Mathematics. — *"On the final Integral occurring in* Prof. WIND's *paper: "Diffraction of a single pulse wave by a slit, according to* KIRCHOFF's *theory."* By Prof. W. KAPTEYN.

1. In Prof. WIND's paper the problem is reduced to the integral

$$I = \int_0^\infty \left[R_e \, as \frac{\mu}{\sqrt{x}} - R_e \, as \frac{\mu}{\sqrt{x-1}} - R_e \, as \frac{b}{\sqrt{x}} + R_e \, as \frac{b}{\sqrt{x-1}} \right]^2 dx$$

wherein $-b$ is written instead of ν. R_e means the real part of the function which follows and *as* represents the function sin^{-1}. The object of this paper is to reduce the preceding integral so that it is ready for numerical computation.

Let

$$R_e \, as \frac{\mu}{\sqrt{x}} = A, \quad R_e \, as \frac{\mu}{\sqrt{x-1}} = B \quad R_e \, as \frac{b}{\sqrt{x}} = A' \quad R_e \, as \frac{b}{\sqrt{x-1}} = B'$$

we have

$$I = I(\mu, b) = \int_0^\infty (A-B)^2 \, dx + \int_0^\infty (A'-B')^2 \, dx -$$

$$- 2\int_0^\infty (AA' - AB' - A'B + BB') \, dx \quad . \quad . \quad . \quad . \quad (1)$$

For $b = 0$ and $\mu = 0$ this reduces to

$$I(\mu, 0) = I(\mu) = \int_0^\infty (A - B)^2 \, dx$$

$$I(0, b) = I(0, -b) = I(b) = \int_0^\infty (A' - B')^2 \, dx$$

thus, if we put

$$K = AA' - AB' - A'B + BB'$$

the equation (1) may be written

$$I(\mu, b) = I(\mu) + I(b) - 2\int_0^\infty K \, dx \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If we call μ, b and $\mu, -b$ corresponding points, it is evident that the values of the integral in corresponding points may be deduced from one another by the relation

$$I(\mu, -b) = 2\, I(\mu) + 2\, I(b) - I(\mu, b) \quad . \quad . \quad . \quad . \quad (3)$$

for, $I(b)$ being the same as $I(-b)$, we have

$$I(\mu, -b) = I(\mu) + I(b) + 2\int_0^\infty K\,dx$$

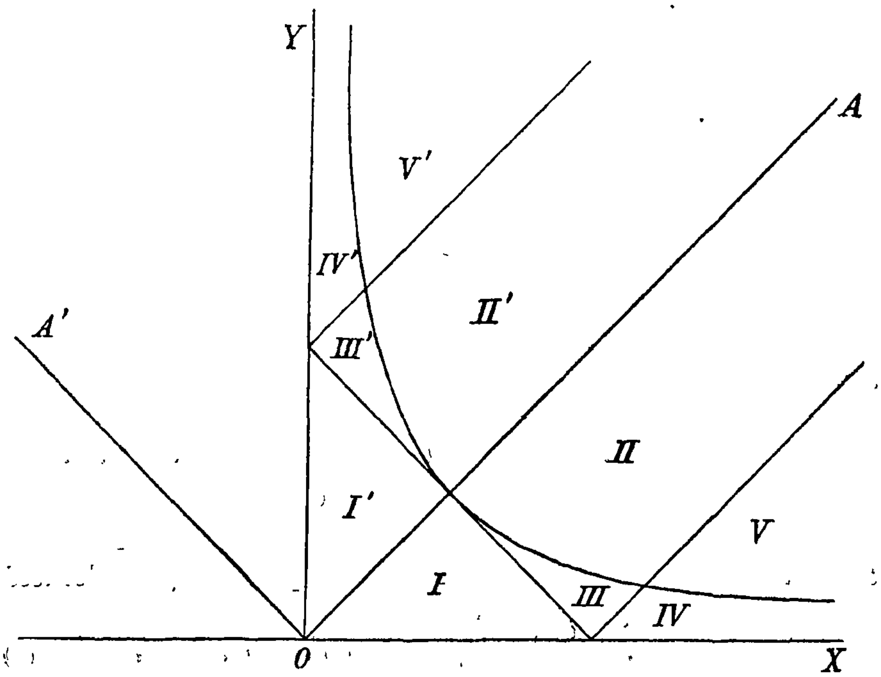
and this equation together with (2) gives the equation (3).

We may therefore limit our investigation to positive values of b only.

2. We must now distinguish the following cases

I.	$\mu^2 < 1$	$b^2 < 1$	$\mu^2 < b^2 + 1$
II.	$\mu^2 > 1$	$b^2 < 1$	$\mu^2 > b^2 + 1$
III.	$\mu^2 > 1$	$b^2 < 1$	$\mu^2 < b^2 + 1$
IV.	$\mu^2 > 1$	$b^2 > 1$	$\mu^2 < b^2 + 1$
V.	$\mu^2 > 1$	$b^2 > 1$	$\mu^2 > b^2 + 1.$

To represent these cases by a figure, we draw from the origin of a rectangular system of axes XOY the lines OA and OA' so that $\angle AOY = \angle A'OY = \varphi$. Considering these lines as the limits of the slit and remembering the signification of μ and b, we get for the coordinates of a point P of the plane



$$x = \frac{1}{4}(\mu + b) \qquad y = \frac{1}{4}(\mu - b)\, ctg\, \varphi .$$

The limits

$$\mu^2 = 1 \qquad b^2 = 1 \qquad \mu^2 = b^2 + 1$$

may therefore be represented by the lines

$$4\,(x + y\, tg\, \varphi) = 1, \qquad 4\,(x - x\, tg\, \varphi) = 1$$

and the hyperbola

$$16\, xy\, tg\, \varphi = 1.$$

These limits divide the plane in different regions which are represented in our figure by the numbers corresponding with the preceding cases. The accentuated numbers are inscribed in the regions where b is negative.

3. In the first place we shall consider the integral $I(\mu)$.

If $z > 1$, we have

$$as\, z = \frac{\pi}{2} - il\,(z + \sqrt{z^2 - 1})$$

so

$$R_e\; as\, z = \frac{\pi}{2}.$$

Therefore, according to the values of μ

			$\mu < 1$					$\mu > 1$	
	x		A	B		x		A	B
0	until	μ^2	$\frac{\pi}{2}$	0	0	until	1	$\frac{\pi}{2}$	0
μ^2	,,	1	$as \frac{\mu}{\sqrt{x}}$	0	1	,,	μ^2	0	0
1	,,	$1+\mu^2$	$as \frac{\mu}{\sqrt{x}}$	$\frac{\pi}{2}$	μ^2	,,	$1+\mu^2$	$as \frac{\mu}{\sqrt{x}}$	$\frac{\pi}{2}$
$1+\mu^2$	,,	∞	$as \frac{\mu}{\sqrt{x}}$	$as \frac{\mu}{\sqrt{x-1}}$	$1+\mu^2$	,,	∞	$as \frac{\mu}{\sqrt{x}}$	$as \frac{\mu}{\sqrt{x-1}}$

So if $\mu < 1$ the value of the integral reduces to

$$I(\mu) = \frac{\pi^2}{2}\mu^2 - \pi \int_1^{1+\mu^2} as \frac{\mu}{\sqrt{x}}\, dx + 2\int_{\mu^2}^{\infty} \left(as \frac{\mu}{\sqrt{x}}\right)^2 dx - 2\int_{1+\mu^2}^{\infty} as \frac{\mu}{\sqrt{x}}\, as \frac{\mu}{\sqrt{x-1}}\, dx \quad (4)$$

and if $\mu > 1$, to

$$I(\mu) = \frac{\pi^2}{2} - \pi \int_{\mu^2}^{1+\mu^2} as \frac{\mu}{\sqrt{x}}\, dx + 2\int_{\mu^2}^{\infty} \left(as \frac{\mu}{\sqrt{x}}\right)^2 dx - 2\int_{1+\mu^2}^{\infty} as \frac{\mu}{\sqrt{x}}\, as \frac{\mu}{\sqrt{x-1}}\, dx \quad (5)$$

In both these equations the second and third integrals of the second members are infinite, their sum however is finite. We may escape this difficulty by considering the infinite limits of those integrals as $Lim \frac{1}{\varepsilon^2}$ for ε approaching to zero.

Differentiating both members of (4) we find

$$\frac{\partial}{\partial\mu}\int_1^{1+\mu^2} as\frac{\mu}{\sqrt{x}}\,dx = 2\mu\, as\frac{u}{\sqrt{1+\mu^2}} + 2 - 2\sqrt{1-\mu^2}$$

$$\frac{\partial}{\partial\mu}\int_{\mu^2}^{\infty}\left(as\frac{\mu}{\sqrt{x}}\right)^2 dx = -\frac{\pi^2}{2}\mu + 4\mu - 4\mu\, l\,\mu\varepsilon$$

$$\frac{\partial}{\partial\mu}\int_{1+\mu^2}^{\infty} as\frac{\mu}{\sqrt{x}}\, as\frac{\mu}{\sqrt{x-1}}\,dx = -\pi\mu\, as\frac{\mu}{\sqrt{1+\mu^2}} +$$

$$+\int_{1+\mu^2}^{\infty} as\frac{\mu}{\sqrt{x-1}}\,\frac{dx}{\sqrt{x-\mu^2}} + \int_{1+\mu^2}^{\infty} as\frac{\mu}{\sqrt{x}}\,\frac{dx}{\sqrt{x-1-\mu^2}}$$

