Mathematics. — *"A quadruple involution in the plane and a triple involution connected with it."* By Prof. JAN DE VRIES.

(Communicated in the meeting of May 28, 1910).

1. In a paper entitled *"An arrangement of the pointfield in involutory groups"* (Versl. van de Kon. Akad. v. Wet., series 3 vol. VI, p. 92—102, 1888; Archives Néerlandaises, vol. XXIII, p. 355—366) I have considered the involutions, the groups of which consist of basepoints of pencils comprised in a net of plane curves of degree n with $\frac{1}{2} n(n+3)-2$ fixed basepoints. Lately Dr. W. VAN DER WOUDE (These Proceedings of March 26th 1910) has investigated a special cubic involution of the first rank in the plane. In the following paper I shall treat the involution, each group of which consists of the points of intersection of two conics α^2 and β^2 belonging to two pencils (α) and (β) with the basepoints A_k and B_k $(k=1,2,3,4)$. Let this quadruple involution be indicated by $(P)^4$.

2. The eight basepoints are evidently *singular points of* $(P)^4$. For on the conic β^2_k which can be brought through A_k the conics α^2 describe a cubic involution, of which each triplet forms with A_k a quadruple of $(P)^4$; I call β^2_k and α^2_k the *singular conics.*

On an arbitrary right line (α) and (β) determine two involutions; their common pair belongs to a quadruple (P). The lines $a_{kl} \equiv A_kA_l$ and $b_{kl} \equiv B_kB_l$ contain an infinite number of pairs; on a_{12} we find that (β) determines an involution of which the pairs are completed to quadruples by the pairs which (β) describes on a_{34}. Lateron (§ 4) it will be evident that these 12 lines are not the only singular lines.

Each α^2 contains 6 quadruples with a double point (coincidence $P_1 \equiv P_2$) belonging to the biquadratic involution which is described by (β); the points P_3, P_4 which still appear in such a quadruple I call branchpoints of $(P)^4$. In each singular point we find such a coincidence, where the corresponding singular conic is touched by a conic of the second system. The locus of the coincidences has therefore with α^2 ten points in common; the *curve of coincidence* is therefore a curve of *order five*, γ^5, passing through the *eight singular points.*

The cubic involution on the singular conic β^2_k has four groups with a double point; so A_k is branchpoint in four quadruples, so the locus of the branchpoints passes four times through each singular point. As an arbitrary α^2 contains in six quadruples twelve branchpoints, the *branchcurve* is a curve of *order fourteen*, φ^{14}, with eight fourfold points in A_k and B_k.

3. If we regard A_2, A_3, A_4 as principal points of a quadratic transformation, then (α) passes into a pencil of lines (A'_1), (β) into a pencil (β') of biquadratic curves through the four points B'_k, with double points in A_2, A_3, A_4. Evidently the curve of coincidence γ^5 is transformed into the polar curve π^7 of A'_1 with relation to (B'), i. e. the locus of the points of contact of the curves B'^4 with lines through A'_1.

In A_2 the polar curve π^7 has a threefold point, of which a tangent passes through A'_1, because the curve β'^4 touching A'_1A_2 in A_2 is touched there at the same time by the polar cubic of A'_1 [1]).

In B'_k the polar curve π^7 touches the line $A'_1B'_k$. It is easy to see that in A'_1 15 tangents of π^7 concur; these right lines, inflectional tangents of curves β'^4, are changed by the quadratic transformation into conics α^2, each osculated by a β^2.

The quadruple involution $(P)^4$ possesses consequently fifteen threefold points $P_1 \equiv P_2 \equiv P_3$.

In each of these points γ^5 and φ^{14} will have to touch each other. So besides their 32 sections lying in the singular points, they have 8 points more in common. These must form four pairs R'_k, R''_k, each consisting of *two coincidences*; i. o. w. $(P)^4$ contains four quadruples, where $P_1 \equiv P_2$ and $P_3 \equiv P_4$. Evidently R'_k and R''_k are the points of contact of two conics α^2, β^2 touching each other twice; the contact chord $R'_k R''_k$ is indicated by c_k.

4. If we make a line l to rotate around a point T_1, the pair P_1, P_2 of $(P)^4$ lying on it describes a curve τ^5 with threefold point T_1, the tangents of which are directed to the points forming a quadruple with T_1. This curve passes through the singular points A_k, B_k and through the points which T_1A_k and T_1B_k have in common with the singular conics β^2_k and α^2_k. Each of the eight tangents t which τ^5 sends through T_1 bears a coincidence $P_1 \equiv P_2$. The lines t, containing two coincided points of P^4, envelop therefore a *curve of the eighth class*, τ_8.

