The experimental verification of LORENTZ's deductions, formulated in § 23 above, gives a new proof of the rational connexion established by VOIGT's theory of the inverse magnetic effect between diverse phenomena.

A more accurate measurement of ϑ_1, the vapour density and the field being chosen, must be postponed.

46. The new type of magnetic separation, with some components polarized, the other ones unpolarized, which returns to the ordinary separation by decrease of vapour density, we were able to observe also with D_2. Since the density of the vapour must be great in the present experiment, the effects observed with D_2, which splits up into a pseudo-triplet, are less clear and characteristic than with D_1. We, therefore, restricted the detailed description of our observations to the case of D_1.

Mathematics. — *"On continuous vector distributions on surfaces"* (3rd communication)[1]. By Dr. L. E. J. BROUWER. (Communicated by Prof. D. J. KORTEWEG).

(Communicated in the meeting of May 28, 1910).

§ 1.

The irrigating field on the sphere.

In order to get an insight into the structure of an arbitrary finite continuous vector field with a finite number of singular points on the sphere over its entire extent, we begin by investigating a particular case characterized by the *absence of simple closed tangent curves.*

In a field which possesses this property, and which we shall call an *irrigating field,* no spirals can appear as tangent curves and no rotation points as singular points. As farthermore a singular point can neither possess elliptic sectors or leaves, it is either a source point without leaves, or a vanishing point without leaves, or it possesses exclusively hyperbolic and parabolic sectors without leaves, in which case we shall speak of a *stroking point.*

The singular points of an irrigating field cannot all be stroking points. This follows from theorem 8 of the second communication [2]) in

[1]) For the first and second communication see these Proceedings Vol. XI 2, p. 850 and Vol. XII 2, p. 716.

[2]) l. c. p. 734.

connection with the observation, that the reduction of stroking points can lead only to reflexion points.

So there are certainly source points or vanishing points; to fix our thoughts we shall start from the existence of source points $B_1, B_2, \ldots\ldots B_m$.

In B_1 we start an arbitrary tangent curve which when pursued indefinitely can neither close itself, nor become a spiral. So it must stop at a singular point, which can be nothing but a vanishing point V_1.

If possible we then start in B_1 a second tangent curve, not crossing the first and stopping at an *other* vanishing point V_2.

If possible then in each of the two sectors generated in B_1 a tangent curve not crossing the two already existing ones and stopping either at a third vanishing point V_3, differing from V_1 and V_2, or, if that is excluded, stopping e.g. at V_1, but then in such a way that in B_1 a sector is determined limited by two tangent curves stopping at V_1, inside which we can draw a tangent curve not crossing the existing ones, starting from B_1 and stopping at V_2.

We continue this process of insertion as often as possible, whereby every time in each sector is inserted a tangent curve not crossing the existing ones which either stops at an other vanishing point as the two tangent curves limiting the sector, or, if that is excluded, determines a new sector, in which such an insertion is possible.

In this way it is impossible that at some moment a sector should appear limited by two tangent curves stopping at the same vanishing point, and within which no other vanishing point should lie.

So the number of tangent curves stopping at one and the same vanishing point, and appearing in this process of insertion, must remain smaller than the total number of vanishing points and from this ensues that the process of insertion ends after a finite number of insertions.

Of the then constructed finite system of tangent curves starting from B_1, which we shall call a *system of skeleton curves of* B_1, no two consecutive ones have the same vanishing point as their end-point.

Let for a certain sense of circuit those skeleton curves be consecutively $r_1, r_2, \ldots\ldots r_n$, stopping respectively at the vanishing points $V_1, V_2, \ldots\ldots V_n$, which of course need not be all different.

We then if possible introduce between every r_p and r_{p+1} a tangent curve starting in B_1 and stopping at a certain vanishing point, not crossing the already existing ones and reaching a distance as great as possible from r_p and r_{p+1}. In each of the sectors thereby generated at B_1 we repeat such an insertion, in each of the sectors thereby generated again and so on; finally after having repeated this process of insertion ω times, we add the limiting curves, which are likewise tangent curves starting from B_1 and stopping at certain vanishing points. After that, as ensues from the reasoning followed in § 2 of the second communication [1]), no new tangent curves starting from B_1 can be inserted, whilst the constructed tangent curves cover on the sphere a closed coherent set of points, to which belong all possible tangent curves starting from B_1, and which we shall call *the irrigation territory of* B_1.