We shall now transform the two last integrals. Integrating by parts we obtain

$$\int_{1+\mu^2}^{\infty} as\frac{\mu}{\sqrt{x-1}}\,\frac{dx}{\sqrt{x-\mu^2}} = 2\mu - \pi + \mu\int_{1+\mu^2}^{\infty}\frac{\sqrt{x-\mu^2}}{(x-1)\sqrt{x-1-\mu^2}}\,dx$$

$$\int_{1+\mu^2}^{\infty} as\frac{\mu}{\sqrt{x}}\,\frac{dx}{\sqrt{x-1-\mu^2}} = 2\mu + \mu\int_{1+\mu^2}^{\infty}\frac{\sqrt{x-1-\mu^2}}{x\sqrt{x-\mu^2}}\,dx$$

where

$$\int_{1+\mu^2}^{\infty}\frac{\sqrt{x-\mu^2}}{(x-1)\sqrt{x-1-\mu^2}}\,dx = \int_{1+\mu^2}^{\infty}\frac{dx}{\sqrt{(x-\mu^2)(x-1-\mu^2)}} +$$

$$+ (1-\mu^2)\int_{1+\mu^2}^{\infty}\frac{dx}{(x-1)\sqrt{(x-\mu^2)(x-1-\mu^2)}}$$

$$\int_{1+\mu^2}^{\infty}\frac{\sqrt{x-1-\mu^2}}{x\sqrt{x-\mu^2}}\,dx = \int_{1+\mu^2}^{\infty}\frac{dx}{(\sqrt{x-\mu^2)(x-1-\mu^2)}} -$$

$$- (1+\mu^2)\int_{1+\mu^2}^{\infty}\frac{dx}{x\sqrt{(x-\mu^2)(x-1-\mu^2)}}$$

and

$$\int_{1+\mu^2}^{\infty} \frac{dx}{\sqrt{(x-\mu^2)(x-1-\mu^2)}} = l\,\frac{4}{\varepsilon^2}$$

$$\int_{1+\mu^2}^{\infty} \frac{dx}{(x-1)\sqrt{(x-\mu^2)(x-1-\mu^2)}} = \frac{2}{\mu\sqrt{1-\mu^2}}\, as\, \sqrt{1-\mu^2}$$

$$\int_{1+\mu^2}^{\infty} \frac{dx}{x\sqrt{(x-\mu^2)(x-1-\mu^2)}} = \frac{2}{\mu\sqrt{1+\mu^2}}\, l\,(\sqrt{1+\mu^2} + \mu)$$

Hence finally

$$\frac{dI(\mu)}{d\mu} = -\,8\mu\, l\, 2\mu + 2\mu\sqrt{1-\mu^2} - 4\sqrt{1-\mu^2}\, as\, \sqrt{1-\mu^2} + $$
$$+\, 4\sqrt{1+\mu^2}\, l\,(\sqrt{1+\mu^2} + \mu)$$

or

$$\frac{dI(\mu)}{d\mu} = -\,8\mu\, l\, 2\mu + 4\sqrt{1-\mu^2}\, as\, \mu + 4\sqrt{1+\mu^2}\, l\,(\sqrt{1+\mu^2}+\mu).$$

If now we integrate this equation again we shall obtain $I(\mu)$ in the required form. For

$$\int \mu\, l\, 2\mu\, d\mu = \frac{\mu^2}{2}\, l\,(2\mu) - \frac{\mu^2}{4}$$

$$\int \sqrt{1-\mu^2}\, as\, \mu\, d\mu = \int \left(\frac{1}{2\sqrt{1-\mu^2}} + \frac{1-2\mu^2}{2\sqrt{1-\mu^2}}\right) as\, \mu\, d\mu =$$
$$= \frac{1}{4}\,(as\,\mu)^2 + \frac{1}{2}\,\mu\sqrt{1-\mu^2}\, as\, \mu - \frac{1}{4}\,\mu^2$$

$$\int \sqrt{1+\mu^2}\, l\,(\sqrt{1+\mu^2} + \mu)\, d\mu =$$
$$\int \left(\frac{1}{2\sqrt{1+\mu^2}} + \frac{1+2\mu^2}{2\sqrt{1+\mu^2}}\right) l\,(\sqrt{1+\mu^2} + \mu)\, d\mu =$$
$$= \frac{1}{4}\, l^2\,(\sqrt{1+\mu^2} + \mu) + \frac{1}{2}\,\mu\sqrt{1+\mu^2}\, l\,(\sqrt{1+\mu^2} + \mu) - \frac{1}{4}\,\mu^2$$

So because $I(0) = 0$

$$I(\mu) = -\,4\mu^2\, l\, 2\mu + 2\mu\sqrt{1-\mu^2}\, as\, \mu + (as\,\mu)^2 +$$
$$+\, 2\mu\sqrt{1+\mu^2}\, l\,(\mu+\sqrt{\mu^2+1}) + l^2\,(\mu+\sqrt{\mu^2+1}) \qquad (\mu < 1) \qquad (6)$$

In the same way differentiating (5) we get

$$\frac{dI(\mu)}{d\mu} = -\,8\mu\, l\, 2\mu + 4\sqrt{\mu^2-1}\, l\,(\mu+\sqrt{\mu^2-1}) + 4\sqrt{\mu^2+1}\, l\,(\mu+\sqrt{\mu^2+1}),$$

which gives by integration

$$I(\mu)=\frac{\pi^2}{4}-4\mu^2\, l\, 2\mu+2\mu\sqrt{\mu^2-1}\; l\,(\mu+\sqrt{\mu^2-1})-l^2\,(\mu+\sqrt{\mu^2-1})+$$
$$+2\mu\sqrt{1+\mu^2}\; l\,(\mu+\sqrt{\mu^2+1})+l^2\,(\mu+\sqrt{\mu^2+1}).\qquad(\mu>1)\qquad(7)$$

4. In order to simplify the computation of the numercal, values, it seems important to expand $I(\mu)$ in a convergent series.

Writing therefore in equation (6)

$$lg\,(\mu-\sqrt{1+\mu^2})=i\;as\,\frac{\mu}{i}$$

we may introduce the known expansions of $as\,z$ and $(as\,z)^2$. In this way we easily obtain, if $(\mu<1)$

$$I(\mu)=-4\mu^2\, l\, 2\mu+6\mu^2-\frac{8}{15}\left[\frac{\mu^6}{3}+\frac{4.6}{7.9}\frac{\mu^{10}}{5}+\frac{4.6.\;8.10}{7.9.11.13}\cdot\frac{\mu^{14}}{7}+\cdot\right]\;(8)$$

a series which is sufficiently convergent for all values from $\mu=0$ to $\mu=0.7$.

If $\mu>1$, we have, putting: $\mu=\frac{1}{m}$

$$I(\mu)=\frac{\pi^2}{4}-\frac{4}{m^2}\, l\,\frac{2}{m}+\frac{2}{m^2}\sqrt{1+m^2}\; l\;\frac{1+\sqrt{1+m^2}}{m}+$$
$$+\frac{2}{m^2}\sqrt{1-m^2}\; l\;\frac{1+\sqrt{1-m^2}}{m}-l^2\,\frac{1+\sqrt{1-m^2}}{m}+l^2\,\frac{1+\sqrt{1+m^2}}{m}$$

and differentiating this equation

$$\frac{dI(\mu)}{dm}=\frac{8}{m^3}\, l\,\frac{2}{m}-\frac{4\sqrt{1+m^2}}{m^3}\, l\,\frac{1+\sqrt{1+m^2}}{m}-\frac{4\sqrt{1-m^2}}{m^3}\, l\,\frac{1+\sqrt{1-m^2}}{m}$$

or

$$\frac{dI(\mu)}{dm}=\frac{4}{m^3}\, l\,\frac{2}{m}\,[2-\sqrt{1+m^2}-\sqrt{1-m^2}]-$$
$$-\frac{4}{m^3}\left[\sqrt{1+m^2}\; l\,\frac{1+\sqrt{1+m^2}}{2}+\sqrt{1-m^2}\; l\;\frac{1+\sqrt{1-m^2}}{2}\right].$$

Writing

$$\varphi(m)=l\,\frac{1+\sqrt{1+m^2}}{2}\qquad \psi(m)=l\,\frac{1+\sqrt{1-m^2}}{2}$$

we get by differentation

$$\varphi'(m)=\frac{1}{m}\left(1-\frac{1}{\sqrt{1+m^2}}\right)=\frac{1}{2}m-\frac{1.3}{2\,4}m^3+\frac{1.3.5}{2.4.6}m^5-\frac{1.3.5\,7}{2.4.6.8}m^7+.$$

$$\psi'(m)=\frac{1}{m}\left(1-\frac{1}{\sqrt{1-m^2}}\right)=-\frac{1}{2}m-\frac{1.3}{2.4}m^3-\frac{1\,3\,5}{2.4.6}m^5-\frac{1.3\,5.7}{2\,4.6.8}m^7-\,.\,.$$

and by integrating, the constants being zero

$$\varphi(m) = \frac{1}{2}\cdot\frac{m^2}{2} - \frac{1.3}{2.4}\cdot\frac{m^4}{4} + \frac{1.3.5}{2.4.7}\cdot\frac{m^6}{6} - \frac{1.3.5.7}{2.4.6.8}\cdot\frac{m^8}{8} + \dots$$

$$\psi(m) = -\frac{1}{2}\cdot\frac{m^2}{2} - \frac{1.3}{2.4}\cdot\frac{m^4}{4} - \frac{1.3.5}{2.4.6}\cdot\frac{m^6}{6} - \frac{1.3.5.7}{2.4.6.8}\cdot\frac{m^8}{8} - \dots$$

Thus

$$\frac{dI(\mu)}{dm} = 8\left[\frac{1.1}{2.4}m + \frac{1.1.3.5}{2.4.6.8}m^5 + \frac{1.1.3.5.\ 7.\ 9}{2.4\ 6.8.10.12}m^9 + \dots\right] l\,\frac{2}{m} -$$

$$- 8\left[\frac{1}{32}m + \frac{59}{3072}m^5 + \frac{1417}{122880}m^9 + \dots\right]$$

$$I(\mu) = \frac{\pi^2}{4} + \left(\frac{m^2}{2} + \frac{5}{96}m^6 + \frac{21}{1280}m^{10} + \dots\right) lg\,\frac{2}{m} +$$

$$+ \frac{1}{8}m^2 - \frac{13}{768}m^6 - \frac{233}{30720}m^{10} + \dots$$

and finally

$$I(\mu) = \frac{\pi^2}{4} + \left(\frac{1}{2\mu^2} + \frac{5}{96\mu^6} + \frac{21}{1280\mu^{10}} + \dots\right) lg\,2\mu +$$

$$+ \frac{1}{8\mu^2} - \frac{13}{768\mu^6} - \frac{233}{30720\mu^{10}} \dots \dots \quad (9)$$

This series is sufficiently convergent for all values of $\mu \geqq 2$.