As each t is conjugated to a definite point of γ^5 we find that τ_8 is of genus six just as the former; so it possesses 15 double tangents. To this belong the 12 singular lines a_{kl} and b_{kl} indicated above containing each an involution (P_1, P_2), hence two coincidences.

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[1]) The other two tangents of the threefold point are the tangents to both β'^4, containing a cusp in A_2.

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If d is one of the remaining three double tangents the quadratic involutions determined on them by (α) and (β) have the double points in common; so they are identical. Therefore the *three* lines d are singular too and $(P)^4$ possesses *fifteen singular lines.*

If we conjugate to each other any two conics α^2 and β^2 cutting d in the same pair of points P_1, P_2, the pencils (α) and (β) are projective and generate a cubic curve δ^3 on which they describe a selfsame central involution (P_3, P_4). The lines $P_3 P_4$ concur in the opposite point D of the groups A_k and B_k.

The locus τ^5 of the pairs lying collinear with D evidently breaks up into δ^3 and a conic with double point D not passing through A_k, B_k, consisting therefore of the other two singular right lines d. The three points D are therefore the vertices of a triangle having the lines d as sides; this is lateron confirmed in a different way (§ 7).

5. We find that (α) and (β) determine on a_{12} and a_{34} two projective involutions in half-perspective position; for the point of intersection $A_{12,34} \equiv S$ of a_{12} and a_{34} appears in two pairs belonging to one and the same quadruple.

From this ensues that the lines connecting the points of a pair P_1, P_2 with the points P_3, P_4 of the corresponding pair envelop a *curve of class three* having a_{12} and a_{34} as tangents. If $Q_{1,2}$ and $R_{1,2}$ are the coincidences of the involution described on a_{12}, Q_3, Q_4 and R_3, R_4 the points of a_{34} forming with them two quadruples, then $Q_3 Q_{1,2}$ is the tangent in Q_3, etc. As the indicated curve is cut by a_{34} in Q_3, Q_4, R_3, R_4 and is touched in the point S_4 belonging to $S_{1,3}$, it is a curve of *order six.*

The lines P_1P_3 and P_2P_4 are conjugated tangents; P_2P_3 and P_1P_4 form a pair of the same system. From this ensues that the locus of the diagonal points $N' \equiv (P_1P_3, P_2P_4)$ and $N'' \equiv (P_1P_4, P_2P_3)$ of quadrangle (P) is a *cubic curve,* $\alpha^3_{12,34}$; its points of intersection with a_{13} are $Q_{1,2}$, $R_{1,2}$ and $S_{1,3}$.

The line $n \equiv N'N''$ describes a pencil; for, the lines n are the polar lines of $S_{1,3}$ with respect to the pencil (β). The pairs (N', N'') form thus on the curve $\alpha^3_{12,34}$ a central involution. Where in future we speak of one of the three points $A_{kl,mn}$ and the cubic curve $\alpha^3_{kl,mn}$ conjugated to it, these will be indicated by A^* and α^*; an analogous signification have B^* and β^*.

6. When P_1 describes the line l, the points P_2, P_3, P_4 describe a *curve of order seven,* λ^7; with l it has in common the pair lying on that line besides the points in which l cuts the curve of coin-

cidence γ^6. As l has two points in common with each of the singular conics β^2_k, α^2_k, A_k and B_k are double points of λ^7. With the curve λ^7 belonging to l' it has 32 sections in the singular points; of the remaining seventeen three form a quadruple with the point (ll'), 14 belong to 7 quadruples, each having a point on l, a second on l'.

To find the class of the curve enveloped by the lines P_2P_3, P_3P_4, P_4P_2, we determine the number of the lines passing through A_1. In the first place belong to these the lines through P_2 and P_3, which form triplets with the two points P_1 of β^2_1 lying on l. As A_1A_k contains a pair of $(P)^4$ lying with the point of intersection of l and $A_m A_n$ in a quadruple, A_1A_k is also one of the indicated tangents. The lines $p \equiv P_kP_l$ envelop therefore *a curve of class seven*, π_7, having the 12 singular lines a_{kl}, b_{kl} as tangents, the three singular lines d as *threefold* tangents; for, l has with δ^3 three points P_1 in common. The curves π_7 and π'_7 belonging to l and l' have thus in the singular lines 39 common tangents; of the remaining ten three belong to the point of intersection of l and l', 7 to as many quadruples, of which one point lies on l, an other point on l'.