The method according to which the skeleton curves have been constructed implies furthermore that between every r_p and r_{p+1} two tangent curves r''_p and r'_{p+1} appear, between which no further tangent curves starting from B_1 can be constructed, whilst all tangent curves, which have been constructed between r_p and r''_p, end in V_p, and all tangent curves, which have been constructed between r'_{p+1} and r_{p+1}, end in V_{p+1}.

From this ensues that these curves r''_p and r'_{p+1} coincide from B_1 up to a certain stroking point S_p, beyond which they diverge for good.

For, when diverging either in a non-singular point or immediately in B_1, insertion of new tangent curves starting from B_1 would be possible.

And also when rejoining after having previously diverged, an insertion of a new tangent curve starting from B_1 would be possible, namely of such a one that had with r''_p as well as with r'_{p+1} an arc in common.

So the irrigation territory of B_1, consisting of n sectors Σ_p, each limited by a tangent curve r'_p and a tangent curve r''_p, possesses an outer circumference $V_1 S_1 V_2 \ldots . V_n S_n V_1$, consisting of $2n$ tangent arcs, which we shall call its "sides". It may happen here, that an even side $S_p V_{p+1}$, and an odd side $S_q V_q$ (p and q different) *touch each other outwardly* along an arc $P V_{p+1}$ resp. $P V_q$ (which can expand to an entire side $S_p V_{p+1}$ or $S_q V_q$, or reduce itself to a point V_{p+1} resp. V_q) but not in an other way.

For, when two such sides $S_p V_{p+1}$ and $S_q V_q$ have collided somewhere outwardly, they cannot leave each other any more before V_{p+1} resp. V_q has been reached. Otherwise a tangent curve coinciding partially with $S_p V_{p+1}$ and partially with $S_q V_q$ might be inserted, which would separate $S_p V_{p+1}$ and $S_q V_q$, so that these could not have collided with each other, but only with the newly inserted tangent curve. The sectors Σ_p connecting in this way B_1

[1]) l. c. p. 723.

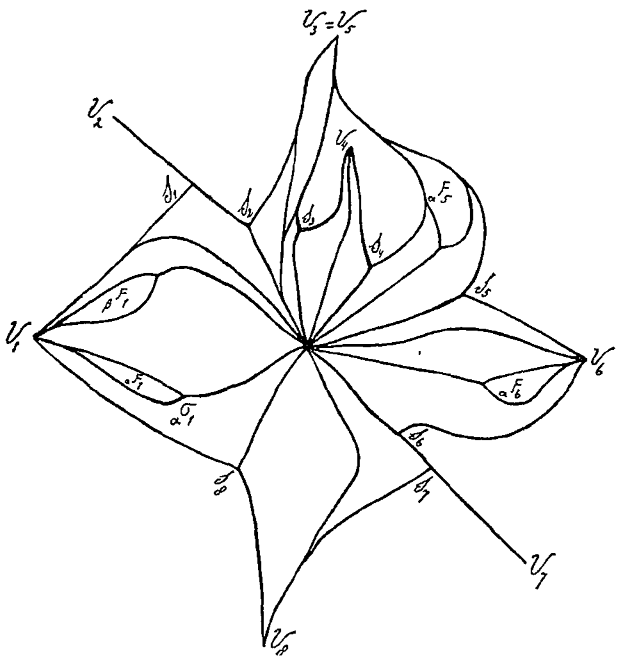

Fig. 1. Irrigation territory.

with one and the same vanishing point possess round about that vanishing point the same cyclic order as about B_1.

Let us consider a sector Σ_p. The limiting tangent curves r'_p and r''_p can collide inwardly in an arbitrary closed set of points (which in particular can entirely cover those curves). Farthermore it is not necessary that the entire inner domain determined by r'_p and r''_p belongs to Σ_p. However for each region ${}_\alpha F_p$ between r'_p and r''_p not belonging to Σ_p the property holds that it is limited by two tangent curves ${}_\alpha\varrho_p$ and ${}_\alpha\varrho'_p$ running from B_1 to V_p (between which no further tangent curves starting from B_1 can be constructed), which coincide from B_1 up to a certain stroking point ${}_\alpha\sigma_p$, then diverge, and finally after rejoining in a point ${}_\alpha H_p$ (which can also coincide with V_p) remain united to their end in V_p. If namely the latter property were lacking, then a new tangent curve starting from B_1 could be inserted. As finally the stroking point ${}_\alpha\sigma_p$ must give inside the region ${}_\alpha F_p$ two (and not more than two) hyperbolic sectors, only a finite number of points ${}_\gamma\sigma_p$ can coincide in one and the same stroking point, and from this ensues that there is only a finite number of regions ${}_\alpha F_p$.