5. Proceeding now to the integral $\int_0^\infty K\,dx$ in the different cases we immediately get the following results.

Case I. $b^2 < 1 \quad \mu^2 < 1 \quad \mu^2 < b^2 + 1$.

x	A	A'	B	B'
0 *until* b^2	$\frac{\pi}{2}$	$\frac{\pi}{2}$	0	0
b^2 ,, μ^2	$\frac{\pi}{2}$	*as* $\frac{b}{\sqrt{x}}$	0	0
μ^2 ,, 1	*as* $\frac{\mu}{\sqrt{x}}$	*as* $\frac{b}{\sqrt{x}}$	0	0
1 ,, $1 + b^2$	*as* $\frac{\mu}{\sqrt{x}}$	*as* $\frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
$1 + b^2$,, $1 + \mu^2$	*as* $\frac{\mu}{\sqrt{x}}$	*as* $\frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	*as* $\frac{b}{\sqrt{x-1}}$
$1 + \mu^2$,, ∞	*as* $\frac{\mu}{\sqrt{x}}$	*as* $\frac{b}{\sqrt{x}}$	*as* $\frac{\mu}{\sqrt{x-1}}$	*as* $\frac{b}{\sqrt{x-1}}$

$$\int_0^\infty K\,dx = \frac{\pi^2}{2} b^2 + \pi \int_{b^2}^{\mu^2} as \frac{b}{\sqrt{x}}\,dx - \frac{\pi}{2}\int_1^{1+\mu^2} as \frac{b}{\sqrt{x}}\,dx - \frac{\pi}{2}\int_1^{1+b^2} as \frac{\mu}{\sqrt{x}}\,dx +$$

$$+ 2\int_{\mu^2}^\infty as \frac{\mu}{\sqrt{x}}\, as \frac{b}{\sqrt{x}}\,dx - \int_{1+\mu^2}^\infty as \frac{b}{\sqrt{x}}\, as \frac{\mu}{\sqrt{x-1}}\,dx - \int_{1+b^2}^\infty as \frac{\mu}{\sqrt{x}}\, as \frac{b}{\sqrt{x-1}}\,dx.$$

Case II. $b^2 < 1 \quad \mu^2 > 1 \quad \mu^2 > b^2 + 1.$

x	A	A'	B	B'
0 *until* b^2	$\frac{\pi}{2}$	$\frac{\pi}{2}$	0	0
b^2 ,, 1	$\frac{\pi}{2}$	$as \frac{b}{\sqrt{x}}$	0	0
1 ,, $1 + b^2$	$\frac{\pi}{2}$	$as \frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
$1 + b^2$,, μ^2	$\frac{\pi}{2}$	$as \frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	$as \frac{b}{\sqrt{x-1}}$
μ^2 ,, $1 + \mu^2$	$as \frac{\mu}{\sqrt{x}}$	$as \frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	$as \frac{b}{\sqrt{x-1}}$
$1 + \mu^2$,, ∞	$as \frac{\mu}{\sqrt{x}}$	$as \frac{b}{\sqrt{x}}$	$as \frac{\mu}{\sqrt{x-1}}$	$as \frac{b}{\sqrt{x-1}}$

$$\int_0^\infty K\,dx = \frac{\pi^2}{4} b^2 + \frac{\pi}{2}\int_{b^2}^{1} as \frac{b}{\sqrt{x}}\,dx - \frac{\pi}{2}\int_{\mu^2}^{1+\mu^2} as \frac{b}{\sqrt{x}}\,dx + \frac{\pi}{2}\int_{\mu^2}^{1+\mu^2} as \frac{b}{\sqrt{x-1}}\,dx +$$

$$+ 2\int_{\mu^2}^\infty as \frac{\mu}{\sqrt{x}}\, as \frac{b}{\sqrt{x}}\,dx - \int_{1+\mu^2}^\infty as \frac{b}{\sqrt{x}}\, as \frac{\mu}{\sqrt{x-1}}\,dx - \int_{\mu^2}^\infty as \frac{\mu}{\sqrt{x}}\, as \frac{b}{\sqrt{x-1}}\,dx.$$

Case III $b^2 < 1 \qquad \mu^2 > 1 \qquad \mu^2 < 1 + b^2.$

x	A	A'	B	B'
0 *until* b^2	$\frac{\pi}{2}$	$\frac{\pi}{2}$	0	0
b^2 ,, 1	$\frac{\pi}{2}$	$as \frac{b}{\sqrt{x}}$	0	0
1 ,, μ^2	$\frac{\pi}{2}$	$as \frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$

μ^2 *until* $1+b^2$	$as\,\frac{\mu}{\sqrt{x}}$	$as\,\frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
$1+b^2$,, $1+\mu^2$	$as\,\frac{\mu}{\sqrt{x}}$	$as\,\frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	$as\,\frac{b}{\sqrt{x-1}}$
$1+\mu^2$,, ∞	$as\,\frac{\mu}{\sqrt{x}}$	$as\,\frac{b}{\sqrt{x}}$	$as\,\frac{\mu}{\sqrt{x-1}}$	$as\,\frac{b}{\sqrt{x-1}}$

$$\int_0^\infty K dx = \frac{\pi^2}{4}(1+2b^2-\mu^2) + \frac{\pi}{2}\int_{b^2}^{1} as\,\frac{b}{\sqrt{x}}\,dx - \frac{\pi}{2}\int_{\mu^2}^{1+\mu^2} as\,\frac{b}{\sqrt{x}}\,dx - \frac{\pi}{2}\int_{\mu^2}^{1+b^2} as\,\frac{\mu}{\sqrt{x}}\,dx +$$

$$+\frac{\pi}{2}\int_{1+b^2}^{1+\mu^2} as\,\frac{b}{\sqrt{x-1}}\,dx + 2\int_{\mu^2}^{\infty} as\frac{\mu}{\sqrt{x}}as\frac{b}{\sqrt{x}}dx - \int_{1+\mu^2}^{\infty} as\frac{b}{\sqrt{x}}as\frac{\mu}{\sqrt{x-1}}dx - \int_{1+b^2}^{\infty} as\frac{\mu}{\sqrt{x}}as\frac{b}{\sqrt{x-1}}dx.$$

Case IV $b^2 > 1 \quad \mu^2 > 1 \quad \mu^2 < b^2+1.$

x	A	A'	B	B'
0 *until* 1	$\frac{\pi}{2}$	$\frac{\pi}{2}$	0	0
1 ,, b^2	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
b^2 ,, μ^2	$\frac{\pi}{2}$	$as\,\frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
μ^2 ,, b^2+1	$as\,\frac{\mu}{\sqrt{x}}$	$as\,\frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
b^2+1 ,, μ^2+1	$as\,\frac{\mu}{\sqrt{x}}$	$as\,\frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	$as\,\frac{b}{\sqrt{x-1}}$
μ^2+1 ,, ∞	$as\,\frac{\mu}{\sqrt{x}}$	$as\,\frac{b}{\sqrt{x}}$	$as\,\frac{\mu}{\sqrt{x-1}}$	$as\,\frac{b}{\sqrt{x-1}}$

$$\int^\infty K dx = \frac{\pi^2}{4}(2+b^2-\mu^2) - \frac{\pi}{2}\int_{\mu^2}^{1+\mu^2} as\,\frac{b}{\sqrt{x}}dx - \frac{\pi}{2}\int_{\mu^2}^{b^2+1} as\,\frac{\mu}{\sqrt{x}}dx + \frac{\pi}{2}\int_{b^2+1}^{\mu^2+1} as\,\frac{b}{\sqrt{x-1}}dx +$$

$$+2\int_{\mu^2}^{\infty} as\,\frac{\mu}{\sqrt{x}}\;as\,\frac{b}{\sqrt{x}}dx - \int_{\mu^2+1}^{\infty} as\,\frac{b}{\sqrt{x}}\;as\,\frac{\mu}{\sqrt{x-1}}dx - \int_{b^2+1}^{\infty} as\,\frac{\mu}{\sqrt{x}}\;as\,\frac{b}{\sqrt{x-1}}\,dx.$$

Case V $b^2 > 1 \quad \mu^2 > 1 \quad \mu^2 > b^2 + 1.$

x	A	A'	B	B'
0 *until* 1	$\frac{\pi}{2}$	$\frac{\pi}{2}$	0	0
1 ,, b^2	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
b^2 ,, $1+b^2$	$\frac{\pi}{2}$	$as\,\frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
$1+b^2$,, μ^2	$\frac{\pi}{2}$	$as\,\frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	$as\,\frac{b}{\sqrt{x-1}}$
μ^2 ,, $1+\mu^2$	$as\,\frac{\mu}{\sqrt{x}}$	$as\,\frac{b}{\sqrt{x}}$	$\frac{\pi}{2}$	$as\,\frac{b}{\sqrt{x-1}}$
$1+\mu^2$,, ∞	$as\,\frac{\mu}{\sqrt{x}}$	$as\,\frac{b}{\sqrt{x}}$	$as\,\frac{\mu}{\sqrt{x-1}}$	$as\,\frac{b}{\sqrt{x-1}}$

$$\int_0^\infty K dx = \frac{\pi^2}{4} - \frac{\pi}{2}\int_{\mu^2}^{1+\mu^2} as\,\frac{b}{\sqrt{x}}\,dx + \frac{\pi}{2}\int_{\mu^2}^{1+\mu^2} as\,\frac{b}{\sqrt{x-1}}\,dx +$$

$$+ 2\int_{\mu^2}^\infty as\,\frac{\mu}{\sqrt{x}}\;as\,\frac{b}{\sqrt{x}}\,dx - \int_{1+\mu^2}^\infty as\,\frac{b}{\sqrt{x}}\;as\,\frac{\mu}{\sqrt{x-1}}\,dx - \int_{\mu^2}^\infty as\,\frac{\mu}{\sqrt{x}}\;as\,\frac{b}{\sqrt{x-1}}.$$

6. The integrals containing only one function *as* may be easily determined. We shall therefore consider only those integrals which contain the product of two functions *as*.