If l passes through a singular point λ^7 breaks up into the corresponding singular conic (α^2_k or β^2_k) and a λ^5. For $l \equiv a_{12}$ we find that λ^7 consists of the conics β^2_1 and β^2_2, the line a_{12} and twice the line a_{34}. For $l \equiv A_1B_1$ we find the conics β^2_1 and α^2_1 with a cubic curve through the remaining six singular points. For $l \equiv d$ we find that λ^7 consists of d and twice δ^3.

The system consisting of l and λ^7 is invariant with respect to the transformation which makes the points of a quadruple to correspond to each other. In general we shall have an invariant curve by assuming a correspondence (m, n) between the pencils (α) and (β). With projective conjugation we find a general biquadratic curve.

7. The conics which can be laid through the quadruples (P) form a linear system of order three (∞^3) which can be represented by an equation

$$\alpha a^2_x + \beta b^2_x + \gamma c^2_x + \delta d^2_x = 0.$$

A pair of lines with double point in O_3 belonging to it has as equation

$$(\sum_4 \alpha a_{11})\, x^2_1 + 2\,(\sum_4 \alpha a_{12})\, x_1x_2 + (\sum_4 \alpha a_{22})\, x^2_2 = 0,$$

where the parameters α, β, γ, δ are connected by the equations

$$a_{13}\alpha + b_{13}\beta + c_{13}\gamma + d_{13}\delta = 0,$$
$$a_{23}\alpha + b_{23}\beta + c_{23}\gamma + d_{23}\delta = 0,$$
$$a_{33}\alpha + b_{33}\beta + c_{33}\gamma + d_{33}\delta = 0,$$

which furnish in general but *one* solution.

We may conclude from this that an arbitrary point bears but *one* pair of opposite sides of a quadrangle (P) so that the diagonal points N, N', N'', of the quadrangles (P) can be arranged in the groups of a triple involution which will be indicated by $(N)^3$.

The lines $s \equiv P_k P_l$ and $s' \equiv P_m P_n$ are consequently conjugated to each other in an *involutory correspondence*, of which we can easily show that it is *quadratic*. For, if we make s to rotate round a point O, then s' will pass in two of its positions through O, namely when it coincides with one of the lines of the pair of lines (s, s') having O as double point; i. e. s' envelops a conic when s describes a pencil.

According to § 4, the *quadratic involution* (s, s') has the lines d as *principal lines*; for d forms a pair with each line through D. From this ensues again that the three points D are the vertices of the triangle formed by the lines d.

The *coincidences* (double lines) of the involution are the *chords of contact* c_k indicated above (§ 3). According to a wellknown property of the quadratic involution the principal lines d are the diagonals of the quadrangle formed by the double lines. In connection with this we put $c_k\, c_l \equiv C_{kl}$ and $C_{kl}\, C_{mn} \equiv d_{kl,mn}$. Apparently C_{kl} is the centre of an involution of rays, having c_k and c_l as double rays.

8. The triple involution $(N)^3$ has 6 *singular points* in C_{kl}; for, C_{kl} bears ∞^1 pairs of rays (s, s'), so it is a diagonal point N of ∞^1 quadrangles (P). Lateron it will be evident, that the locus of the corresponding pairs N', N'' is a *biquadratic curve* (§ 10).

Also the poles C_k of the four lines c_k with respect to the pairs of conics α^2, β^2 of which they form the chords of contact, are *singular points* of $(N)^3$. Each point C_k is as diagonal point N conjugated to the pairs N', N'' of an involution placed on c_k having R'_k and R''_k (§ 3) as double points.

Finally A^* and B^* are also *singular points*. As was evident in § 5, the corresponding *singular curves* α^* and β^* are of *order three*.

The involution $(N)^3$ has thus *sixteen singular points.*

Singular lines of $(N)^3$ are evidently the three right lines d and the four right lines c.

The triplets (N) determined by the quadruples (P) of a conic α^2, lie on a cubic curve ν^3, which cuts α^2 in the six coincidences of the biquadratic involution $(P)^4$. For the singular conic α^2_k (passing through B_k) ν^3 has a double point in B_k, of which the tangents are directed to the branchpoints of the coincidence lying in B_k.