The preceding shows that the residual regions determined on the sphere by the irrigation territory of B_1, are each bounded by a single inner circumference $V_{\alpha_1} S_{\alpha_1} V_{\alpha_2} S_{\alpha_2} \ldots . V_{\alpha_n} S_{\alpha_n} V_{\alpha_1}$, whose sides each join a stroking point and a vanishing point, in such a way that two successive sides concurring in a vanishing point V_{α_p} can touch each other inwardly from a certain point P up to V_{α_p}, but other inner contacts are excluded, and farthermore that each stroking point S_{α_p} possesses in the considered residual region *two* hyperbolic sectors.

The irrigation territory s_1 of B_1 possesses a finite distance from all the remaining source points.

If we construct for B_2 the irrigation territory analogously as for B_1, these two irrigation territories can partially penetrate into each other. This can however, when constructing the irrigation territory of B_2, be prevented by enforcing on its tangent curves starting from B_2 the condition that they may neither cross each other nor any tangent curve starting from B_1, whilst for the rest we act in the same way as before.

In that manner we have the *irrigation territory s_2 of B_2, independent of B_1*, containing all those tangent curves starting from B_2 which do not cross any tangent curve starting from B_1. The structure of s_2 is entirely the same as of s_1. Between s_1 and s_2 outward contact may take place on account of the coincidence of an even (resp. odd) side $S_\alpha V_\gamma$ of s_1 and an odd (resp. even) side $S_\beta V_\gamma$ of s_2 along an arc PV_γ, which can expand to an entire side $S_\alpha V_\gamma$ or $S_\beta V_\gamma$ or can reduce itself to the point V_γ. Farthermore s_2 lies entirely in *one* of the residual regions determined by s_1, however in such a way, that between two successive sides of this region which are inwardly pressed together, s_2 can very well penetrate to the vanishing point in which those sides concur. Together s_1 and s_2 contain all tangent curves starting from B_1 or B_2. For the residual regions which are determined on the sphere by s_1 and s_2 together the same properties hold as for the residual regions of s_1 alone.

In one of those residual regions lies B_3 at a finite distance from s_1 and s_2, and in that region we construct the *irrigation territory s_3 of B_3, independent of B_1 and B_2*, containing all those tangent curves starting from B_3 which do not cross any tangent curve starting from B_1 or B_2. Together s_1, s_2, and s_3 contain then all the tangent curves starting from B_1, B_2 or B_3. Outward contact between s_3 and s_1 or s_2 can take place in the same way as between s_1 and s_2.

In a quite analogous way we construct s_4 in one of the residual regions determined by s_1, s_2, and s_3. And in this way we go on. When we have constructed $s_1, s_2, \ldots . s_{m-1}$, then the sphere is not yet quite covered. For, the system of the tangent curves starting from

B_1, B_2, ... B_{m-1} cannot approach B_m within a certain finite distance. But after insertion of s_m the sphere is completely covered, for the set of the tangent curves starting from B_1, B_2, ... B_{m-1}, B_m is identical with the set of *all* tangent curves, so must cover the sphere entirely, and we have proved:

THEOREM 1. *An irrigating field divides the sphere into a finite number of irrigation territories each of which contains in its interior one of the source points.*

A clear example of an irrigating field is the force field of a finite number of positive and negative divergency points [1]).

The notion of irrigating field can be extended in the following manner:

Let be given on the sphere a multiply connected region γ, bounded by a finite number of coherent boundaries, and in γ a finite, continuous vector distribution, which continuity is uniform with the exception of a finite number of points. We then can construct of the region γ exclusive of its boundaries a continuous one-one representation on a sphere β in such a way that to the boundaries of γ correspond on β single points. The tangent curves of γ are thereby represented on a set of simple curves ϱ described in a certain sense. If among these curves ϱ no simple closed curves appear, they determine on β the structure of an irrigating field. In that case we shall call the given vector field in γ likewise an irrigating field.