In the first place we have

$$\int as\,\frac{\mu}{\sqrt{x}}\,as\,\frac{b}{\sqrt{x}}\,dx = x\;as\,\frac{\mu}{\sqrt{x}}\,as\,\frac{b}{\sqrt{x}} + b\sqrt{x-b^2}\;as\,\frac{\mu}{\sqrt{x}} +$$

$$+ \mu\sqrt{x-\mu^2}\;as\,\frac{b}{\sqrt{x}} + 2b\mu l\,(\sqrt{x-\mu^2} + \sqrt{x-b^2}) -$$

$$- (b^2+\mu^2)\,l\,\frac{\mu\sqrt{x-b^2} + b\sqrt{x-\mu^2}}{\sqrt{x}}$$

so in each case

$$\int_{\mu^2}^\infty as\,\frac{\mu}{\sqrt{x}}\,as\,\frac{b}{\sqrt{x}}\,dx = 3b\mu + 2b\mu\,lg\,\frac{2}{\varepsilon} - \frac{\pi}{2}\mu^2\,as\,\frac{b}{\mu} - \frac{\pi b}{2}\sqrt{\mu^2-b^2} +$$

$$+ \frac{(\mu-b)^2}{2}\,l\,(\mu-b) - \frac{(\mu+b)^2}{2}\,l\,(\mu+b).$$

The second integral containing the product of two functions *as* is different according to the value of b.

If $b < 1$, as in the three cases I, II, III.

$$\int as\frac{b}{\sqrt{x}}\,as\frac{\mu}{\sqrt{x-1}}\,dx = x\,as\frac{b}{\sqrt{x}}\,as\frac{\mu}{\sqrt{x-1}} + \mu\,as\frac{b}{\sqrt{x}}\cdot\sqrt{x-1-\mu^2}$$

$$+ b\,as\frac{\mu}{\sqrt{x-1}}\cdot\sqrt{x-b^2} + 2b\mu\,l\,(\sqrt{x-b^2}+\sqrt{x-1-\mu^2})$$

$$+ b\sqrt{1-b^2}\,as\frac{\sqrt{1-b^2}\,\sqrt{x-1-\mu^2}}{\sqrt{1+\mu^2-b^2}\,\sqrt{x-1}}$$

$$-\frac{\mu\sqrt{1+\mu^2}}{2}\,l\,\frac{\sqrt{1+\mu^2}\,\sqrt{x-b^2}+b\sqrt{x-1-\mu^2}}{\sqrt{1+\mu^2}\,\sqrt{x-b^2}-b\sqrt{x-1-\mu^2}}$$

$$+\frac{\mu}{2}\int as\frac{b}{\sqrt{x}}\,\frac{dx}{(x-1)\sqrt{x-1-\mu^2}}.$$

Putting

$$x = 1 + \frac{\mu^2}{\sin^2 v}$$

we have

$$\frac{\mu}{2}\int_{1+\mu^2}^{\infty} as\frac{b}{\sqrt{x}}\,\frac{dx}{(x-1)\sqrt{x-1-\mu^2}} = \int_0^{\frac{\pi}{2}} as\frac{b\sin v}{\sqrt{\mu^2+\sin^2 v}}\,dv = T'$$

so introducing the limits

$$\int_{1+\mu^2}^{\infty} as\frac{b}{\sqrt{x}}\,as\frac{\mu}{\sqrt{x-1}}\,dx = 3b\mu + 2b\mu\,l\,\frac{2}{\varepsilon} - \frac{\pi}{2}b\sqrt{1+\mu^2-b^2}$$

$$-\frac{\pi}{2}(1+\mu^2)\,as\frac{b}{\sqrt{1+\mu^2}} + b\sqrt{1-b^2}\,as\frac{\sqrt{1-b^2}}{\sqrt{1+\mu^2-b^2}}$$

$$-\frac{\mu\sqrt{1+\mu^2}}{2}\,l\,\frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b} - b\mu\,l\,(1+\mu^2-b^2) + T'.$$

If $b > 1$, as in the cases IV and V

$$\int as\frac{b}{\sqrt{x}}\,as\frac{\mu}{\sqrt{x-1}}\,dx = x\,as\frac{b}{\sqrt{x}}\,as\frac{\mu}{\sqrt{x-1}} + \mu\,as\frac{b}{\sqrt{x}}\cdot\sqrt{x-1-\mu^2}$$

$$+ b\,as\frac{\mu}{\sqrt{x-1}}\cdot\sqrt{x-b^2} + 2b\mu\,l\,(\sqrt{x-b^2}+\sqrt{x-1-\mu^2})$$

$$- b\sqrt{b^2-1}\,l\,\frac{\mu\sqrt{x-b^2}+\sqrt{b^2-1}\,\sqrt{x-1-\mu^2}}{\sqrt{x-1}}$$

$$-\frac{\mu\sqrt{1+\mu^2}}{2}\,l\,\frac{\sqrt{1+\mu^2}\sqrt{x-b^2}+b\sqrt{x-1-\mu^2}}{\sqrt{1+\mu^2}\sqrt{x-b^2}-b\sqrt{x-1-\mu^2}}$$
$$+\frac{\mu}{2}\int as\frac{b}{\sqrt{x}}\,\frac{dx}{(x-1)\sqrt{x-1-\mu^2}}$$

and

$$\int_{1+\mu^2}^{\infty} as\frac{b}{\sqrt{x}}\,as\frac{\mu}{\sqrt{x-1}}\,dx = 3b\mu + 2b\mu\,l\frac{2}{\varepsilon} - \frac{\pi}{2}\,b\sqrt{1+\mu^2-b^2}$$
$$-\frac{\pi}{2}(1+\mu^2)\,as\frac{b}{\sqrt{1+\mu^2}} - \frac{b\sqrt{b^2-1}}{2}\,l\,\frac{\mu+\sqrt{b^2-1}}{\mu-\sqrt{b^2-1}}$$
$$-\frac{\mu\sqrt{1+\mu^2}}{2}\,l\,\frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b} - b\mu\,l\,(1+\mu^2-b^2) + T'.$$

The third integral containing two functions *as*, which has different limits in the two cases II and V, may be easily deduced from the preceding integrals.

When $\mu < 1$ we find

$$\int_{1+b^2}^{\infty} as\frac{\mu}{\sqrt{x}}\,as\frac{b}{\sqrt{x-1}}\,dx = 3b\mu + 2b\mu\,l\frac{2}{\varepsilon} - \frac{\pi}{2}\mu\sqrt{1+b^2-\mu^2}$$
$$-\frac{\pi}{2}(1+b^2)\,as\frac{\mu}{\sqrt{1+b^2}} + \mu\sqrt{1-\mu^2}\,as\frac{\sqrt{1-\mu^2}}{\sqrt{1+b^2-\mu^2}}$$
$$-\frac{b\sqrt{1+b^2}}{2}\,l\,\frac{\sqrt{1+b^2}+\mu}{\sqrt{1+b^2}+\mu} - b\mu\,l\,(1+b^2-\mu^2)+T$$

and when $\mu > 1$

$$\int_{1+b^2}^{\infty} as\frac{\mu}{\sqrt{x}}\,as\frac{b}{\sqrt{x-1}}\,dx = 3b\mu + 2b\mu\,l\frac{2}{\varepsilon} - \frac{\pi}{2}\,\mu\sqrt{1+b^2-\mu^2}$$
$$-\frac{\pi}{2}(1+b^2)\,as\frac{\mu}{\sqrt{1+b^2}} - \frac{\mu\sqrt{\mu^2-1}}{2}\,l\,\frac{b+\sqrt{\mu^2-1}}{b-\sqrt{\mu^2-1}}$$
$$-\frac{b\sqrt{1+b^2}}{2}\,l\,\frac{\sqrt{1+b^2}+\mu}{\sqrt{1+b^2}-\mu} - b\mu\,l\,(1+b^2-\mu^2) + T$$

wherein

$$T = \int_0^{\frac{\pi}{2}} as\frac{\mu\sin v}{\sqrt{b^2+\sin^2 v}}\,dv.$$

Similarly we get

$$\int_{\mu^2}^{\infty} as\frac{\mu}{\sqrt{x}}\,as\frac{b}{\sqrt{x-1}}\,dx = 3b\mu + 2b\mu\, l\frac{2}{\varepsilon} - \frac{\pi}{2}\, b\sqrt{\mu^2-1-b^2}$$

$$-\mu\sqrt{\mu^2-1}\, l(b+\sqrt{\mu^2-1}) + \frac{\mu\sqrt{\mu^2-1}-2b\mu}{2}\, l(\mu^2-1-b^2) -$$

$$-\frac{b\sqrt{1+b^2}}{2}\, l\frac{\mu+\sqrt{1+b^2}}{\mu-\sqrt{1+b^2}} - \frac{\pi}{2}\mu^2\, as\frac{b}{\sqrt{\mu^2-1}} + T'''$$

where

$$T''' = \frac{b}{2}\int_{\mu^2}^{\infty} as\frac{\mu}{\sqrt{x}}\,\frac{dx}{(x-1)\sqrt{x-1-b^2}}.$$

This integral may be transformed as follows·
Putting

$$x = \frac{\mu^2}{sin^2\, v}$$

we obtain

$$T''' = b\,\mu^2\int_0^{\frac{\pi}{2}} \frac{v\, cos\, v\, dv}{(\mu^2 - sin^2 v)\sqrt{\mu^2-(1+b^2)sin^2 v}} = \int_0^{\frac{\pi}{2}} vd\,.\, as\frac{b\, sin\, v}{\sqrt{\mu^2 - sin^2 v}}$$

$$T''' = \frac{\pi}{2} as\frac{b}{\sqrt{\mu^2-1}} - \int_0^{\frac{\pi}{2}} as\frac{b\, sin\, v}{\sqrt{\mu^2-sin^2 v}}\, dv$$

or

$$T''' = \frac{\pi}{2} as\frac{b}{\sqrt{\mu^2-1}} - U$$

where

$$U = \int_0^{\frac{\pi}{2}} as\frac{b\, sin\, v}{\sqrt{\mu^2 - sin^2 v}}\, dv.$$

7. It is evident from the preceding article that in all cases $\int_0^{\infty} K\, dx$ may be reduced to the three definite integrals T, T'' and U. Introducing these we obtain, after some slight reductions, the following results.