9. When $N \equiv ss'$ describes the line l, then s and s' envelop a *curve of class three* touched by l in the point N which it has in common with the line l' conjugated to it. This curve λ_3 intersects l on the four right lines c, is thus a curve of *order six.*

We now determine the order of the locus of the quadruples (P) lying on the pairs of tangents of λ^6_3. The curve ν^3 belonging to a definite conic α^2 determines on l three points N, so it contains three quadruples of the locus. This passes three times through each point A_k, because the rational ν_3 belonging to β^2_k determines on l three diagonal points N of quadruples, in which A_k appears. It passes through the eight points R'_k, R''_k lying on the lines c_k and touches there the lines $C_kR'_k$, $C_kR''_k$.

The curves π_1^{12} and π_2^{12} belonging to l_1 and l_2 have $8 \times 9 = 72$ sections in A_k, B_k, $4 \times 4 = 16$ in the points R'_k, R''_k and 4 in the quadruple (P) for which $(l_1 l_2)$ is one of the diagonal points. The remaining 52 sections form 13 quadruples (P), of which one diagonal point lies on l_1, and a second on l_2.

From this ensues that to the points N of a line l correspond the pairs N', N'' of a *curve of order thirteen,* λ^{13}. With l the curve λ^{13} has five points of the curve of coincidence γ^5 in common, which is at the same time curve of coincidence of the involution $(N)^3$; the remaining eight form four pairs (N, N'). Each line bears thus four pairs of $(N)^3$.

10. The curve λ^{13} passes three times through each of the six singular points A^*, B^*, because l has three points in common with the corresponding singular curve α^*, respect. β^*. It also passes through the four singular points C_k and with a number of branches to be determined more closely through each of the six singular points C_{kl}.

The curves λ_1^{13} and λ_2^{13}, belonging to l_1 and l_2 have $6 \times 9 = 54$ sections in A^* and B^*, 4 in the points C_k; fartheron they have in common the pair of points conjugated to $N \equiv l_1 l_2$ besides the 13 points N forming each with a point of l_1 and a point of l_2 a triplet of $(N)^3$. As the remaining 96 sections must lie in the 6 points C_{kl}, we find that λ^{13} passes four times through each point C_{kl}. To the *singular point* C_{kl} belongs therefore a *singular biquadratic curve* γ^4_{kl}.

When l coincides with c_1, we find that λ^{13} breaks up into the line c_1 and the three singular curves γ^4_{12}, γ^4_{13}, γ^4_{14}. These pass all through the singular points A^*, B^*, because these points are threefold on λ^{13}. As the three curves γ^4_{1k} pass together four times through the points C_{23}, C_{24} and C_{34}, we find that γ^4_{1k} has a double point in C_{mn}. That γ^4_{kl} must have at least *one* double point, was deducible from the fact, that on a general biquadratic curve no involutions of pairs appear. On the uninodal $\gamma^4{}_{kl}$ exists but *one* involution of pairs; the pairs (N', N'') belonging to $N \equiv C_{kl}$ lie thus collinear with the double point C_{mn} of $\gamma^4{}_{kl}$.

As $\gamma^4{}_{12}$ passes through C_{12} we find that C_{12} is a coincidence of $(N)^3$ and at the same time of $(P)^4$. The third point N'' of the corresponding triplet must lie on $C_{12}C_{34} \equiv d_{12,34}$. From this ensues that C_{12} is one of the double points of the quadratic involution determined by (α) and (β) on $d_{12,34}$; then the second double point is C_{34}. The curve γ^5 cuts $d_{12,34}$ in C_{12}, C_{34} and in the three points which $d_{12,34}$ has in common with $\delta^3{}_{12,34}$.

The curves $\gamma^4{}_{12}$ and $\gamma^4{}_{34}$ have six points of intersection in A^*, B^*, 4 points of intersection in C_{13}, C_{14}, C_{23}, C_{24} and 4 points of intersection in C_{12} and C_{34} (for, C_{12} is double point of γ^4). The remaining two points of intersection are diagonal points of two quadruples having each a diagonal point in C_{12} and in C_{34}; the lines connecting these two points N with C_{34}, are apparently the tangents in the double point C_{34} of $\gamma^4{}_{12}$.

The curves $\gamma^4{}_{12}$ and $\gamma^4{}_{13}$ have 8 sections in the points C_{kl}, 6 in the points A^*, B^* and both of them pass through C_1; their 16th point of intersection is a point N forming a triplet with C_{12} and C_{13}. We see that N as point of $\gamma^4{}_{12}$ must lie on the line $C_{13}C_{34}$, so this point must coincide with C_{23}. So the two curves must touch each other in C_{23}.

