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In drawing charts of the magnetic fields in sun-spots, showing the intensity, the direction and the polarity of the magnetic force, the determination of the direction of the force will give some difficulties.

The value of the correction to the indications of the elementary theory necessary in some cases shall be given on another occasion.

The rule, which determines the direction of the deviation, may be indicated here.

The direction of rotation in the vibration ellipses of the outer components towards the red and towards the violet shows whether ϑ is acute or obtuse. If ϑ is obtuse (Fig. 3), then the relative position of the directions of the magnetic force, of the major axis of the vibration ellipses and of the vibration of the middle component is shown in Fig. 5.

From any point O draw a line OB parallel to the major axis of the vibration ellipses of the outer components and a line OM parallel to the vibration of the middle component, the angle BOM being always chosen acute. The projection OF of the magnetic force on a plane normal to the line of sight then makes a positive acute angle with OB, the angle BOF being greater than BOM, the positive direction being reckoned from OB to OM.

By ascertaining whether or not the major axes of the ellipses and the vibrations of the middle component are perpendicular to each other we can make sure whether the elementary theory may be applied or not.

Mathematics. — "*On linear polar groups belonging to a biquadratic plane curve*". By Prof. JAN DE VRIES.

(Communicated in the meeting of April 29, 1910).

1. With respect to a biquadratic curve γ^4, with the symbolic equation $a_x^4 = 0$, the points X, Y, Z, W form a polarquadruple when the relation $a_x a_y a_z a_w = 0$ is satisfied. If we take X, Y, Z arbitrarily on a line r and if we take as fourth point W the point of intersection of r with the "triple polar line" p_{xyz} of X, Y, Z, we get a "linear" polarquadruple. The linear polarquadruples on r evidently form an involution I_3^4, its "*principal points*" (fourfold elements) are the points of intersection of γ^4 with r.

If we assume for the points on r such a parameter representation that two of its principal points are indicated by $\lambda = 0$ and $\lambda = \infty$ we find for the groups of the I_3^4 the relation

$$\sum_4 \lambda_1\lambda_2\lambda_3 + p\sum_6 \lambda_1\lambda_2 + q\sum_4 \lambda_1 = 0, \quad \ldots \quad (1)$$

by which to each triplet $\lambda_1, \lambda_2, \lambda_3$ one value λ_4 belongs, unless at the same time

$$\lambda_1\lambda_2\lambda_3 + p(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) + q(\lambda_1 + \lambda_2 + \lambda_3) = 0$$

and

$$(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) + p(\lambda_1 + \lambda_2 + \lambda_3) + q = 0,$$

are satisfied.

So there is a *"neutral"* involution I_1^3, the groups of which are completed by each point to quadruples. The right line r is thus triple polar line for ∞^1 triplets of poles lying on it.

2. If we put $z_k = x_k + \lambda y_k$ and $w_k = x_k + \mu y_k$ we find for the linear polarquadruples on the line XY the relation

$$a_x a_y (a_x + \lambda a_y)(a_x + \mu a_y) = 0$$

or

$$a_x^3 a_y + (\lambda + \mu) a_x^2 a_y^2 + \lambda\mu a_x a_y^3 = 0.$$

If we choose X and Y in such a way, that at the same time the three relations

$$a_x^3 a_y = 0, \qquad a_x^2 a_y^2 = 0, \qquad a_x a_y^3 = 0$$

are satisfied, then X and Y form a *"neutral pair"* of the I_3^4; i. e. a pair forming with each two points of r a quadruple of the I_3^4 [1]).

In this case the points of intersection of γ^4 with r are indicated by

$$a_x^4 + \lambda^4 a_y^4 = 0.$$

So for them the relation [2])

$$(\lambda_1\lambda_2\lambda_3\lambda_4) = -1,$$

holds, so that the principal points form a harmonic group.

The neutral points of such a *"harmonic"* I_3^4 we shall call *"associated"* with respect to γ^4. They form with each point of their connecting line r a triplet of poles, of which r is the triple polar line.

The line connecting two points U and V of γ^4 intersects it in two points more determined by

$$4a_u^3 a_v + 6\lambda a_u^2 a_v^2 + 4\lambda^2 a_u a_v^3 = 0.$$

They lie harmonically with respect to U and V when $a_u^2 a_v^2 = 0$ is satisfied.

