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Mathematics. — *"On the centra of the integral curves which satisfy differential equations of the first order and the first degree."* By Prof. W. Kapteyn.

1. Considering x and y as the coordinates of a point in the plane, the real curves which satisfy a differential equation of the form $\frac{dy}{dx}=\frac{Q}{P}$, Q and P being polynomia in x and y with real coefficients, present different singularities. Between these we meet with points (foci) which are asymptotic points for the integral curves which present themselves as spirals in the neigbourhood of such points. These spirals sometimes change in closed curves and then the corresponding focus is called a centrum, and it is a question of great interest to determine the conditions when this happens. This question has been solved theoretically by Poincaré, but meets with great difficulties in practice.

The object of this paper now is to examine the differential equation, supposing P and Q to be polynomia of the second degree, and to determine all cases when centra may be expected instead of foci.

2. When the origin of coordinates is the point which must be examined, the differential equation may be written

$$\frac{dy}{dx}=\frac{-x+a'x^2+2b'xy+c'y^2}{y+ax^2+2bxy+cy^2}$$

where a, b, c, a', b', c', are real constants.

By substituting

$$\xi = hx + ky \qquad \eta = -kx + hy$$

the form of this equation is not changed, for we get

$$\frac{d\eta}{d\xi}=\frac{-\xi+\alpha'\xi^2+2\beta'\xi\eta+\gamma'\eta^2}{\eta+\alpha\xi^2+2\beta\xi\eta+\gamma\eta^2}$$

where

$$(h^2+k^2)^2\,\alpha = ah^3+(a'+2b)\,h^2k+(2b'+c)\,hk^2+c'k^3$$
$$(h^2+k^2)^2\,\beta = bh^3-(a-b'-c)\,h^2k-(a'+b-c')\,hk^2-b'k^3$$
$$(h^2+k^2)^2\,\gamma = ch^3-(2b-c')\,h^2k+(a-2b')\,hk^2+a'k^3$$
$$(h^2+k^2)^2\,\alpha' = a'h^3-(a-2b')\,h^2k-(2b-c')\,hk^2-ck^3$$
$$(h^2+k^2)^2\,\beta' = b'h^3-(a'+b-c')\,h^2k+(a-b'-c)\,hk^2-bk^3$$
$$(h^2+k^2)^2\,\gamma' = c'h^3-(2b'+c)\,h^2k+(a'+2b)\,hk^2-ak^3$$

Now h and k may be chosen so that the six coefficients $\alpha\,\beta\,\gamma$, $\alpha'\,\beta'\,\gamma'$ satisfy two conditions. Adopting

$$\alpha+\gamma=\lambda \qquad \alpha'+\gamma'=0$$

we have

$$(h^2 + k^2)\,\lambda = (a + c)\,h + (a' + c')\,k$$
$$0 = (a' + c')\,h - (a + c)\,k$$

so

$$h = \frac{a+c}{\lambda}\ ,\quad k = \frac{a'+c'}{\lambda} \quad \ldots \ldots \ldots \quad (1)$$

λ being a real number whatever, except zero.

From this it is evident that we may write

$$\frac{dy}{dx} = \frac{-x + a'x^2 + 2b'xy - a'y^2}{y + ax^2 + 2bxy + cy^2} = \frac{-x + Y}{y + X}$$

where still c could be replaced by $a - \lambda$. As we do not want this condition we will retain this coefficient in the old form.

Now after POINCARÉ's[1] theory here the origin is a centrum when it is possible to construct an infinity of homogeneous functions F_i of order i, satisfying the following series of partial differential equations

$$\left.\begin{aligned} x\frac{\partial F_3}{\partial y} - y\frac{\partial F_3}{\partial x} &= 2xX + 2yY \\ x\frac{\partial F_4}{\partial y} - y\frac{\partial F_4}{\partial x} &= X\frac{\partial F_3}{\partial x} + Y\frac{\partial F_3}{\partial y} \\ x\frac{\partial F_5}{\partial y} - y\frac{\partial F_5}{\partial x} &= X\frac{\partial F_4}{\partial x} + Y\frac{\partial F_4}{\partial y} \\ \ldots\ldots\ldots\ldots \end{aligned}\right\} \quad \ldots\ldots \quad (2)$$

This leads to an infinity of conditions for the five constants a, b, c, a', b' and if these are all fulfilled the origin is a centrum and the general integral may be written

$$x^2 + y^2 + F_3 + F_4 + F_5 + \ldots = Const.$$

where the series converges until the closed curves, represented by this equation, pass through the nearest singular point.