This more general irrigating field differs thereby from the particular kind first considered that a boundary can play the part of a singular point. We accordingly distinguish *source boundaries, vanishing boundaries,* and *stroking boundaries.* From this ensues that in the more general irrigating field also spirals can appear as tangent curves, namely such whose windings converge uniformly to a source boundary or to a vanishing boundary.

§ 2.

The most general field with a finite number of singular points.

Let there be given an arbitrary finite continuous vector field on the sphere with a finite number of singular points. Let N be one of the singular points, then we shall say that a closed tangent curve *flows round about* N, if it does not contain a singular point, and encloses a region in which lies N but no other singular point. Fartheron we shall say that a closed tangent curve *flows round*

[1]) Compare my paper: "*The force field of the non-Euclidean spaces with positive curvature*", these Proceedings Vol. IX 1, p. 250.

against N, if it contains N but no other singular point, and encloses a region in which no singular point lies.

If there is neither a tangent curve flowing round about N nor a tangent curve flowing round against N, then we shall call N a *naked singular point,* otherwise a *wrapped singular point.*

We shall assume that N is a wrapped singular point and we shall distinguish two cases:

First case. There is no tangent curve flowing round about N. Let then ϱ be a tangent curve flowing round against N and let us agree about an arbitrary tangent curve r inside ϱ, that, when it reaches ϱ, we shall pursue resp. recur it along ϱ, until it reaches N; in this way r also becomes a tangent curve flowing round against N. We can thus fill the inner domain of ϱ with tangent curves flowing round against N and not crossing each other in the same way as in the second communication p. 727 was executed for an elliptic sector.

If we now construct a well-ordered series continued as far as possible of tangent curves flowing round against N, enclosing ϱ and bounding outside ϱ an ever increasing area, then it converges either to a tangent curve flowing round against N, or to a *circumference* consisting of simple closed tangent curves which can contain besides N still other singular points and which possesses all the properties deduced in the second communication p. 720 and 721 for the limiting

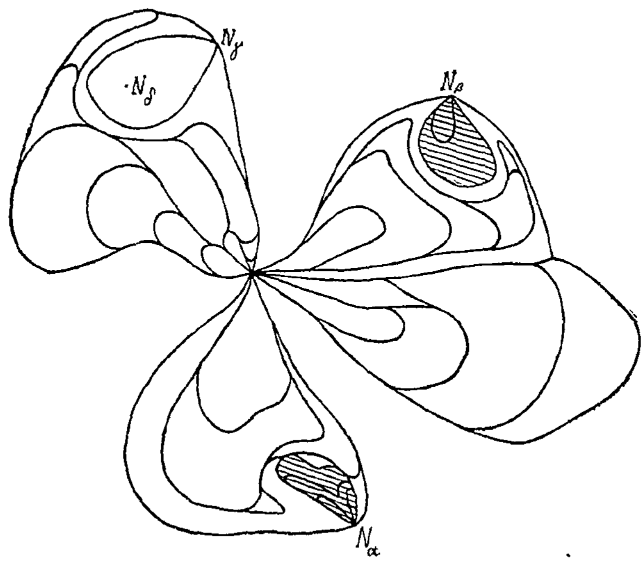

Fig 2. Circumfluence territory with (shaded) additional territories. First case.

circumference of a spiral tangent curve. The inner region of that circumference, which can be entirely filled with tangent curves flowing round against N and not crossing each other, we shall call a *circumfluence sector of* N.

The singular point N can possess an infinite number of circumfluence sectors lying outside each other, but amongst these there are only a finite number, which reach an arbitrarily assumed finite distance from N.

The set of regions covered by the different circumfluence sectors of N we shall call the *circumfluence territory* of N.

We shall now regard of this circumfluence territory those residual regions which are bounded by a tangent curve flowing round against an other singular point N_α, and we shall fill them with tangent curves flowing round against N_ν and not crossing each other. The set of regions filled in this way with tangent curves possesses at each of the points N_α entirely the structure of a circumfluence territory, and we shall call it an *additional circumfluence territory of* N. The point N possesses then only a finite number of additional circumfluence territories.

The circumfluence territory of N determines with its additional territories together a finite number of residual regions on the sphere.