For a given point U this equation represents a polar conic of U. As this conic touches the curve γ^4 in U and intersects it in six

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[1]) Since $a_x(\rho a_x + \sigma a_y)^3 = 0$ has the same scope as $\rho^3 = 0$, XY is the tangent in the point of inflection Y of the polar cubic of X. So XY forms a part of the polar conic of X and Y and it is therefore polar line of any point Z on XY.

[2]) We have namely

$$\frac{\lambda_1 - \lambda_3}{\lambda_2 - \lambda_3} : \frac{\lambda_1 - \lambda_4}{\lambda_2 - \lambda_4} = \frac{1-i}{-1-i} : \frac{1+i}{-1+i} = \frac{(1-i)^2}{(1+i)^2} = -1.$$

points V, each point of γ^4 bears six right lines cut by γ^4 into harmonic groups. With this we have proved the known property according to which there is a curve of *class six* of which the tangents are cut by γ^4 into *harmonic* groups.

3. If we write the equation (1) in the form

$$(\lambda_1+\lambda_2)+p]\lambda_3\lambda_4+[\lambda_1\lambda_2+p(\lambda_1+\lambda_2)+q](\lambda_3+\lambda_4)+[p\lambda_1\lambda_2+q(\lambda_1+\lambda_2)]=0,$$

it is evident that I_3^4 will have a *neutral pair* when at the same time the three relations

$$(\lambda_1+\lambda_2)+p=0,\quad \lambda_1\lambda_2+p(\lambda_1+\lambda_2)+q=0,\quad p\lambda_1\lambda_2+q(\lambda_1+\lambda_2)=0$$

are satisfied, so when we have $2pq=p^3$.

If $p=0$, the neutral pair is determined by $\lambda^2=q$, if $q=\frac{1}{2}p^2$, we have for it $\lambda^2+p\lambda+\frac{1}{2}p^2=0$.

From this ensues that the neutral points coincide when we have $p=0$ and $q=0$. But then we find the principal points out of $\lambda^3=0$, the neutral points out of $\lambda^2=0$. Two associated points can therefore only coincide in a point of inflection of γ^4, and the 24 inflectional tangents are the only right lines on which the I_3^4 of the polarquadruples have a neutral double point.

The degree of the locus of the associated points must therefore be a multiple of six.

To determine that degree we make X to describe a line r. Out of the relations found above

$$a_x^3\, a_y=0,\quad a_x^2\, a_y^2=0,\quad a_x\, a_y^3=0 \quad . \quad . \quad . \quad . \qquad (2)$$

is evident, that the polar cubic p_x^3 describes a pencil, the polar conic p_x^2 a system with index 2, the polar line p_x a system with index 3. From this it is easy to deduce that the points of intersection of p_x^2 and p_x^3 describe a curve γ^8 which has double points in the 9 basepoints of the pencil (p_x^3), whilst the points of intersection of p_x^3 with p_x describe a curve γ^{10} possessing in the basepoints just mentioned threefold points.

In each point of intersection of r with γ^4 the curves p_x^3, p_x^2, and p_x touch each other; those four points are therefore at the same time points of contact of γ^8 and γ^{10}. As those two curves have 54 points of intersection in the 9 basepoints mentioned above there will be 18 points Y, where three corresponding polar lines p_x^3, p_x^2, and p_x concur.

The locus of the associated points is therefore a curve of order 18 which osculates γ^4 in its points [1] of inflection.

[1]) According to a wellknown rule we find out of (2) by elimination of y_k an equation of order 26 in x_k. From the above follows that this can be broken up into an equation of order 18 and two times the equation of γ^4.

4. The principal points of a I_3^4 can in general not be united to a group of the involution. If again I_3^4 is represented by (1), then $4\lambda^3 + 6p\lambda^2 + 4q\lambda = 0$ furnishes its principal points; so these are determined by $\lambda_1 = 0$, $\lambda_2 = \infty$, $\lambda_3 + \lambda_4 = -\frac{3}{2}p$, $\lambda_3 \lambda_4 = q$.

If we put in (1) $\lambda_1 = 0$ and $\lambda_2 = \infty$, we find

$$\lambda_3 \lambda_4 + p(\lambda_3 + \lambda_4) + q = 0.$$

This is satisfied by the principal points λ_3 and λ_4, when we have $3p^2 = 4q$. But then the four principal points form an *equianharmonic* group, as is easily evident by substitution into the wellknown condition.