3. The equations (2) may be transformed as follows. If we suppose F_n to be a homogeneous function of degree n, and $x\frac{\partial F_n}{\partial y} - y\frac{\partial F_n}{\partial x}$ to be divisible by $xX + yY$, the function $X\frac{\partial F_n}{\partial x} + Y\frac{\partial F_n}{\partial y}$ will also be divisible by $xX + yX$. For eliminating the differential quotients between

$$x\frac{\partial F_n}{\partial y} - y\frac{\partial F_n}{\partial x} = (xX + yY)\,P_{n-3}$$

[1]) Journ. de Math. (1885) p. 173.

$$Y\frac{\partial F_n}{\partial y}+X\frac{\partial F_n}{\partial x}=U$$

$$y\frac{\partial F_n}{\partial y}+x\frac{\partial F_n}{\partial x}=nF_n$$

we obtain

$$(xX+yY)(xY-yX)P_{n-3}-U(x^2+y^2)+n(xX+yY)F_n=0$$

which proves that U is divisible by $xX+yY$.

If therefore

$$U=(xX+yY)P_{n-2}$$

we have

$$(xY-yX)P_{n-3}-(x^2+y^2)P_{n-2}+nF_n=0 \quad . \quad . \quad . \quad (3)$$

and the conditions for a centrum may be written

$$x\frac{\partial F_3}{\partial x}-y\frac{\partial F_3}{\partial x}=2(xX+yY)$$

$$x\frac{\partial F_4}{dy}-y\frac{\partial F_4}{\partial x}=X\frac{\partial F_3}{\partial x}+Y\frac{\partial F_3}{\partial y}=(xX+yY)P_1$$

$$x\frac{\partial F_5}{\partial y}-y\frac{\partial F_5}{\partial x}=X\frac{\partial F_4}{\partial x}+Y\frac{\partial F_4}{dy}=(xX+yY)P_2$$

. . . .

where evidently P_i represents a homogeneous function of order i.

These conditions may be further reduced, for

$$X\frac{\partial F_{n+2}}{\partial x}+Y\frac{\partial F_{n+2}}{\partial y}=(xX+yY)P_n$$

$$x\frac{\partial F_{n+2}}{\partial y}-y\frac{\partial F_{n+2}}{\partial x}=(xX+yY)P_{n-1}$$

give

$$\frac{\partial F_{n+2}}{\partial x}=xP_n-YP_{n-1}$$

$$\frac{\partial F_{n+2}}{\partial y}=XP_{n-1}+yP_n$$

hence

$$\frac{\partial}{\partial y}\left\{xP_n-YP_{n-1}\right\}=\frac{\partial}{\partial x}\left\{XP_{n-1}+yP_n\right\}$$

or

$$x\frac{\partial P_n}{\partial y}-y\frac{\partial P_n}{\partial x}=X\frac{\partial P_{n-1}}{\partial x}+Y\frac{\partial P_{n-1}}{\partial y}+\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)P_{n-1}.$$

Remarking that $P_0=2$, the conditions (2) may finally be replaced by these

$$\left.\begin{aligned}
x\frac{\partial P_1}{\partial y} - y\frac{\partial P_1}{\partial x} &= \left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)P_0 \\
x\frac{\partial P_2}{\partial y} - y\frac{\partial P_2}{\partial x} &= X\frac{\partial P_1}{\partial x} + Y\frac{\partial P_1}{\partial y} + \left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)P_1 \\
x\frac{\partial P_3}{\partial y} - y\frac{\partial P_3}{\partial x} &= X\frac{\partial P_2}{\partial x} + Y\frac{\partial P_2}{\partial y} + \left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)P_2
\end{aligned}\right\} \quad . \quad . \quad (4)$$

etc.