Case I. $\mu^2 < 1 \quad b^2 < 1 \quad \mu^2 < b^2 + 1$

$$\int_0^\infty K\,dx = -T - T' + \frac{\pi}{2}\,as\,b + \frac{\pi}{2}\,as\,\mu$$

$$+ b\sqrt{1-b^2}\,as\,\frac{\mu}{\sqrt{1+\mu^2-b^2}} + \mu\sqrt{1-\mu^2}\,as\,\frac{b}{\sqrt{1+b^2-\mu^2}}$$

$$+ (\mu-b)^2\,l\,(\mu-b) - (\mu+b)^2\,l\,(\mu+b)$$

$$+ b\mu\,l\,(1+\mu^2-b^2) + b\mu\,l\,(1+b^2-\mu^2)$$

$$+ \frac{b\sqrt{1+b^2}}{2}\,l\,\frac{\sqrt{1+b^2}+\mu}{\sqrt{1+b^2}-\mu} + \frac{\mu\sqrt{1+\mu^2}}{2}\,l\,\frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b}$$

Case II $\mu^2 > 1 \quad b^2 < 1 \quad \mu^2 > b^2 + 1.$

$$\int_0^\infty K dx = U - T' + \frac{\pi}{2}\,as\,b + b\sqrt{1-b^2}\,as\,\frac{\mu}{\sqrt{1+\mu^2-b^2}}$$

$$+ (\mu-b)^2\,l\,(\mu-b) - (\mu+b)^2\,l\,(\mu+b)$$

$$+ b\mu\,l\,(\mu^2+1-b^2) + b\mu\,l\,(\mu^2-1-b^2)$$

$$+ \frac{b\sqrt{1+b^2}}{2}\,l\,\frac{\mu+\sqrt{1+b^2}}{\mu-\sqrt{1+b^2}} + \frac{\mu\sqrt{1+\mu^2}}{2}\,l\,\frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b}$$

$$+ \frac{\mu\sqrt{\mu^2-1}}{2}\,l\,\frac{\sqrt{\mu^2-1}+b}{\sqrt{\mu^2-1}-b}.$$

Case III $\mu^2 > 1 \quad b^2 < 1 \quad \mu^2 < b^2 + 1.$

$$\int_0^\infty K dx = \frac{\pi^2}{4} - T - T' + \frac{\pi}{2}\,as\,b + b\sqrt{1-b^2}\,as\,\frac{\mu}{\sqrt{1+\mu^2-b^2}}$$

$$+ (\mu-b)^2\,l(\mu-b) - (\mu+b)^2\,l(\mu+b)$$

$$+ b\mu\,l(1+\mu^2-b^2) + b\mu\,l(1+b^2-\mu^2)$$

$$+ \frac{b\sqrt{1+b^2}}{2}\,l\,\frac{\sqrt{1+b^2}+\mu}{\sqrt{1+b^2}-\mu} + \frac{\mu\sqrt{1+\mu^2}}{2}\,l\,\frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b}$$

$$+ \frac{\mu\sqrt{\mu^2-1}}{2}\,l\,\frac{b+\sqrt{\mu^2-1}}{b-\sqrt{\mu^2-1}}.$$

Case IV $\mu^2 > 1 \quad b^2 > 1 \quad \mu^2 < b^2 + 1.$

$$\int_0^\infty K dx = \frac{\pi^2}{2} - T - T'$$

$$+ (\mu-b)^2\,l\,(\mu-b) - (\mu+b)^2\,l\,(\mu+b)$$

$$+ b\mu\,l\,(1+\mu^2-b^2) - b\mu\,l\,(1+b^2-\mu^2)$$

$$+\frac{b\sqrt{1+b^2}}{2}\,l\,\frac{\sqrt{1+b^2}+\mu}{\sqrt{1+b^2}-\mu}+\frac{\mu\sqrt{1+\mu^2}}{2}\,l\,\frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b}$$

$$+\frac{b\sqrt{b^2-1}}{2}\,l\,\frac{\mu+\sqrt{b^2-1}}{\mu-\sqrt{b^2-1}}+\frac{\mu\sqrt{\mu^2-1}}{2}\,l\,\frac{b+\sqrt{\mu^2-1}}{b-\sqrt{\mu^2-1}}.$$

Case V $\mu^2>1 \quad b^2>1 \quad \mu^2>b^2+1.$

$$\int_0^\infty K dx=\frac{\pi^2}{4}+U-T''$$

$$+(\mu-b)^2\,l\,(\mu-b)-(\mu+b)^2\,l\,(\mu+b)$$
$$+b\mu\,l\,(1+\mu^2-b^2)+b\mu\,l\,(\mu^2-1-b^2)$$
$$+\frac{b\sqrt{1+b^2}}{2}\,l\,\frac{\mu+\sqrt{b^2+1}}{\mu-\sqrt{b^2+1}}+\frac{\mu\sqrt{1+\mu^2}}{2}\,l\,\frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b}$$
$$+\frac{b\sqrt{b^2-1}}{2}\,l\,\frac{\mu+\sqrt{b^2-1}}{\mu-\sqrt{b^2-1}}+\frac{\mu\sqrt{\mu^2-1}}{2}\,l\,\frac{\sqrt{\mu^2-1}+b}{\sqrt{\mu^2-1}-b}.$$

8. After these reductions it seems to be necessary to expand the three integrals T, T'', and U in convergent series [1]). It is however preferable to get an expansion for the general integrals $\int_0^\infty K\,dx$ and $I(\mu, b)$. To this we shall now proceed, beginning with the two cases II and V which are different from the rest.

Case II.

Differentiating U and T'' with respect to b, we have

$$\frac{\partial U}{\partial b}=\int_0^{\frac{\pi}{2}}\frac{\sin v\,dv}{\sqrt{\mu^2-(1+b^2)\sin^2 v}}=\frac{1}{2\sqrt{1+b^2}}\,l\,\frac{\mu+\sqrt{1+b^2}}{\mu-\sqrt{1+b^2}},$$

$$\frac{\partial T''}{\partial b}=\int_0^{\frac{\pi}{2}}\frac{\sin v\,dv}{\sqrt{\mu^2+(1-b^2)\sin^2 v}}=\frac{1}{\sqrt{1-b^2}}\,as\,\frac{\sqrt{1-b^2}}{\sqrt{1+\mu^2-b^2}}$$

thus generally

$$\frac{\partial}{\partial b}\int_0^\infty K\,dx=\frac{\partial L}{\partial b}=\sqrt{1+b^2}\,l\frac{\mu+\sqrt{1+b^2}}{\mu-\sqrt{1+b^2}}+2\sqrt{1-b^2}\,as\,\frac{\mu}{\sqrt{1+\mu^2-b^2}}$$

$$-2(\mu-b)\,l\,(\mu-b)-2(\mu+b)\,l\,(\mu+b)$$
$$+\mu\,l\,(\mu^2+1-b^2)+\mu\,l\,(\mu^2-1-b^2).$$

[1]) The expansions for T and T'' are to be found: Nieuw Archief voor Wiskunde (2) Vol. IX,

Now the different terms of this equation may be expanded as follows

$$2\sqrt{1-b^2}\,as\frac{\mu}{\sqrt{1+\mu^2-b^2}}=2\sqrt{1-b^2}\left(\frac{\pi}{2}-as\frac{\sqrt{1-b^2}}{\sqrt{1+\mu^2-b^2}}\right)$$

$$=\pi\sqrt{1-b^2}-\frac{\sqrt{1-b^2}}{i}\,l\frac{\mu+i\sqrt{1-b^2}}{\mu-i\sqrt{1-b^2}}$$

$$-\frac{\sqrt{1-b^2}}{i}\,l\,\frac{\mu+i\sqrt{1-b^2}}{\mu-i\sqrt{1-b^2}}=$$

$$=-2\left\{\frac{1-b^2}{\mu}-\frac{1}{3}\frac{(1-b^2)^2}{\mu^3}+\frac{1}{5}\frac{(1-b^2)^3}{\mu^5}-\frac{1}{7}\frac{(1-b^2)^4}{\mu^7}+\ldots\right\}$$

$$\sqrt{1+b^2}\,l\,\frac{\mu+\sqrt{1+b^2}}{\mu-\sqrt{1+b^2}}=$$

$$=2\left\{\frac{1+b^2}{\mu}+\frac{1}{3}\frac{(1+b^2)^2}{\mu^3}+\frac{1}{5}\frac{(1+b^2)^3}{\mu^5}+\frac{1}{7}\frac{(1+b^2)^4}{\mu^7}+\ldots\right\}$$

$$\mu\,l\,(\mu^2+1-b^2)=$$

$$=2\mu\,l\mu+\frac{1-b^2}{\mu}-\frac{1}{2}\frac{(1-b^2)^2}{\mu^3}+\frac{1}{3}\frac{(1-b^2)^3}{\mu^5}-\frac{1}{4}\frac{(1-b^2)^4}{\mu^7}+\ldots$$

$$\mu\,l(\mu^2-1-b^2)=$$

$$=2\mu\,l\mu-\frac{1+b^2}{\mu}-\frac{1}{2}\frac{(1+b^2)^2}{\mu^3}-\frac{1}{3}\frac{(1+b^2)^3}{\mu^5}-\frac{1}{4}\frac{(1+b^2)^4}{\mu^7}-\ldots$$

$$-2(\mu-b)\,l(\mu-b)-2(\mu+b)\,l(\mu+b)=$$

$$=-4\mu\,l\mu-2\frac{b^2}{\mu}-\frac{2}{2.3}\frac{b^4}{\mu^3}-\frac{2}{3.5}\frac{b^6}{\mu^5}-\frac{2}{4.7}\frac{b^8}{\mu^7}-\ldots$$

thus by addition

$$\frac{\partial L}{\partial b}=\pi\sqrt{1-b^2}+\frac{2}{3\mu^3}+2\,\frac{3b^2}{3.5\,\mu^5}+2\,\frac{1+6b^4}{4.7\,\mu^7}+$$

$$+2\,.\,\frac{5b^2+10b^6}{5.9\,\mu^9}+2\,.\,\frac{1+15b^4+15b^8}{6.11\,\mu^{11}}+\ldots$$

or by arranging this series according to ascending powers of b

$$\frac{\partial L}{\partial b}=\pi\sqrt{1-b^2}+2\sum_1^{\infty}\frac{1}{(2n)(4n-1)}\frac{1}{\mu^{4n-1}}$$

$$+2b^2\sum_1^{\infty}\frac{1}{4n+1}\frac{1}{\mu^{4n+1}}$$

$$+\frac{2b^4}{2!}\sum_1^{\infty}\frac{2n+1}{4n+3}\frac{1}{\mu^{4n+3}}$$

$$+\frac{2b^6}{3!}\sum_1^{\infty}\frac{(2n+2)(2n+1)}{4n+5}\frac{1}{4^{4n+5}}$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots$$