If U and V are two principal points of such a particular I_3^4, the other two are determined by

$$4a_u^3 a_v + 6\lambda a_u^2 a_v^2 + 4\lambda^2 a_u a_v^3 = 0. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

with the condition

$$a_u a_v (a_u + \lambda_1 a_v)(a_u + \lambda_2 a_v) = 0$$

or

$$a_u^3 a_v + (\lambda_1 + \lambda_2)\, a_u^2 a_v^2 + \lambda_1 \lambda_2 a_u a_v^3 = 0 \quad . \quad . \quad . \quad . \quad (4)$$

Out of (3) follows however

$$\frac{1}{2b_u b^3_v} = \frac{\lambda_1 + \lambda_2}{-3\, b^2_u b^2_v} = \frac{\lambda_1 \lambda_2}{2\, b^3_u b_v}.$$

By substitution in (4) we now find

$$4\, a^3_u\, a_v\, b_u\, b^3_v = 3\, a^2_u\, a^2_v\, b^2_u\, b^2_v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This relation can be interpreted in a peculiar way. The points of intersection of the polar cubic of U, $a_u a_w^3 = 0$, with the line through U and the point V taken arbitrarily, are found out of

$$3\, a^3_u\, a_v + 3\, \lambda\, a^2_u\, a^2_v + \lambda^2\, a_u\, a^3_v = 0.$$

If now (5) is satisfied, this equation has two equal roots λ and UV touches the polar cubic of U.

The tangents out of the point U lying on γ^4 to its polar cubic are therefore cut by γ^4 into equianharmonic groups.

With this we have likewise proved the wellknown property according to which the lines divided *equianharmonically* by γ^4 envelop a curve of *class four*. The tangents of this curve cut γ^4 into *linear polarquadruples*.

If besides (5) also $a^2_u\, a^2_v = 0$ is satisfied, we have either $a^3_u\, a_v = 0$ or $a_u\, a^3_v = 0$.

In both cases the equation $(a_u + \lambda\, a_v)^4 = 0$ has three equal roots ($\lambda^3 = 0$ or $\lambda^3 = \infty$). By this the well known property is confirmed according to which the 24 *inflectional tangents* of γ^4 are the common tangents of a curve of class 4 with a curve of class 6.

The polar conic of a point of inflection possessing the inflectional tangent as component part, this replaces according to § 2 two of the tangents out of the inflectional point to the envelope of the harmonic quadruples; this curve therefore touches the inflectional tangents of γ^4 in the inflectional points.

Indeed, this follows also from the fact, that no tangent of γ^4 can bear a harmonic group unless its point of contact is inflectional point.

5. If U and V are the points of contact of a double tangent of γ^4, then as $(a_u + \lambda a_v)^4 = 0$ shows, $a^3_u a_v = 0$ and $a_u a^3_v = 0$ are satisfied; each of those points is then the point of intersection of the polar line and the polar cubic of the other point

If we allow U to describe the curve γ^4 then p_u and p^3_u touch each other in U and *their point of intersectian describes a curve of order* 32. For, p_u and p^3_u describe respectively systems with index 12 and 4, since the poles of the polar lines and of the polar cubics passing through a point V are generated on γ^4 by p^3_v and p_v. On a right line the two systems determine a (4,36) correspondence and as γ^4 belongs twice to the generated locus, the locus of the point of intersection of p_u and p^3_u is a curve γ^{32}.

In each point of contact of γ^4 the line p_u and curve p^3_u have three points in common; therefore γ^4 is osculated there by γ^{32}. The remaining 56 common points of the two curves are evidently the points of contact of the 28 double tangents of γ^4.

Physiology. — *"About exchange of gases in cold-blooded animals in connection with their size."* By F. J. J. BUYTENDIJK. (Communicated by Prof. H. ZWAARDEMAKER).

In a previous communication [1]) I have been able to prove that in fishes as well as in a number of invertebrate sea-animals the consumption of oxygen of the smaller individuals is considerably larger than that of the larger ones of the same kind.

Through the kindness of the Director of the Royal Zoological Society "Natura Artis Magistra" at Amsterdam I have been enabled to examine the gas-exchange of a great number of cold-blooded animals, in order to see whether the phenomenon stated in sea-animals occurs also in amphibia and reptilia.

The older investigations of REIGNAULT and REISET [2]), MOLESCHOTT,

[1]) These Proc. XII p. 48.

[2]) REIGNAULT and REISET, Annales de Chimie et de Phys 1849. Vol. 26. p. 299.