4. Examining now the possibility of a series of homogeneous functions P_i satisfying the conditions (4), we introduce the values

$$X = ax^2 + 2b\,xy + cy^2 \qquad\qquad Y = a'x^2 + 2b'\,xy - a'y^2$$

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 2\,(a+b')\,x + 2\,(b-a')\,y.$$

Putting

$$P_1 = p'_0 x + p'_1 y$$

the first condition gives immediately

$$p'_0 = 4\,(a'-b) \qquad\qquad p'_1 = 4\,(a+b').$$

Thus the function P_1 divided by 4

$$P_1 = (a'-b)\,x + (a+b')\,y = p_0 x + p_1 y$$

always exists. Proceeding now to

$$P_2 = q_0 x^2 + q_1 xy + q_2 y^2$$

we find that this function exists when the coefficients satisfy the following equations

$$\begin{aligned}
q_1 &= (3a+2b')\,p_0 + a'p_1 \\
2q_2 - 2q_0 &= (4b-2a')\,p_0 + 2\,(a+2b')\,p_1 \\
-q_1 &= cp_0 + (2b-3a')\,p_1.
\end{aligned}$$

Therefore it is necessary that

$$(3a+2b'+c)\,p_0 + 2\,(b-a')\,p_1 = 0$$

or that

$$(a'-b)\,(a+c) = 0.$$

By hypothesis $a + c \neq 0$, thus the first condition is

$$a' - b = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

If this condition is fulfilled q_0 may be chosen arbitrarily, for instance $q_0 = 0$, and we have

$$P_2 = (a+b')\,[bxy + (a+2b')\,y^2].$$

From this form it is evident that this function and all the following functions P_3 $P_4 \ldots$ vanish when

$$a + b' = 0. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

The origin is therefore a centrum if both the conditions (5) and (6) are satisfied.

In this case the integral of the differential equation

$$\frac{dy}{dx} = \frac{-x + bx^2 - 2axy - by^2}{y + ax^2 + 2bxy + cy^2}$$

takes the finite form

$$x^2 + y^2 + F_3 = const.$$

where F_3 may be determined from (3).

The integral curve

$$x^2 + y^2 - \frac{2}{3} bx^3 + 2ax^2y + 2bxy^2 + \frac{2}{3} cy^3 = const.$$

thus represents a series of closed curves round the origin of coordinates.

5. Assuming now

$$a' = b \text{ and } a + b' \neq o$$

we may omit the factor $a + b'$ and write

$$P_2 = bxy + (a+2b')\, y^2 = q_1\, xy + q_2\, y^2.$$

Now it is always possible to find a homogeneous function

$$P_3 = r_0\, x^3 + r_1\, x^2y + r_2\, xy^2 + r_3 y^3$$

satisfying the condition

$$x \frac{\partial P_3}{\partial y} - y \frac{\partial P_3}{\partial x} = X \frac{\partial P_2}{\partial x} + Y \frac{\partial P_2}{\partial y} + \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) P_2$$

and the coefficients are found to be

$$r_0 = -\frac{b}{3}(a + 2c)$$

$$r_1 = b^2$$

$$r_2 = b\,(2a + 4b' - c)$$

$$r_3 = \frac{1}{3}(2a^2 + 10ab' + 3b^2 + 12b'^2).$$

Proceeding to

$$P_4 = s_0\, x^4 + s_1\, x^3\, y + s_2\, x^2\, y^2 + s_3\, xy^3 + s_4\, y^4$$

we find that the following relations between the coefficients of P_4 and P_3 must exist

$$s_1 = (5a+2b')\, r_0 + br_1$$

$$2s_2 - 4s_0 = 6br_0 + 4\,(a+b')\, r_1 + 2br_2$$

$$3s_3 - 3s_1 = 3cr_0 + 3br_1 + (3a+6b')\, r_2 + 3br_3$$

$$4s_4 - 2s_2 = 2cr_1 + (2a+8b')\, r_3$$

$$- s_3 = cr_2 - 3br_3$$

which are impossible unless

$$(5a+2b'+c)\,r_0 + 2br_1 + (a+2b'+c)\,r_2 - 2br_3 = 0$$

or

$$b\,(a+c)\,(2b' - 3a - 5c) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This condition breaks up into three conditions which will be considered separately.