11. As $\gamma^4{}_{kl}$ passes through the singular points A^* and B^* the singular points C_{kl} lie on the singular curves α^* and β^*. The curves $\alpha^3{}_{12,34}$ and $\alpha^3{}_{13,24}$ intersect each other in the 6 points C_{kl} and in the 3 points B^*; the last follows from the consideration of the quadruple that is determined by the right lines b_{kl}, b_{mn} on the right lines a_{pq}, a_{rs}.

The curves α^* and β^* have therefore in common the 6 points C_{kl}, the two points A^*, B^* and finally the point N forming a triplet with the last two points.

For the *singular points* and *lines* we have therefore the following orientation:

c_k contains the three points C_{kl};

$\alpha^3{}_{kl,mn}$ contains $A_{kl,mn}$, the three points B^* and the six points C_{kl};

$\beta^3{}_{kl,mn}$ contains $B_{kl,mn}$, the three points A^* and the six points C_{kl};

$\gamma^4{}_{kl}$ has a double point in C_{mn} and passes through the remaining points C_{pq}, through the points C_k and C_l and through the six points A^*, B^*;

$d_{kl,mn}$ contains the points C_{kl} and C_{mn}.

For the singular line $d_{12,34}$ the curve λ^{13} consists of $\gamma^4_{12}, \gamma^4_{34}$ and a curve $\delta^5_{12,34}$ passing through the points A^*, B^*, C_{12} and C_{34}, and having double points in the remaining four points C_{kl}. It is a curve of *genus two*, so it contains only one involution of pairs; the pairs (N', N'') are determined by the conics containing the four double points, and the lines $n \equiv N'N''$ envelop a conic δ^2, touching δ^5 in five points[1]).

We find that $d_{12,34}$ is cut by $\delta^5_{12,34}$ in C_{12}, C_{34} and in the three points which $d_{12,34}$ has in common with $\delta^3_{12,34}$; these five points lie also on the curve of coincidence γ^5 (§ 10).

12. If N describes the line $d_{12,34}$, the line n envelops a figure of class four composed of the points C_{12}, C_{34} and a conic $\delta^2_{12,34}$. From this we may conclude that n will envelop *a curve of class four*, λ_4, when N describes the line l.

Between N and n there is no birational correspondence; N does determine in general *one* right line n, but on a non-singular n lie (§ 9) four pairs (N', N''), so that to n belong four points N.

The lines $s \equiv P_1P_2$ bearing the coincidences of $(P)^4$ envelop (§ 4) a curve of class eight, τ_8, having a_{kl}, b_{kl} and the principal lines d of the quadratic involution (s, s') as double tangents. Therefore the line $s' \equiv P_3P_4$ envelops a curve of class ten, τ'_{10}, possessing three fourfold tangents d and six double tangents a_{kl} and b_{kl} (a_{kl} corresponds in the involution to a_{mn}). The point of intersection of s' with s is the branchpoint N'' belonging to the points N and N' coinciding with P_1 and P_2. As none of the lines s coincides with the s' conjugated to them, the locus of the point (s, s') is a curve of order 18. The *branchcurve* of the involution $(N)^3$ is therefore of *order eighteen*, has double points in C_{kl}, fourfold points in A^*, B^* and of course passes through the fifteen threefold points of $(N)^3$.

13. If the basepoints A_4 and B_4 coincide in the point E, then the quadruple involution $(P)^4$ passes into a *triple involution* with the *singular points* A_k, B_k $(k = 1, 2, 3)$ and E. If to each conic α^2 the conic β^2 is conjugated which it touches in E then the biquadratic curve ε^4 containing the points of intersection of corresponding curves has with an arbitrarily chosen α^2 three points, A_k and two points P_1, P_2, in common; so it passes three times through E. So to the singular point E belongs a singular biquadratic curve with threefold

[1]) See my paper *"Ueber Curven fünfter Ordnung mit vier Doppelpunkten"*. (Sitz Ber. Akad. Wien, CIV, 46).

point in E bearing the pairs of points forming triplets with E; it of course passes through the six singular points A_k, B_k.

The curve of coincidence γ^5 possesses now a *threefold point* in E; for, with an α^2 it has in common the three points A_k and the four coincidences of the cubic involution lying on α^2. The curves γ^5 and ε^4 have in E the same three tangents.