Finally integrating and remarking that the constant is zero, we have this result

$$L=\int_0^\infty K\,dx=\frac{\pi}{2}\left(as\,b+b\sqrt{1-b^2}\right)$$

$$+\,2b\sum_1^\infty\frac{1}{(2n)(4n-1)}\,\frac{1}{\mu^{4n-1}}$$

$$+\,\frac{2b^3}{3}\sum_1^\infty\frac{1}{4n+1}\,\frac{1}{\mu^{4n+1}}$$

$$+\,\frac{2b^5}{2!5}\sum_1^\infty\frac{2n+1}{4n+3}\,\frac{1}{\mu^{4n+3}}$$

$$+\,\frac{2b^7}{3!7}\sum_1^\infty\frac{(2n+2)(2n+1)}{4n+5}\,\frac{1}{\mu^{4n+5}}$$

$$+\,\frac{2b^9}{3!9}\sum_1^\infty\frac{(2n+3)(2n+2)(2n+1)}{4n+7}\,\frac{1}{\mu^{4n+7}}$$

$$+\,\ldots\ldots\ldots\ldots\ldots\ldots$$

This series is sufficiently convergent for values of b between 0 and 0.4, and for values of μ from 1.2 upwards.

Case V.

Here again we have the same value for $\frac{\partial U}{\partial b}$ as in the preceding and further

$$\frac{\partial T''}{\partial b}=\frac{1}{2\sqrt{b^2-1}}\;l\,\frac{\mu+\sqrt{b^2-1}}{\mu-\sqrt{b^2-1}}$$

thus

$$\frac{\partial L}{\partial b}=\sqrt{1+b^2}\,l\,\frac{\mu+\sqrt{b^2+1}}{\mu-\sqrt{b^2+1}}+\sqrt{b^2-1}\;l\,\frac{\mu+\sqrt{b^2-1}}{\mu-\sqrt{b^2-1}}$$

$$-\,2\,(\mu-b)\,l\,(\mu-b)-2\,(\mu+b)\,l\,(\mu+b)$$

$$+\,\mu\,l\,(\mu^2+1-b^2)+\mu\,l\,(\mu^2-1-b^2).$$

Expanding now $\sqrt{b^2-1}\;l\,\frac{\mu+\sqrt{b^2-1}}{\mu-\sqrt{b^2-1}}$ we get the same series as that for $-\frac{\sqrt{1-b^2}}{i}\;l\,\frac{\mu+i\sqrt{1-b^2}}{\mu-i\sqrt{1-b^2}}$ in the former case, therefore we may write

$$L = 2b \sum_1^\infty \frac{1}{(2n)(4n-1)} \frac{1}{\mu^{4n-1}}$$
$$+ \frac{2b^3}{3} \sum_1^\infty \frac{1}{4n+1} \frac{1}{\mu^{4n+1}}$$
$$+ \frac{2b^5}{2!5} \sum_1^\infty \frac{2n+1}{4n+3} \frac{1}{\mu^{4n+3}}$$
$$+ \frac{2b^7}{3!7} \sum_1^\infty \frac{(2n+2)(2n+1)}{4n+5} \cdot \frac{1}{\mu^{4n+5}}$$
$$+ \cdots \cdots$$
$$+ \text{const.}$$

It is evident that this series and the series for L in case II must agree for $b = 1$. Therefore the value of the constant must be $\frac{\pi^2}{4}$ so that finally

$$L = \int_0^\infty K\,dx = \frac{\pi^2}{4} + 2b \sum_1^\infty \frac{1}{(2n)(4n-1)} \frac{1}{\mu^{4n-1}}$$
$$+ \frac{2b^3}{3} \sum_1^\infty \frac{1}{4n+1} \frac{1}{\mu^{4n+1}}$$
$$+ \frac{2b^5}{2!5} \sum_1^\infty \frac{2n+1}{4n+3} \frac{1}{\mu^{4n+3}}$$
$$+ \frac{2b^7}{3!7} \sum_1^\infty \frac{(2n+2)(2n+1)}{4n+5} \frac{1}{\mu^{4n+5}}$$
$$+ \frac{2b^9}{4!9} \sum_1^\infty \frac{(2n+3)(2n+2)(2n+1)}{4n+7} \frac{1}{\mu^{4n+7}}$$
$$+ \cdots \cdots$$

The values of this series have been computed for $b = 1.2$ $\mu \geq 1.8$; $b = 2.4$ $\mu \geq 3.4$, $b = 3.6$ $\mu \geq 3.8$, it has been found to be sufficiently convergent for these values.

9. In the three cases I, III, IV we will try to expand at once $I(\mu,b)$ according to ascending powers of $\mu - b = a$.

Let

$$I(\mu, b) + 4b^2\, l\, 2b + 4\mu^2\, l\, 2\mu + 2(\mu-b)^2\, l\,(\mu-b) - 2(\mu+b)^2\, l\,(\mu+b) = f(\mu)$$

and

$$f(\mu) = f(b) + a f'(b) + \frac{a^2}{2!} f''(b) + \frac{a^3}{3!} f'''(b) + \ldots.$$

then it is evident that $f(b) = 0$ for $I(b,b) = 0$ and $(\mu-b)\, l(\mu-b) = 0$ if $\mu = b$, Dividing $f(\mu)$ in two parts $\varphi(\mu)$ and $\psi(\mu)$ we may write:

Case I.

$$\varphi(\mu) = 2\mu\sqrt{1-\mu^2}\, as\, \mu + (as\, \mu)^2 + 2\mu\sqrt{1+\mu^2}\; l\,(\mu+\sqrt{\mu^2+1}) +$$
$$+ l^2(\mu+\sqrt{\mu^2+1}) + 2b\sqrt{1-b^2}\, as\, b + (as\, b)^2 +$$
$$+ 2b\sqrt{1+b^2}\, l\,(b+\sqrt{b^2+1}) + l^2(b+\sqrt{b^2+1})$$

$$\psi(\mu) = 2T + 2T' - \pi\, as\, \mu - \pi\, as\, b - 2\mu\sqrt{1-\mu^2}\, as \frac{b}{\sqrt{1+b^2-\mu^2}} -$$
$$- 2b\sqrt{1-b^2}\, as \frac{\mu}{\sqrt{1+\mu^2-b^2}} - 2b\,\mu\, l\,(1+\mu^2-b^2) -$$
$$- 2b\,\mu\, l\,(1+b^2-\mu^2) - \mu\sqrt{1+\mu^2}\, l \frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b} -$$
$$- b\sqrt{1+b^2}\, l \frac{\sqrt{1+b^2}+\mu}{\sqrt{1+b^2}-\mu}$$

$$\frac{\partial\varphi}{\partial\mu} = 4\sqrt{1+\mu^2}\, l\,(\mu+\sqrt{\mu^2+1}) + 4\sqrt{1-\mu^2}\, as\, \mu + 4\mu$$

$$\frac{\partial\psi}{\partial\mu} = -\,2\sqrt{1+\mu^2}\, l \frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b} - 4\sqrt{1-\mu^2}\, as \frac{b}{\sqrt{1+b^2-\mu^2}} -$$
$$- 2b\, l\,(1+\mu^2-b^2) - 2\, b\, l\,(1+b^2-\mu^2) - 4b$$

and if $\frac{\partial\varphi}{\partial\mu} = \frac{\partial\varphi_1}{\partial\mu} + \frac{\partial\varphi_2}{\partial\mu}$, $\frac{\partial\psi}{\partial\mu} = \frac{\partial\psi_1}{\partial\mu} + \frac{\partial\psi_2}{\partial\mu}$

$$\frac{\partial\varphi_1}{\partial\mu} = 4\sqrt{1+\mu^2}\, l\,(\mu + \sqrt{\mu^2+1}) + 2\mu\ ; \quad \frac{\partial\varphi_2}{\partial\mu} = 4\sqrt{1-\mu^2}\, as\, \mu + 2\mu$$

$$\frac{\partial\psi_1}{\partial\mu} = -\,2\sqrt{1+\mu^2}\, l \frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b} - 2b\, l\,(1+\mu^2-b^2) - 2b;$$

$$\frac{\partial\psi_2}{\partial\mu} = -\,4\sqrt{1-\mu^2}\, as \frac{b}{\sqrt{1+b^2-\mu^2}} - 2b\, l\,(1+b^2-\mu^2) - 2b$$