Supposing in the first place

$$a' = b \text{ and } b = 0$$

the differential equation may be solved. For putting

$$\frac{x^2}{2} = t \qquad 1 - 2b'y = z$$

we obtain the linear differential equation

$$\frac{dt}{dz} - \frac{a}{b'z}\,t = \frac{1}{8b'^3} \cdot \frac{2b'+c - 2\,(b'+c)\,z + cz^2}{z}.$$

A particular integral of this equation being

$$t = \alpha + \beta z + \gamma z^2$$

where

$$\alpha = -\frac{2b'+c}{8ab'^2} \quad \beta = -\frac{2\,(b'+c)}{8b'^2\,(b'-a)} \quad \gamma = \frac{c}{8b'^2\,(2b'-a)}.$$

the general integral of the original differential equation takes the form

$$\{x^2 - 2\,(\alpha+\beta+\gamma) + 4b'\,(\beta+2\gamma)\,y - 8b'^2\,\gamma y^2\}\,(1-2b'y)^{-\frac{a}{b'}} = const.$$

which for small values of x and y may be expanded in the form

$$x^2 + y^2 + F_3 + F_4 \ldots = const.$$

In this case therefore the origin is a centrum.

7. If, in the second place

$$a' = b \text{ and } a + c = 0$$

the corresponding differential equation

$$\frac{dy}{dx} = \frac{-x + bx^2 + 2b'xy - by^2}{y + ax^2 + 2bxy - ay^2}$$

has three particular integrals of the form

$$y = Ax + B$$

for substituting this value and equalling the coefficients of the different powers of x in both members, we have

$$A(2bA - aA^2 + a) = 2b'A - bA^2 + b$$
$$A(2bB + A - 2aAB) = 2b'B - 2bAB - 1$$
$$AB(1 - aB) = - bB^2$$

which are satisfied by the roots of the cubic

$$aA^3 - 3bA^2 + (2b' - a)A + b = 0,$$

and by

$$B = \frac{A}{aA-b}.$$

In this case, the general integral may be written

$$(y - y_1)^{\lambda_1}(y - y_2)^{\lambda_2}(y - y_3)^{\lambda_3} = const.$$

where $y_1\ y_2\ y_3$ stand for the three particular integrals and $\lambda_1\ \lambda_2\ \lambda_3$ are certain constants. To prove this we will show that the necessary and sufficient condition [1], that

$$(y - y_1)(y - y_2)(y - y_3)\left\{\frac{\lambda_1}{y-y_1} + \frac{\lambda_2}{y-y_2} + \frac{\lambda_3}{y-y_3}\right\}$$

is divisible by $y + ax^2 + 2bxy - ay^2$, may be fulfilled by choosing properly the constants $\lambda_1\ \lambda_2\ \lambda_3$.

This condition may be written, τ being a constant factor

$$\lambda_1 + \lambda_2 + \lambda_3 = - a\tau$$
$$\lambda_1(y_2 + y_3) + \lambda_2(y_1 + y_3) + \lambda_3(y_1 + y_2) = -(1 + 2bx)\tau$$
$$\lambda_1 y_2 y_3 + \lambda_2 y_1 y_3 + \lambda_3 y_1 y_2 = ax^2\tau$$

or, replacing $y_1\ y_2\ y_3$ by their values

(a) $\lambda_1 + \lambda_2 + \lambda_3 = - a\tau$

(b) $\lambda_1(A_2 + A_3) + \lambda_2(A_1 + A_3) + \lambda_3(A_1 + A_2) = - 2b\tau$

(c) $\lambda_1(B_2 + B_3) + \lambda_2(B_1 + B_3) + \lambda_3(B_1 + B_2) = - \tau$

(d) $\lambda_1 A_2 A_3 + \lambda_2 A_1 A_3 + \lambda_3 A_1 A_2 = a\tau$

(e) $\lambda_1(A_2 B_3 + A_3 B_2) + \lambda_2(A_1 B_3 + A_3 B_1) + \lambda_3(A_1 B_2 + A_2 B_1) = 0$

