e^{2i\pi\nu})}\} + \gamma}{\delta\{2B' + \sqrt{k + e^{i\pi\nu}\omega(e^{2i\pi\nu})} + \sqrt{k - e^{i\pi\nu}\omega(e^{2i\pi\nu})}\} + \varepsilon}, \\ y_n &= \frac{\alpha\{B' + \sqrt{k + e^{i\pi(\nu+n)}\omega(e^{2i\pi(\nu+n)})}\} + \beta\{B' + \sqrt{k - e^{i\pi(\nu+n)}\omega(e^{2i\pi(\nu+n)})}\} + \gamma}{\delta\{2B' + \sqrt{k + e^{i\pi(\nu+n)}\omega(e^{2i\pi(\nu+n)})} + \sqrt{k - e^{i\pi(\nu+n)}\omega(e^{2i\pi(\nu+n)})}\} + \varepsilon}. \end{aligned}\right\} \quad (20)$$

If $S(x,y)$ is of order *three*, then the two derivatives are of order one, therefore of the form $p_1(x+y) + p_0$. The equation (18a) becomes therefore likewise of order one, e.g.

$$P_1(x+y) + P_0 = 0.$$

As this relation must hold for all values of $x + y$, we have to satisfy

$$P_1 = 0 \quad , \quad P_0 = 0,$$

so that we have now obtained *two* relations between the constants of the equation and the *two* constants α and β of the transformation. So we conclude from this:

The general symmetrical cubic equation can be brought by a *finite number of projective transformations* into the form $\psi(x) + \psi(y) = 2k$.

If we put e.g.

$$S(x.y) \equiv a_3(x+y)^3 + 3b_3(x+y)xy + 3a_2(x+y)^2 + 6b_2xy + 3a_1(x+y) + a_0 = 0,$$

we have

$$S_{xx} = 6\{a_3(x+y) + b_3y + a_2\} \ , \ S_{xy} = 6\{(a_3+b_3)(x+y) + a_2 + b_2\},$$
$$S_{yy} = 6\{a_3(x+y) + b_3x + a_2\} \ , \ S_{xz} = 6\{a_2(x+y) + b_2y + a_1\} \ ,$$
$$S_{yz} = 6\{a_2(x+y) + b_2x + a_1\} \ , \ S_{zz} = 6\{a_1(x+y) + a_0\}.$$

So equation (18a) now becomes

$$[a_1 + (2a_2+b_2)(\alpha+\beta) + (a_3+b_3)(\alpha+\beta)^2 - b_3\alpha\beta](x+y) +$$
$$+ [a_0 + 2a_1(\alpha+\beta) + (a_2+b_2)(\alpha+\beta)^2 - 2b_2\alpha\beta] = 0,$$

so that α and β are determined by

$$a_1 + (2a_2+b_2)(\alpha+\beta) + (a_3+b_3)(\alpha+\beta)^2 - b_3\alpha\beta = 0 \; , \quad . \quad (21)$$
$$a_0 + 2a_1(\alpha+\beta) + (a_2+b_2)(\alpha+\beta)^2 - 2b_2\alpha\beta = 0 . \quad . \quad . \quad . \quad (19)$$

Out of these equations we find two values for $\alpha+\beta$ and two corresponding values of $\alpha\beta$, thus two sets (α,β) or (β,α). So in general *two* projective transformations are possible transferring the symmetrical cubic equation into the standardform desired by us. This is

$$(A\xi^3+B\xi^2+C\xi) + (A\eta^3+B\eta^2+C\eta) = 2D.$$

We can modify the constants in such a way that we find

$$\{(\xi-B')^3 + 3\mu(\xi-B')\} + \{(\eta-B')^3 + 3\mu(\eta-B')\} = 2k,$$

so that

$$\psi(\xi-B') = (\xi-B')^3 + 3\mu(\xi-B') = k + \sigma . \omega(\sigma^2) = k+e^{i\pi\nu}\omega(e^{2i\pi\nu}),$$
$$\psi(\eta-B') = (\eta-B')^3 + 3\mu(\eta-B') = k - \sigma . \omega(\sigma^2) = k-e^{i\pi\nu}\omega(e^{2i\pi\nu}),$$

hence

$$\xi = B' + \psi_{-1}\{k+e^{i\pi\nu}\omega(e^{2i\pi\nu})\} \; , \; \eta = B' + \psi_{-1}\{k-e^{i\pi\nu}\omega(e^{2i\pi\nu})\},$$