Second case. There exists a tangent curve flowing round about N. Let ϱ be that curve, we then construct from ϱ outwards a well-ordered series continued as far as possible of tangent curves flowing round about N, enclosing ϱ and bounding outside ϱ an ever increasing area. The limit τ_1 to which this series converges is either a tangent curve flowing round about N, or a *circumference* containing singular points, consisting of simple closed tangent curves, and possessing all the properties deduced in the second communication p. 720, 721 for the limiting circumference of a spiral tangent curve.

Let us construct likewise from ϱ inwards a well-ordered series continued as far as possible of tangent curves flowing round about N, enclosed by ϱ, and limiting around N an ever decreasing area, then the limit τ_0 to which this series converges is either the point N, or a tangent curve flowing round about N, or a circumference consisting of a finite or countable set of tangent curves flowing round against N.

If τ_0 is a circumference containing N, we can fill up its inner regions with tangent curves flowing round against N and not crossing each other.

If τ_0 is a tangent curve flowing round about N, there can exist no tangent curve flowing round against N and having with a τ_0

point in common. For then in the terminology of § 3 of the second communication we should possess between N and τ_0 a positive as well as a negative curve of the third kind, from which we could start to fill the inner region of τ_0 with tangent curves not crossing each other. We should then have to find there the number of elliptic sectors equal to the number of hyperbolic sectors; so there would have to be at least *one* hyperbolic sector inside τ_0; this would however give rise to tangent curves flowing round about N and lying inside τ_0, which is excluded.

So if τ_0 is a tangent curve flowing round about N, then there exists inside τ_0 at a finite distance from τ_0 a circumference τ'_0 containing N, consisting of a finite or countable set of tangent curves flowing round against N, and inside which lie all existing tangent curves flowing round against N. If τ'_0 does not reduce itself to the single point N, its inner regions can be filled with tangent curves flowing round against N and not crossing each other.

The tangent curves not crossing each other with which the annular region between τ_0 and τ'_0 can be filled, must on one side either all enter into τ_0 or all converge spirally to τ_0, and on the other side either all enter into τ'_0 or all converge spirally to τ'_0.

In order to fill up the annular region between τ_0 and τ_1 with tangent curves not crossing each other, we construct in it a tangent curve $r_{\frac{1}{2}}$ flowing round about N and reaching from τ_0 and τ_1 a distance as great as possible. Between τ_0 and $r_{\frac{1}{2}}$ we then if possible insert a tangent curve $r_{\frac{1}{4}}$ flowing round about N and reaching from τ_0 and $r_{\frac{1}{2}}$ a distance as great as possible; likewise between $r_{\frac{1}{2}}$ and τ_1 if possible a tangent curve $r_{\frac{3}{4}}$ flowing round about N and reaching from $r_{\frac{1}{2}}$ and τ_1 a distance as great as possible. This inserting process we repeat as often as possible, eventually ω times, and finally we add the limiting curves. We are then sure that no more tangent curves flowing round about N can be inserted, so that eventually the regions between τ_0 and τ_1 remained empty of tangent curves must be annular regions.

Let α be such an annular region bounded by the tangent curves r_p and r_q flowing round about N, then α can be filled with tangent curves not crossing each other, which on one side either all enter into r_p or all converge spirally to r_p, and on the other side either all enter into r_q or all converge spirally to r_q.

The inner region of τ_1, in this manner entirely filled with tangent curves not crossing each other, we shall call the *circumfluence territory* of N.

We shall farther of this circumfluence territory fill each residual

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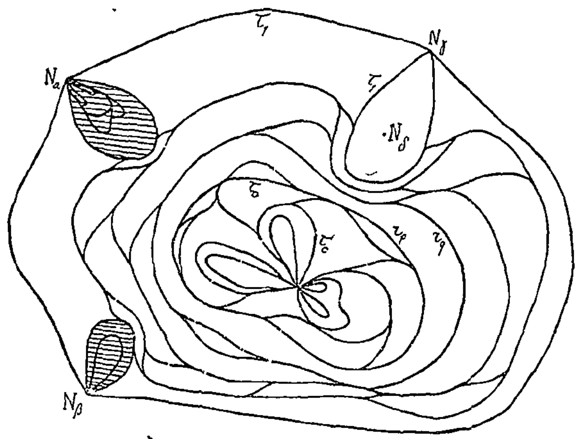

Fig. 3. Circumfluence territory with (shaded) additional territories. Second case.

region, bounded by a tangent curve flowing round against a singular point N_ν, with tangent curves flowing round against N_α and not crossing each other. In this manner we add to the circumfluence territory of N a finite number of *additional circumfluence territories*, after which there remain on the sphere only a finite number of residual regions.