(f) $\lambda_1 B_2 B_3 + \lambda_2 B_1 B_3 + \lambda_3 B_1 B_2 = 0$

As

$$A_1 + A_2 + A_3 = \frac{3b}{a},\quad A_1A_2 + A_1A_3 + A_2A_3 = \frac{2b' - a}{a},\quad A_1A_2A_3 = -\frac{b}{a}$$

the conditions (a) (b) and (d) may be written

[1]) Korkine, Math. Ann. 48, p. 350.

$$\lambda_1 + \lambda_2 + \lambda_3 = -a\tau$$
$$\lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 A_3 = -b\tau$$
$$\lambda_1 A_2 A_3 + \lambda_2 A_1 A_3 + \lambda_3 A_1 A_2 = a\tau.$$

Hence

$$\lambda_1 = \frac{\tau}{N}(A_2 - A_3)(aA_1^2 - 2bA_1 - a)$$

$$\lambda_2 = \frac{\tau}{N}(A_3 - A_1)(aA_2^2 - 2bA_2 - a)$$

$$\lambda_3 = \frac{\tau}{N}(A_1 - A_3)(aA_3^2 - 2bA_3 - a)$$

where

$$N = A_1^2(A_3 - A_2) + A_2^2(A_1 - A_3) + A_3^2(A_1 - A_2)$$

and it is still to be proved that the three conditions (c) (e) and (f) are satisfied by these values of $\lambda_1\ \lambda_2\ \lambda_3$. Before proving this, we will write $\lambda_1\ \lambda_2\ \lambda_3$ in another form.

The values of $B_1\ B_2\ B_3$ expressed in the values $A_1\ A_2\ A_3$ are found to be

$$B_1 = \sigma(aA_1^2 - 2bA_1 - a)$$
$$B_2 = \sigma(aA_2^2 - 2bA_2 - a)$$
$$B_3 = \sigma(aA_3^2 - 2bA_3 - a)$$

and it is not difficult to find the value of σ. For introducing

$$A = \frac{bB}{aB-1}$$

in the cubic, it is evident that the values of B are the roots of the cubic

$$2a(ab' - b^2)B^3 + (3b^2 - 4ab' - a^2)B^2 + 2(a + b')B - 1 = 0$$

Therefore

$$B_1 + B_2 + B_3 = -\frac{3b^2 - 4ab' - a^2}{2a(ab' - b^2)}$$

and

$$\sigma = \frac{1}{2(b^2 - ab')}.$$

Now

$$aA_1 - b)(a A_2 - b)(aA_3 - b) = N_1 N_2 N_3 =$$
$$= a^3 A_1 A_2 A_3 - a^2 b(A_1 A_2 + A_1 A_3 + A_2 A_3) + ab^2(A_1 + A_2 + A_3) - b^3 =$$
$$= 2b(b^2 - ab') = \frac{b}{\sigma}$$

and finally

$$\lambda_1 = \frac{\tau}{bN} A_1(A_2-A_3)N_2N_3$$

$$\lambda_2 = \frac{\tau}{bN} A_2(A_3-A_1)N_1N_3$$

$$\lambda_3 = \frac{\tau}{bN} A_3(A_1-A_2)N_1N_2.$$

With these values and $B_i = \frac{A_i}{N_i}$ the first member of (c) reduces to

$$\frac{\tau}{bN} A_1 (A_2-A_3) [2aA_2A_3 - b(A_2+A_3)] +$$

$$+ \frac{\tau}{bN} A_2 (A_3-A_1) [2aA_1A_3 - b(A_1+A_3)] +$$

$$+ \frac{\tau}{bN} A_3 (A_1-A_2) [2aA_1A_2 - b(A_1+A_2)] =$$

$$= -\frac{\tau}{N} [A_1 (A_2^2-A_3^2) + A_2 (A_3^2-A_1^2) + A_3 (A_1^2-A_2^2)] = -\tau.$$

Further, the first member of (e) takes the form

$$\frac{\tau}{bN} A_1 (A_2-A_3) A_2A_3 [a(A_2+A_3) - 2b] +$$

$$+ \frac{\tau}{bN} A_2 (A_3-A_1) A_1A_3 [a(A_1+A_3) - 2b] +$$

$$+ \frac{\tau}{bN} A_3 (A_1-A_2) A_1A_2 [a(A_1+A_2) - 2b] = 0$$

and the first member of (f)

$$\frac{\tau}{bN} \{A_1A_2A_3 (A_2-A_3) + A_1A_2A_3 (A_3-A_1) + A_1A_2A_3 (A_1-A_2)\} = 0.$$