$$\left.\begin{aligned} x &= \frac{\alpha[B'+\psi_{-1}\{k+e^{i\pi\nu}\omega(e^{2i\pi\nu})\}]+\beta[B'+\psi_{-1}\{k-e^{i\pi\nu}\omega(e^{2i\pi\nu})\}]+\gamma}{\delta[2B'+\psi_{-1}\{k+e^{i\pi\nu}\omega(e^{2i\pi\nu})\}+\psi_{-1}\{k-e^{i\pi\nu}\omega(e^{2i\pi\nu})\}]+\varepsilon}, \\ y_n &= \frac{\alpha[B'+\psi_{-1}\{k+e^{i\pi(\nu+n)}\omega(e^{2i\pi(\nu+n)})\}]+\beta[B'+\psi_{-1}\{k-e^{i\pi(\nu+n)}\omega(e^{2i\pi(\nu+n)})\}]+\gamma}{\delta[2B'+\psi_{-1}\{k+e^{i\pi(\nu+n)}\omega(e^{2i\pi(\nu+n)})\}+\psi_{-1}\{k-e^{i\pi(\nu+n)}\omega(e^{2i\pi(\nu+n)})\}]+\varepsilon}. \end{aligned}\right\} (23)$$

If we now regard the *affine* transformation, we have but to put in equation (18) $\delta = 0$; we then find

$$\alpha\beta(S_{xx} + S_{yy}) + (\alpha^2 + \beta^2)S_{xy} = 0$$

or

$$\frac{S_{xy}}{S_{xx} + S_{yy}} = - \frac{\alpha\beta}{\alpha^2+\beta^2} = const. \quad . \quad . \quad . \quad . \quad . \quad (24)$$

For the quadratic equation this can always be satisfied and that by two values of the ratio $\alpha : \beta$; hence:

the *general symmetrical quadratic equation* can be brought by *two affine transformations into* the form $\psi(x) + \psi(y) = 2k$.

For the cubic equation, the equation (24) demands

$$\frac{(a_3 + b_3)(x+y) + a_2 + b_2}{(2a_3 + b_3)(x + y) + 2a_2} = const.,$$

therefore

$$\frac{a_3 + b_3}{2a_3 + b_3} = \frac{a_2 + b_2}{2a_2}$$

or

$$2a_3 b_2 + b_2 b_3 - a_2 b_3 = 0. \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The *general symmetrical cubic equation* can be brought by an *affine transformation* into the form $\psi(x) + \psi(y) = 2k$ *only* when condition (25) is satisfied.

This condition expresses that the three asymptotes of the cubic curve represented by the given equation pass through *one* point.

In connection with this we might have obtained equation (25) also in a geometrical way. Of a cubic curve which has as equation

$$A\xi^3 + B\xi^2 + C\xi + A\eta^3 + B\eta^2 + C\eta = 2D,$$

the three asymptotes pass namely through *one* point, a property which can stand an affine transformation.

Chemistry. — "*On the appearance of a maximum and minimum pressure with heterogeneous equilibria at a constant temperature*". By Dr. F. E. C. Scheffer. (Communicated by Prof. A. F. Holleman.)

(Communicated in the meeting of April 29, 1910).

In the spacial figure of a binary system in which occurs a complete miscibility in the liquid condition, a complete separation in the solid condition and where the vapour pressures of the liquid fall continuously from $x = 0$ to $x = 1$, two three-phase lines appear at the place where one of the two components in the solid condition coexists with liquid and vapour. Whereas the pressure values on the three-phase line of the first component increase continuously with the temperature, this is not the case with the line of the second component; Roozeboom suspected that the latter in its P-T-projection always possessed a maximum [1] Later, Kohnstamm [2]) showed that this maximum need not appear always; from the equation of the three-phase line deduced in 1897 by van der Waals [3]), the condition could be deduced when a maximum appeared and when not, because in the former case the value of $(\eta_v - \eta_s) - \frac{x_v}{x_l}(\eta_l - \eta_s)$ must be 0. This condition, however, may point to the appearance of a minimum as well as that of a maximum.

The appearance of a minimum pressure on the three-phase line of the second component becomes even very probable when a minimum occurs in the P-x-lines of the liquid-vapour plane. For this case the

1) Bakhuis Roozeboom, Heterogene Gleichgewichte. II. 331.
2) Kohnstamm, Proc. 1907, Febr. 23.
3) Van der Waals, Proc. 1897, April 21.