Let us now consider on the sphere a finite and, with the exception of a finite number of points, uniformly continuous vector distribution in a multiply connected region γ with a finite number of coherent boundaries. By a *closed tangent curve* we shall understand here, besides each tangent curve to which we have formerly given this name, each system of n simple tangent arcs not meeting each other and n cyclically ordered boundaries or singular points not contained in a boundary $N_1, N_2, N_3 \ldots N_n$, between which those tangent arcs run consecutively from N_1 to N_2, from N_2 to $N_3, \ldots$ and from N_n to N_1. In particular thus a simple tangent arc whose endpoints lie on one and the same boundary forms together with that boundary a closed tangent curve. Fartheron we shall understand by the *boundaries* of such a field for shortness' sake also the singular points which are not contained in a boundary. Finally we shall call a closed tangent curve not containing a boundary, and enclosing in γ a region in which lies N but no other boundary, *a tangent curve flowing round about* N, and we shall call a closed tangent curve containing N but no other boundary, and enclosing in γ a region in which lies no boundary, *a tangent curve flowing round against* N. *Naked* and *wrapped boundaries* we then define analogously as before naked and wrapped singular points.

For a wrapped boundary the circumfluence territory can be constructed in the same way as was done above for a wrapped singular point; the whole of its structure undergoes in this more general case no change, we only have to replace closed tangent curves in the narrower sense by closed tangent curves in the wider sense. The filling with tangent curves not crossing each other and the completion of the territories by means of its additional territories needs no modification either.

We shall understand by the *order* of the field twice the number of its naked boundaries plus three times the number of its wrapped boundaries.

We shall now start from a finite and, with the exception of only a finite number of points, uniformly continuous vector field in a region of the sphere with a finite number of coherent boundaries, each of which either reduces itself to a single point, or consists of tangent arcs turning one of their sides to the field, whilst in the latter case we assume that each fundamental series of consecutive points in a segment of a boundary determines only *one* limiting point, which property we express by calling the boundary *simple*. So the appearance of spirals in the boundaries is excluded.

We shall indicate two operations, both of which reduce this field to a finite number of fields of the same kind but of a lower order:

First reducing operation: We construct in the given field such a closed tangent curve which together with each of the two partial fields determined by it contains at least two of the boundaries of the given field.

Then namely each of the two partial fields is of a lower order than the original field.

Second reducing operation: we construct to a wrapped boundary the circumfluence territory with its eventual additional territories.

Then namely each of the residual fields is of a lower order than the original field.

It is clear that after a finite number of applications of these reducing operations either nothing of the original field is left or there remain only such fields to which neither of the two operations can any more be applied.

Then however in these residual fields there exists no closed tangent curve, so that they are irrigating fields.

If these last remaining residual fields are lacking, then the original field can be divided by *simple* boundaries consisting of tangent arcs into a finite number of circumfluence territories with additional territories which property we shall express by calling it a *circumfluent field.*

The circumfluent field can be regarded as the counterpiece to the irrigating field analysed in § 1.

A clear example of a circumfluent field is the force field without divergences of a finite number of positive and negative rotation points. [1])

We now have proved:

THEOREM 2. *A finite continuous vector field on the sphere with a finite number of singular points can be divided by simple boundaries consisting of tangent arcs into a finite number of irrigating fields and a finite number of circumfluence territories.*

At the same time we notice that among the tangent curves not crossing each other, with which in the preceding pages we have filled the field, spirals *cannot* appear in the *boundaries* of the irrigating fields or circumfluence territories meant in theorem 2, and in their *interior* exclusively in the following two ways:

1st. A circumfluence territory of the second kind can contain annular regions filled with spirals.

2nd. An irrigating field can possess source boundaries or vanishing boundaries round about which all tangent curves arrive resp. depart spirally.

§ 3.

The theorem of the invariant point on the sphere.