The general integral is therefore

$$(y - A_1x - B_1)^{\lambda_1} (y - A_2x - B_2)^{\lambda_2} (y - A_3x - B_3)^{\lambda_3} = Const.$$

where

$$\lambda_1 : \lambda_2 : \lambda_3 = (A_2-A_3)(aA_1^2-2bA_1-a) :$$
$$(A_3-A_1)(aA_2^2-2bA_2-a) : (A_1-A_2)(aA_3^2-2bA_3-a).$$

When the cubic in A has a pair of imaginary roots the corresponding particular integrals are conjugate imaginary and therefore the general integral is imaginary unless two of the quantities λ are equal. This is only possible if $b(a+b')=0$ and these cases have already been considered in Art. 6 and Art. 4. We must thus suppose that all the roots of the cubic in A are real.

For small values of x and y the general integral may be expanded in the form

$$x^2 + y^2 + F_3 + F_4 + \ldots = Const.$$

which proves that in this case the origin is a centrum.

8. If we assume in the third place

$$a' = b \text{ and } 2b' = 3a + 5c$$

the corresponding differential equation takes the form

$$\frac{dy}{dx} = \frac{-x + bx^2 + (3a + 5c)\, xy - by^2}{y + ax^2 + 2bxy + cy^2} = \frac{-x + Y}{v + X}.$$

Here

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 5(a + c)x$$

and omitting constant factors we have, as before

$$P_0 = 2$$
$$P_1 = y$$
$$P_2 = bxy + (4a + 5c)\, y^2$$
$$P_3 = -\frac{b}{3}(a + 2c)\, x^3 + b^2x^2y + b\,(8a + 9c)\, xy^2 + $$
$$+ \frac{1}{3}(4a^2 + 115ac + 3b^2 + 75c^2)\, y^3.$$
$$P_4 = s_0x^4 + s_1x^3y + s_2x^2y^2 + s_3xy^3 + s_4y^4$$

where

$$s_0 = 0$$
$$s_1 = -\frac{b}{3}[8a^2 + 10ac - 3b^2 + 10c^2]$$
$$s_2 = 12b^2(a + c)$$
$$s_3 = b\,[44a^2 + 107ac + 3b^2 + 66c^2]$$
$$s_4 = \frac{1}{6}[308a^3 + 1245a^2c + 57ab^2 + 1675ac^2 + 69b^2c].$$

Proceeding to

$$P_5 = t_0x^5 + t_1x^4y + t_2x^3y^2 + t_3x^2y^3 + t_4xy^4 + t_5y^5$$

and

$$P_6 = u_0x^6 + u_1x^5y + \ldots + u_6y_6$$

we obtain

$$t_1 = (9a + 5c)s_0 + bs_1$$
$$2t_2 - 5t_0 = 8bs_0 + (11a + 10c)\, s_1 + 2bs_2$$
$$3t_3 - 4t_1 = 4cs_0 + 5bs_1 + (13a + 15c)\, s_2 + 3bs_3$$
$$4t_4 - 3t_2 = 3cs_1 + 2bs_2 + (15a + 20c)\, s_3 + 4bs_4$$
$$5t_5 - 2t_3 = 2cs_2 - bs_3 + (17a + 25a)\, s_4$$
$$-t_4 = cs_3 - 4bs_4.$$

from which always the coefficients t may be determined, and

$$\begin{aligned}
u_1 &= (10a + 5c)\, t_0 + bt_1 \\
2u_2 - 6u_0 &= 10bt_0 + (12a + 10c)\, t_1 + 2bt_2 \\
3u_3 - 5u_1 &= 5ct_0 + 7bt_1 + (14a + 15c)\, t_2 + 3bt_3 \\
4u_4 - 4u_2 &= 4ct_1 + 4bt_2 + (16a + 20c)\, t_3 + 4bt_4 \\
5u_5 - 3u_3 &= 3ct_2 + \ bt_3 + (18a + 25c)\, t_4 + 5bt_5 \\
6u_6 - 2u_4 &= 2ct_3 - 2bt_4 + (20a + 30c)\, t_5 \\
- u_5 &= ct_4 - 5bt_5 .
\end{aligned}$$