Hence

$$(1+\mu^2)\frac{\partial^2\varphi_1}{\partial\mu^2} - \mu\frac{\partial\varphi_1}{\partial\mu} - (4\mu^2+6) = 0,\ (\mu^2-1)\frac{\partial^2\varphi_2}{\partial\mu^2} - \mu\frac{\partial\varphi_2}{\partial\mu} - (4\mu^2-6) = 0$$

$$\frac{\partial^2\psi_1}{\partial\mu^2} = -\frac{2\mu}{\sqrt{1+\mu^2}}\; l\, \frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b},\quad \frac{\partial^2\psi_2}{\partial\mu^2} = \frac{4\mu}{\sqrt{1-\mu^2}}\, as \frac{b}{\sqrt{1+b^2-\mu^2}}$$

$$\mu\,(1+\mu^2)(1+\mu^2-b^2)\frac{\partial^3\psi_1}{\partial\mu^3} - (1+\mu^2-b^2)\frac{\partial^2\psi_1}{\partial\mu^2} - 4b\mu^3 = 0,$$

$$\mu\,(\mu^2-1)(1+b^2-\mu^2)\frac{\partial^3\psi_2}{\partial\mu^3} + (1+b^2-\mu^2)\frac{\partial^2\psi_2}{\partial\mu^2} + 4b\mu^3 = 0$$

so that

$$\left(\frac{\partial\varphi_1}{\partial\mu}\right)_b+\left(\frac{\partial\psi_1}{\partial\mu}\right)_b=0\ ,\quad \left(\frac{\partial\varphi_2}{\partial\mu}\right)_b+\left(\frac{\partial\psi_2}{\partial\mu}\right)_b=0$$

$$\left(\frac{\partial^2\varphi_1}{\partial\mu_2}\right)_b+\left(\frac{\partial^2\psi_1}{\partial\mu^2}\right)_b=6\ ,\quad \left(\frac{\partial^2\varphi_2}{\partial\mu^2}\right)_b+\left(\frac{\partial^2\psi_2}{\partial\mu^2}\right)_b=6$$

$$\left(\frac{\partial^3\varphi_1}{\partial\mu^3}\right)_b+\left(\frac{\partial^3\psi_1}{\partial\mu^3}\right)_b=4b\ ,\quad \left(\frac{\partial^3\varphi_2}{\partial\mu^3}\right)_b+\left(\frac{\partial^3\psi_2}{\partial\mu^3}\right)_b=-4b.$$

Thus in the required expansion the coefficients of a and a^3 will be wanting and the coefficient of $\frac{a^2}{2!}$ will be 12. We will defer the determination of the further coefficients until we have considered also the three first coefficients in the two cases III and IV.

Case III.

Here we have

$$\varphi(\mu)=2\mu\sqrt{\mu^2-1}\,l(\mu+\sqrt{\mu^2-1})-l^2(\mu+\sqrt{\mu^2-1})+$$
$$+\,2\mu\sqrt{1+\mu^2}\,l(\mu+\sqrt{\mu^2+1})+l^2(\mu+\sqrt{\mu^2+1})$$

$$\psi(\mu)=-\frac{\pi^2}{4}+2T+2T'-\pi\,as\,b-2b\sqrt{1-b^2}\,as\frac{\mu}{\sqrt{1+\mu^2-b^2}}$$
$$-\,2b\mu\,l(1+\mu^2-b^2)-2b\mu\,l(1+b^2-\mu^2)$$
$$-\,b\sqrt{1+b^2}\,l\frac{\sqrt{1+b^2}+\mu}{\sqrt{1+b^2}-\mu}-\mu\sqrt{1+\mu^2}\,l\frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}+b}-$$
$$-\,\mu\sqrt{\mu^2-1}\,l\frac{b+\sqrt{\mu^2-1}}{b-\sqrt{\mu^2-1}}$$

$$\frac{\partial\varphi}{\partial\mu}=4\sqrt{\mu^2-1}\,l(\mu+\sqrt{\mu^2-1})+4\sqrt{1+\mu^2}\,l(\mu+\sqrt{\mu^2+1})+4\mu$$

$$\frac{\partial\psi}{\partial\mu}=-2\sqrt{1+\mu^2}\,l\frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b}-2\sqrt{\mu^2-1}\,l\frac{b+\sqrt{\mu^2-1}}{b-\sqrt{\mu^2-1}}$$
$$-\,2b\,l(1+\mu^2-b^2)-2b\,l(1+b^2-\mu^2)-4b.$$

Putting now

$$\frac{\partial\varphi_1}{\partial\mu}=4\sqrt{1+\mu^2}\,l(\mu+\sqrt{\mu^2+1})+2\mu\ ,$$

$$\frac{\partial\varphi_2}{\partial\mu}=4\sqrt{\mu^2-1}\,l(\mu+\sqrt{\mu^2-1})+2\mu$$

$$\frac{\partial\psi_1}{\partial\mu}=-2\sqrt{1+\mu^2}\,l\frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b}-2b\,l(1+\mu^2-b^2)-2b\ ,$$

$$\frac{\partial\psi_2}{\partial\mu}=-2\sqrt{\mu^2-1}\,l\frac{b+\sqrt{\mu^2-1}}{b-\sqrt{\mu^2-1}}-2b\,l(1+b^2-\mu^2)-2b$$

we find

$$(1+\mu^2)\frac{\partial^2\varphi_1}{\partial\mu^2}-\mu\frac{\partial\varphi_1}{\partial\mu}-(4\mu^2+6)=0\,,$$

$$(\mu^2-1)\frac{\partial^2\varphi_2}{\partial\mu^2}-\mu\frac{\partial\varphi_2}{\partial\mu}-(4\mu^2-6)=0$$

$$\frac{\partial^2\psi_1}{\partial\mu^2}=-\frac{2\mu}{\sqrt{1+\mu^2}}\,l\,\frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b}\qquad\frac{\partial^2\psi_2}{\partial\mu^2}=-\frac{2\mu}{\sqrt{\mu^2-1}}\,l\,\frac{b+\sqrt{\mu^2-1}}{b-\sqrt{\mu^2-1}}$$

$$\mu(1+\mu^2)(1+\mu^2-b^2)\frac{\partial^3\psi_1}{\partial\mu^3}-(1+\mu^2-b^2)\frac{\partial^2\psi_1}{\partial\mu^2}-4b\mu^3=0\,,$$

$$\mu(\mu^2-1)(1+b^2-\mu^2)\frac{\partial^3\psi_2}{\partial\mu^3}+(1+b^2-\mu^2)\frac{\partial^2\psi_2}{\partial\mu^2}+4b\mu^3=0$$

thus

$$\left(\frac{\partial\varphi_1}{\partial\mu}\right)_b+\left(\frac{\partial\psi_1}{\partial\mu}\right)_b=0\qquad\left(\frac{\partial\varphi_2}{\partial\mu}\right)_b+\left(\frac{\partial\psi_2}{\partial\mu}\right)_b=0$$

$$\left(\frac{\partial^2\varphi_1}{\partial\mu^2}\right)_b+\left(\frac{\partial^2\psi_1}{\partial\mu^2}\right)_b=6\qquad\left(\frac{\partial^2\varphi_2}{\partial\mu^2}\right)_b+\left(\frac{\partial^2\psi_2}{\partial\mu^2}\right)_b=6$$

$$\left(\frac{\partial^3\varphi_1}{\partial\mu^2}\right)_b+\left(\frac{\partial^3\psi_1}{\partial\mu^3}\right)_b=4b\qquad\left(\frac{\partial^3\varphi_2}{\partial\mu^3}\right)_b+\left(\frac{\partial^3\psi_2}{\partial\mu^3}\right)_b=-4b$$

and evidently the expansion in this case will agree with that in the preceding case.

Case IV.

In this case we may write

$$\begin{aligned}\varphi(\mu)=&\,2\sqrt{\mu^2-1}\,l(\mu+\sqrt{\mu^2-1})-l^2(\mu+\sqrt{\mu^2-1})\\&+2\mu\sqrt{1+\mu^2}\,l(\mu+\sqrt{\mu^2+1})+l^2(\mu+\sqrt{\mu^2+1})\\&+2b\sqrt{1+b^2}\,l(b+\sqrt{b^2+1})+l^2(b+\sqrt{b^2+1})\\&+2b\sqrt{b^2-1}\,l(b+\sqrt{b^2-1})-l^2(b+\sqrt{b^2-1})\end{aligned}$$

$$\begin{aligned}\psi(\mu)=&-\frac{\pi^2}{2}+2T+2T''-2b\mu\,l(1+\mu^2-b^2)-2b\mu\,l(1+b^2-\mu^2)\\&-b\sqrt{1+b^2}\,l\,\frac{\sqrt{1+b^2}+\mu}{\sqrt{1+b^2}-\mu}-\mu\sqrt{1+\mu^2}\,l\,\frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b}\\&-b\sqrt{b^2-1}\,l\,\frac{\mu+\sqrt{b^2-1}}{\mu-\sqrt{b^2-1}}-\mu\sqrt{\mu^2-1}\,l\,\frac{b+\sqrt{\mu^2-1}}{b-\sqrt{\mu^2-1}}\end{aligned}$$

$$\frac{\partial\varphi}{\partial\mu}=4\sqrt{\mu^2-1}\,l\,(\mu+\sqrt{\mu^2-1})+4\sqrt{\mu^2+1}\,l\,(\mu+\sqrt{\mu^2+1})+4\mu$$

$$\begin{aligned}\frac{\partial\psi}{d\mu}=&-2\sqrt{1+\mu^2}\,l\,\frac{\sqrt{1+\mu^2}+b}{\sqrt{1+\mu^2}-b}-2\sqrt{\mu^2-1}\,l\,\frac{b+\sqrt{\mu^2-1}}{b-\sqrt{\mu^2-1}}\\&-2b\,l(1+\mu^2-b^2)-2b\,l(1+b^2-\mu^2)-4b\end{aligned}$$

what is in perfect accordance with the preceding case.

Thus we have the same expansion for all the three cases.

10. To determine the coefficients of this expansion we must differentiate repeatedly the differential equations obtained.