In the first communication on this subject (these Proceedings Vol. XI 2) we have on page 857 brought an arbitrary continuous one-one transformation of the sphere in itself into relation with the vector distribution for which in each point the vector direction is determined by the shortest arc of principal circle joining that point with its image point, for which distribution appear as singular points: 1st. the points invariant for the transformation. 2nd. the points having their antipodic points as their image points. The singular points of the latter kind form for transformations with inversion of the indicatrix as well as for transformations with invariant indicatrix a closed set of points of the most general kind which makes it pretty well

[1]) Compare my paper quoted above: "*The force field of the non-Euclidean spaces with positive curvature*".

impossible to deduce out of the properties of the vector distribution, either by means of theorem 2 of the first communication, or by means of theorem 8 of the second communication, the existence of at least *one* invariant point for transformations with invariant indicatrix.

The difficulty caused by this inconvenient set of points disappears however for an other vector distribution deduced from the transformation.

To construct this distribution we bring through each point P a circle containing its image point P' and a fixed point O, and we determine the vector direction in P by the arc of circle PP' *not* containing O. Let Q be the point having O as its image point, then as singular points of this vector distribution appear 1st. the point O. 2nd. the point Q. 3rd. the points invariant for the transformation.

If this vector distribution has an infinite number of singular points, then there are certainly points invariant for the transformation; so we assume in the following that the number of singular points is finite, and we investigate first the nature of the singularity in O.

For a point P in sufficient proximity of O the vector direction differs indefinitely little from the direction of the geodetic arc of circle OP. So by a circuit of a small circle about O the total angle which the vector turns with respect to the tangent to the small circle is zero, *so that when reduced the singularity gives rise to a radiating point.*

To investigate the nature of the singularity in Q, we represent the sphere stereographically on a Euclidean plane in such a way that O represents the infinite of the plane. Then in this plane the vector distribution is determined in each point by the straight line segment joining the point with its image point.

In the Euclidean plane the image of an infinitesimal circle about Q is an infinitely large circle; the infinitesimal circle and the infinitely large circle possess for transformations with invariant indicatrix *opposite senses of circuit*; for transformations with inversion of the indicatrix *equal senses of circuit.*

In the former case the vector describes in a circuit of the infinitesimal circle an angle 2π *in a sense opposite to the circuit*; in the latter case an angle 2π *in the same sense as that of the circuit.*

So when reduced the singularity in Q gives rise for transformations with invariant indicatrix to a reflexion point, for transformations with inversion of the indicatrix to a radiating point.

Thus the two radiating points, which according to theorem 8 of the second communication (p. 734) must be present in the reduced distribution, appear for a transformation with inversion of the indicatrix in the points O and Q; for a transformation with invariant indicatrix however the second radiating point can be furnished only by a point invariant for the transformation, *which therefore must necessarily exist.*

§ 4.

The index relation on the sphere for a finite number of singular points.

We shall now discuss the questions whether the number of singular points of a finite continuous vector distribution on the sphere, which according to theorem 2 of the first communication cannot be zero, is arbitrary for the rest, and farther whether the structure of the singular points, which according to theorem 8 of the second communication is not entirely free, is liable to still other restrictions than those expressed in that theorem.

These questions can be fully answered by means of the following reasoning, which is analogous to the proof of EULER's law, and which was indicated to me by Prof. HADAMARD.

The total angle which for a finite stereographic representation of the inner region of a simple closed curve enveloping only *one* singular point on a Euclidean plane the vector describes by a circuit in the sense of that circuit, and which according to theorem 5 of the second communication (page 731) is equal to $\pi\ (2 + n_1 - n_2)$, where n_1 represents the number of elliptic sectors, n_2 the number of hyperbolic sectors of the singular point, can be written in the form $2k\pi$, where k is an integer, which we call the *index* [1]) of the singular point.

For a simple closed curve, enveloping n singular points with indices $k_1, k_2, k_3, \ldots\ldots k_n$, the total angle which, for a finite stereographic representation of the inner domain of that curve on a Euclidean plane, the vector describes by a circuit in the sense of that circuit, is equal to $2\pi\,(k_1 + k_2 + \ldots\ldots k_n)$, as is immediately evident when we divide the inner domain under observation by means of arcs of simple curve into n inner domains of such simple closed curves, which each envelop only *one* of the singular points.