On an arbitrarily chosen α^2 lie four branchpoints; as E is branchpoint for triplets, in which the two coincidences lying on ε^4 appear, and also the points A_k and B_k are each branchpoints for two groups, the branchcurve is of order six, φ^6, and it has double points in the *seven* singular points.

The curves γ^5 and φ^6 have in the singular points 18 sections; as fartheron they can only touch each other $(P)^3$ has *six threefold points.*

14. The pairs (P_1, P_2) lying collinear with a point T_1 form a curve τ^4 with double point T_1 where six tangents t concur. The bearers of the coincidences of $(P)^3$ envelop therefore a *curve of class six*, τ_6. As τ^4 has still 5 points in common with γ^5 besides the 6 points of contact of the tangents concurring in T_1 and the 7 singular points, the lines which connect each a point of coincidence with the corresponding branchpoint envelop a *curve of class five.*

The curve τ_6 is, like γ^5, of genus three, so it has seven double tangents. To this belong the six singular lines a_{kl}, b_{kl}; also the *seventh* indicated by d, *is singular* because (α) and (β) determine the same involution on it; the third movable point of intersection of two curves α^2 and β^2 conjugated in this way describes a cubic curve δ^3 with double point in E.

If P_1 describes the line l, then P_2 and P_3 describe a curve λ^7, passing four times through E, twice through A_k, B_k and cutting l into a pair and into five coincidences of $(P)^3$.

The line $p_1 \equiv P_2P_3$ envelops a curve of *class four*, π_4, for the positions of p_1 passing through E are furnished by the lines to the points P_2, which form triplets with E and the points of intersection P_1 of l and ε^4. This π^4 has a_{kl} and b_{kl} as tangents; for, on a_{kl} lies e. g. a pair P_2, P_3 belonging to the point of intersection P_1 of l with A_mE. The singular line d is *threefold tangent* of π_4; the three pairs P_2, P_3 lying on it correspond to the points of intersection P_1 of l and δ^3.

The curves π_4 and π'_4 belonging to l and l' have therefore in the seven singular lines 15 tangents in common; the 16th common tangent p is conjugated to the point of intersection of l and l'. By the birational transformation (P, p) a pencil is therefore transformed into a curve *of class four.*

When p rotates around T_1 the pair P_2, P_3 lying on it describes the above mentioned curve τ^4, possessing with λ^7 sixteen sections in the singular points; four points of intersection form each a pair with a point of l; the remaining ones belong to four pairs P_2, P_3, for which P_1 lies on l. To a pencil described by p corresponds therefore a *biquadratic curve* π^4 described by P.

As τ^4 has with ε^4 in common besides the singular points a point lying on ET_1 and three pairs P_2, P_3 placed on lines through T_1 for which P_1 falls in E, this E is a threefold point on π^4. In an analogous way is evident that A_k, B_k are points of π^4. Two curves π^4 have thus 15 sections in the singular points; the 16th common point corresponds to the common ray of the two pencils.

15. Finally we note the case, in which (α) and (β) have in common the basepoints E_1 and E_2, thus determine an involution of pairs (P_1, P_2).

To the *singular points* A_1, A_2; B_1, B_2 conics β^2_1, β^2_2; α^2_1, α^2_2 are conjugated, of which the points form a pair with the corresponding singular point.

If we conjugate again each α^2 to the β^2 touching it in E_1 its movable point of intersection describes a figure of order four, passing three times through E_1 and twice through E_2, thus composed of the line $e \equiv E_1 E_2$ and a cubic curve ε^3_1, having E_1 as double point and passing through E_2, A_1, A_2, B_1, B_2; it contains moreover the point of intersection C of A_1A_2 and B_1B_2.

As evidently C belongs also to the *singular curve* ε^3_2, of which the points form pairs with E_2, therefore C is also a *singular point*; it corresponds to each point of e.

The *curve of coincidence* has double points in E_1 and E_2; it is biquadratic and passes through the four points A_k, B_k.

If P_1 describes the line l, then P_2 describes a λ^6 through C with four double points A_k, B_k and two threefold points E_1, E_2., So we have here a *birational involutory* transformation of order six and class one (a pair on an arbitrary right line), with 7 principal points of which 2 are threefold, 4 twofold and 1 single.

The pairs on rays through T form a cubic curve τ^3 through the 7 principal points; two curves τ^3 have besides the principal points in common the pair on the line connecting the corresponding points T. As four tangents of τ^3 pass through T, the bearers of the coincidences envelop a *curve of class four.*