From these the coefficients u may be found only when

$$(50a + 30c)t_0 + 12bt_1 + (14a + 18c)t_2 + 4bt_3 + (18a + 30c)t_4 - 20bt_5 = 0$$

or when

$$-\frac{20}{3}\, bt_1 + (10a + 6c)\,(2t_2 - 5t_0) + \frac{4}{3}\, b\,(3t_3 - 4t_1) +$$

$$+ \frac{1}{3}\,(34a + 30c)\,(4t_4 - 3t_2) + 4b\,(5t_5 - 2t_3) + \frac{1}{3}\,(190a + 210c)(-t_4) = 0.$$

Substituting now the values s, we have the condition

$$(a+c)\,[2bs_0 + (11a + 9c)\, s_1 + 6bs_2 + (17a + 27c)\, s_3 - 14bs_4] = 0$$

or

$$b\,(a + c)\,(ac + b^2 + 2c^2) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If $b = 0$, the differential equation reduces to

$$\frac{dy}{dx} = \frac{-x + (3a + 5c)\, xy}{y + ax^2 + cy^2}$$

which has been considered in Art. 5.

If $a + c = 0$ we have

$$\frac{dy}{dx} = \frac{-x + bx^2 - 2axy - by^2}{y + ax^2 + 2bxy - ay^2}$$

which has been treated in Art. 7.

When however $ac = -b^2 - 2c^2$ the differential equation takes a new form

$$\frac{dy}{dx} = \frac{-cx + bcx^2 - (3b^2 + c^2)\, xy - bcy^2}{cy - (b^2 + 2c^2)\, x^2 + 2bc\, xy + c^2y^2} .$$

To solve this we will try to find particular integrals. If the conic

$$x^2 + 2Hxy + By^2 + 2Gy + 2Fy + C = 0$$

satisfies the equation,

$$-\frac{x + Hy + G}{Hx + By + F} = \frac{-cx + bcx^2 - (3b^2 + c^2)\, xy - bcy^2}{cy - (b^2 + 2c^2)\, x^2 + 2bc\, xy + c^2y^2}$$

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must be equivalent with

$$(x^2 + 2H\,xy + By^2 + 2G\,x + 2F\,y + C)\,(\alpha x + \beta y) = 0.$$

This may be done by chosing $\alpha = -(b^2 + c^2)$, $\beta = 0$,

$$H = \frac{c}{b}, \quad B = \frac{c^2}{b^2}, \quad G = 0, \quad F = \frac{c}{b^2}, \quad C = \frac{c^2}{c^2\,(b^2 + c^2)}.$$

Hence a first particular integral is the conic

$$(bx + cy)^2 + 2cy + \frac{c^2}{b^2 + c^2} = 0.$$

In the same way we find that the differential equation is satisfied by the curve of the third degree

$$(bx + cy)^3 + 3cy\,(bx + cy) + 3cy + \frac{c^2}{b^2 + c^2} = 0.$$

Combining these, the general integral is found to be

$$\left\{(bx+cy)^3 + 3cy\,(bx+cy) + 3cy + \frac{c^2}{b^2+c^2}\right\}^2 = Const. \times \left\{(bx+cy)^2 + 2cy + \frac{c^2}{b^2+c^2}\right\}^3$$

which for small values of x and y may be expanded in the form

$$x^2 + y^2 + F_3 + F_4 + \ldots = Const.$$

Therefore the origin is also a centrum in this case.

Resuming we may conclude that when the differential equation is reduced to the form

$$\frac{dy}{dx} = \frac{-x + a'x^2 + 2b'xy - a'y^2}{y + ax^2 + 2bxy + cy^2}$$

the origin is a centrum only in the four following cases

1. $a' = b$, and $a + b' = 0$
2. $a' = b = 0$
3. $a' = b$, $a + c = 0$ and the roots of
   $aA^3 - 3bA^2 + (2b' - a)\,A + b = 0$ all real.
4. $a' = b$, $2b' = 3a + 5c$ and $ac + b^2 + 2c^2 = 0$.

In all other cases the origin will be a focus for the real integral curves.