[1]) This expression is used (not for the singular point itself but for a curve by which it is enclosed) by POINCARÉ: "*Sur les courbes définies par une équation différentielle*", 1er mémoire, Journ. de Math. (3) 7, p. 400. The univalent continuous vector distributions treated there are of a particular algebraic kind, so that only indices $+1$ and -1 appear for the singular points.

We now make on the sphere a circuit along a certain principal circle on which lies no singular point; the total angle, which in the sense of a certain indicatrix on the sphere the vector direction describes by that circuit with respect to the tangent direction, is equal to $2h\pi$, where h is an integer.

The sense of that circuit is with respect to one of the hemispheres, into which the sphere is divided by that principal circle, the same as the sense of the indicatrix, with respect to the other opposite to the sense of the indicatrix; so for a circuit of the first hemisphere the vector describes with respect to the tangent direction an angle $2h\pi$ in the sense of the circuit, for a circuit of the second hemisphere an angle $2h\pi$ opposite to the sense of the circuit.

The total angle which, for finite stereographic representation of the first resp. the second hemisphere on a Euclidean plane, the vector describes by a circuit in the sense of that circuit, is thus equal to $2(1+h)\pi$ resp. $2(1-h)\pi$.

If in the first hemisphere lie m singular points with indices $k_1, k_2, \ldots k_m$, in the second hemisphere $n-m$ singular points with indices $k_{m+1}, k_{m+2}, \ldots k_n$, we have

$$\begin{array}{c} k_1 + k_2 + \ldots k_m = 1 + h, \\ k_{m+1} + k_{m+2} + \ldots k_n = 1 - h, \\ \hline k_1 + k_2 + \ldots + k_{n-1} + k_n = 2, \end{array}$$

so that the sum of the indices of the singular points is equal to 2, a generalisation of the relation deduced by POINCARÉ for the particular case treated by him [1]), whilst the structure of the singular points is submitted to the following restrictive property:

THEOREM 3. *Twice the number of singular points plus the number of elliptic sectors is equal to the number of hyperbolic sectors plus four.*

The necessary existence of at least one singular point before reduction as well as of at least two radiating points after reduction lies included in this theorem and finds there its simplest proof.

We shall finally show that the set of singular points (supposed finite) is submitted to no other restriction than the one expressed in theorem 3.

Let us namely assume an arbitrary finite set of points as singular points, let us enclose them each by a suchlike simple closed curve that these curves do not intersect each other, and let us give inside and on these curves to the vector field a structure satisfying theorem 3 but for the rest arbitrary. We must then show that the outer domain

[1]) l.c. p. 405.

of these curves can be filled up with a finite continuous vector distribution *without* singular points and passing into the already existing ones.

To that end we take for the closed curves a certain cyclic order and join each of them with the succeeding one by such an arc of simple curve that these arcs do not intersect each other, so that on the sphere two free regions γ_1 and γ_2, bounded by simple closed curves, are determined. We then construct along the inserted arcs of curve suchlike finite continuous vector distributions without singular points and passing into the existing ones that the total angle, which for finite stereographic representation of γ_1 on a Euclidean plane the vector describes in a circuit, is zero. Then γ_1 can be filled, in the manner indicated in the second communication p. 732, 733, with a finite continuous vector distribution *without* singular points and passing into the existing ones.

As now however the singularities have been chosen in such a way that they satisfy theorem 3, the vector describes in a circuit of the complementary domain of γ_2, stereographically represented on a finite region, a total angle 4π in the sense of the circuit; thus by a circuit of the region γ_2 itself, when stereographically represented on a finite region, a total angle zero. Therefore γ_2 also can be filled with a finite continuous vector distribution *without* singular points and passing on its boundary into the existing ones, with which the lack of other restrictions than those expressed in theorem 3, has been proved.

As for the singular points (supposed to form a finite set) of a finite continuous vector distribution in the Euclidean plane, neither their number, nor their structure is submitted to any restriction.

ERRATUM.

In the first communication on this subject, these Proceedings Vol. XI 2, p. 856, l. 3 and 7 from top

for: recure it, meets   read: recur it, it meets

Zoology. — "*The saccus vasculosus of fishes a receptive nervous organ and not a gland*". By Prof. J. BOEKE and K. W. DAMMERMAN. (Communicated by Prof. A. A. W. HUBRECHT).

(Communicated in the meeting of May 28, 1910).

In 1901 one of us came to the conclusion, based on the study of the development and of the histological structure of the saccus