From

$$(1+\mu^2)\frac{\partial^2\varphi_1}{\partial\mu^2}-\mu\frac{\partial\varphi_1}{\partial\mu}-(4\mu^2+6)=0$$

we derive successively

$$(1+\mu^2)\frac{\partial^3\varphi_1}{\partial\mu^3}+\mu\frac{\partial^2\varphi_1}{\partial\mu^2}-\frac{\partial\varphi_1}{\partial\mu}-8\mu=0$$

$$(1+\mu^2)\frac{\partial^4\varphi_1}{\partial\mu^4}+3\mu\frac{\partial^3\varphi_1}{\partial\mu^3}-8=0$$

$$(1+\mu^2)\frac{\partial^5\varphi_1}{\partial\mu^5}+5\mu\frac{\partial^4\varphi_1}{\partial\mu^3}+3\frac{\partial^3\varphi_1}{\partial\mu^3}=0$$

$$(1+\mu^2)\frac{\partial^6\varphi_1}{\partial\mu^6}+7\mu\frac{\partial^5\varphi_1}{\partial\mu^5}+8\frac{\partial^4\varphi_1}{\partial\mu^4}=0 \text{ etc.}$$

and from

$$\mu(\mu^2+1)(\mu^2+1-b^2)\frac{\partial^3\psi_1}{\partial\mu^3}-(\mu^2+1-b^2)\frac{\partial^2\psi_1}{\partial\mu^2}-4b\mu^3=0$$

the following

$$(\mu^2+1)(\mu^2+1-b^2)\frac{\partial^4\psi_1}{\partial\mu^4}+[5\mu^3+(5-3b^2)\mu]\frac{\partial^3\psi_1}{\partial\mu^3}-2\frac{\partial^2\psi_1}{\partial\mu^2}-12b\mu=0$$

$$(\mu^2+1)(\mu^2+1-b^2)\frac{\partial^5\psi_1}{\partial\mu^5}+[9\mu^3+(9-5b^2)\mu]\frac{\partial^4\psi_1}{\partial\mu^4}+$$

$$+[15\mu^2+(3-3b^2)]\frac{\partial^3\psi_1}{\partial\mu^3}-12b=0$$

$$(\mu^2+1)(\mu^2+1-b^2)\frac{\partial^6\psi_1}{\partial\mu^6}+[13\mu^3+(13-7b^2)\mu]\frac{\partial^5\psi_1}{\partial\mu^5}+$$

$$+[42\mu^2+(12-8b^2)]\frac{\partial^4\psi_1}{\partial\mu^4}+30\mu\frac{\partial^3\psi_1}{\partial\mu^3}=0 \text{ etc.}$$

If now in the latter equations we put $\mu=b$ and write

$$D^n\psi_1=\left(\frac{\partial^n\psi_1}{\partial\mu^n}\right)_b$$

we obtain

(p) $b(b^2+1)D^3\psi_1-D^2\psi_1-4b^4=0$

(q) $(b^2+1)D^4\psi_1+(2b^3+5b)D^3\psi_1-2D^2\psi_1-12b^2=0$

(r) $(b^2+1)D^5\psi_1+(4b^3+9b)D^4\psi_1+(12b^2+3)D^3\psi_1-12b=0$

(s) $(b^2+1)D^6\psi_1+(6b^3+13b)D^5\psi_1+(34b^2+12)D^4\psi_1+30b\,D^3\psi_1=0$

(t) $(b^2+1)D^7\psi_1+(8b^3+17b)D^6\psi_1+(66b^2+25)D^5\psi_1+114bD^4\psi_1+30D^3\psi_1=0$

etc.

Multiplying the equation (p) by 2 and subtracting this product from the equation (q) we arrive at

$$(b^2+1)\,D^4\psi_1 + 3bD^3\psi_1 = 12b^2 - 8b^4.$$

Now multiplying this equation by $4b$ and subtracting it from the equation (r) we obtain

$$(b^2+1)\,D^5\psi_1 + 5bD^4\psi_1 + 3bD^3\psi_1 = 12b - 48b^3 + 32b^5.$$

Multiplying again this equation by $6b$, the preceding one by 4 and subtracting these from the equation (s) we find

$$(b^2+1)\,D^6\psi_1 + 7bD^5\psi_1 + 8D^4\psi_1 = -120b^2 + 320b^4 - 192b^6.$$

In the same way we may deduce the following equations

$$(b^2+1)D^7\psi_1 + 9bD^6\psi_1 + 15D^5\psi_1 = -120b + 1440b^3 - 2880b^5 + 1536b^7$$

$$(b^2+1)D^8\psi_1 + 11bD^7\psi_1 + 24D^6\psi_1 = 3360b^2 - 20160b^4 + 32256b^6 - 15360b^8$$

$$(b^2+1)\,D^9\psi_1 + 13bD^8\psi_1 + 35D^7\psi_1 = 3360b - 80640b^3 + 322560b^5 - 430080b^7 + 184320b^9$$

$$(b^2+1)D^{10}\psi_1 + 15bD^9\psi_1 + 48D^8\psi_1 = -60480b^2 + 1935360b^4 - 5806080b^6 + 6635520b^8 - 2580480b^{10}$$

etc.

Adding to these

$$(1+b^2)\,D^4\varphi_1 + 3bD^3\varphi_1 - 8 = 0$$
$$(1+b^2)\,D^5\varphi_1 + 5bD^4\varphi_1 + 3D^3\varphi_1 = 0$$
$$(1+b^2)\,D^6\varphi_1 + 7bD^5\varphi_1 + 8D^4\varphi_1 = 0$$
$$(1+b^2)\,D^7\varphi_1 + 9bD^6\varphi_1 + 15D^5\varphi_1 = 0$$
$$(1+b^2)\,D^8\varphi_1 + 11bD^7\varphi_1 + 24D^6\varphi_1 = 0$$
$$(1+b^2)\,D^9\varphi_1 + 13bD^8\varphi_1 + 35D^7\varphi_1 = 0$$
$$(1+b^2)\,D^{10}\varphi_1 + 15bD^9\varphi_1 + 48D^8\varphi_1 = 0$$

etc.,

we get

$$(1+b^2)\,D^4(\varphi_1+\psi_1) + 3bD^3(\varphi_1+\psi_1) = 8 + 12b^2 - 8b^4$$

$$(1+b^2)\,D^5(\varphi_1+\psi_1) + 5bD^4(\varphi_1+\psi_1) + 3D^3(\varphi_1+\psi_1) = 12b - 48b^3 + 32b^5$$

$$(1+b^2)\,D^6(\varphi_1+\psi_1) + 7bD^5(\varphi_1+\psi_1) + 8D^4(\varphi_1+\psi_1) = -120b^2 + 320b^4 - 192\,b^6$$

$$(1+b^2)\,D^7(\varphi_1+\psi_1) + 9bD^6(\varphi_1+\psi_1) + 15D^5(\varphi_1+\psi_1) = -120b + 1440b^3 - 2880b^5 + 1536b^7$$

$$(1+b^2)\,D^8(\varphi_1+\psi_1) + 11bD^7(\varphi_1+\psi_1) + 24D^6(\varphi_1+\psi_1) = 3360b^2 - 20160b^4 + 32256b^6 - 15360b^8$$

$$(1+b^2)\,D^9(\varphi_1+\psi_1) + 13bD^8(\varphi_1+\psi_1) + 35D^7(\varphi_1+\psi_1) = 3360b - 80640b^3 + 332560b^5 - 430080b^7 + 184320\,b^9$$

$$(1+b^2)\,D^{10}(\varphi_1+\psi_1) + 15bD^9(\varphi_1+\psi_1) + 48D^8(\varphi_1+\psi_1) = -181440b^2 + 1935360b^4 - 5806080b^6 + 6635520b^8 - 2580480b^{10}$$

etc.

Between the second members of these equations $P_4 P_5 \dots P_{10}$ there exists a recurrent relation which from $n = 7$ upwards is given by

$$P_n = -2\,(n-3)\,b P_{n-1} - (n-2)\,(n-5)\,P_{n-2}.$$

From the preceding relations the following values are easily found

$D^3\,(\varphi_1 + \psi_1) = 4b$
$D^4\,(\varphi_1 + \psi_1) = 8(1 - b^2)$
$D^5\,(\varphi_1 + \psi_1) = -8b\,(5 - 4b^2)$
$D^6\,(\varphi_1 + \psi_1) = -32\,(2 - 9b^2 + 6b^4)$
$D^7\,(\varphi_1 + \psi_1) = 96b\,(11 - 28b^2 + 16b^4)$
$D^8\,(\varphi_1 + \psi_1) = 192\,(8 - 87\,b^2 + 160\,b^4 - 80\,b^6)$
$D^9\,(\varphi_1 + \psi_1) = -192\,b\,(279 - 1480\,b^2 + 2160\,b^4 - 960\,b^6)$
$D^{10}\,(\varphi_1 + \psi_1) = -4608\,(16 - 325\,b^2 + 1150\,b^4 - 1400\,b^6 + 560\,b^8)$
etc.

In the same way the functions φ_2 and ψ_2 give

$D^3\,(\varphi_2 + \psi_2) = -4b$
$D^4\,(\varphi_2 + \psi_2) = -8\,(1 + b^2)$
$D^5\,(\varphi_2 + \psi_2) = -8b\,(5 + 4\,b^2)$
$D^6\,(\varphi_2 + \psi_2) = -32\,(2 + 9\,b^2 + 6\,b^4)$
$D^7\,(\varphi_2 + \psi_2) = -96\,b\,(11 + 28\,b^2 + 16\,b^4)$
$D^8\,(\varphi_2 + \psi_2) = -192\,(8 + 87\,b^2 + 160\,b^4 + 80\,b^6)$
$D^9\,(\varphi_2 + \psi_2) = -192\,b\,(279 + 1480\,b^2 + 2160\,b^4 + 960\,b^6)$
$D^{10}\,(\varphi_2 + \psi_2) = -4608\,(16 + 325\,b^2 + 1150\,b^4 + 1400\,b^6 + 560\,b^8)$
etc.

the resulting expansion is therefore

$$\begin{aligned} I\,(a,b) = & -2a^2\,l\,a - 4a^2\,l\,2a - 4b^2\,l\,2b + 2\,(a + b)^2\,l\,(a + b) \\ & + 6a^2 - \frac{2}{3}\,b^2\,a^4 - \frac{2}{3}\,b\,a^5 - \frac{8}{45}\,(1 + 3b^4)\,a^6 - \frac{16}{15}\,b^3\,a^7 \\ & - \frac{b^2}{105}\,(87 + 80\,b^4)\,a^8 - \frac{b}{105}\,(31 + 240\,b^4)\,a^9 \\ & - \frac{8}{1575}\,(8 + 575\,b^4 + 280\,b^8)\,a^{10}\,. \quad \dots \end{aligned} \qquad (10)$$

This series has been computed for the following values

$a = 0.04$	$b = 0.2$ to 3.6
$a = 0{,}08$	$b = 0.2$ „ 2.4
$a = 0.14$	$b = 0.2$ „ 2.4
$a = 0.2$	$b = 0.2$ „ 2.4
$a = 0.4$	$b = 0.3$ „ 0.8
$a = 0.6$	$b = 0.2$ „ 0.4

The convergency for all these values proved to be tolerably well.