Citation:

W. van der Woude, Double points of a c6 of genus 0 or 1, in:
KNAW, Proceedings, 13 II, 1910-1911, Amsterdam, 1911, pp. 629-635

Mathematics. — *Double points of a c_6 of genus 0 or 1.* By Dr. W. VAN DER WOUDE. (Communicated by Prof. SCHOUTE.)

(Communicated in the meeting of November 26, 1910).

§ 1. A curve of order n is in general determined by $\frac{1}{2} n (n + 3)$ single conditions. So a curve of order five is determined by 20 points, or — a node counting for 3 single data — by 6 double points and 2 single points. However, it is not possible to take arbitrarily the double points of a rational curve of higher order than the fifth; for the number of double points of this curve amounts to $\frac{1}{2} (n-1) (n-2)$ and for $n > 5$ three times this amount is larger than $\frac{1}{2} n (n + 3)$. Between the double points of a curve of order n whose number of double points is greater than $\frac{1}{6} n (n + 3)$ one or more relations must exist. "So far no attempt seems to have been made to express those relations geometrically"; [1]) in the following paragraphs I intend to do this with reference to the curve of order six.

§ 2. A curve of order six is determined by 27 single data, hence by 27 points. It can possess at most 10 double points; according to the preceding 9 of these can be taken arbitrarily and really a curve of order six is determined by 9 double points taken arbitrarily; this is however degenerated into a cubic curve to be counted double through those 9 points.

If therefore a (non-degenerated) curve of order six has 9 double points, there must be a relation between these already; only 8 can be taken arbitrarily. In future we shall understand by $D_1, D_2, .., D_8$ points taken arbitrarily; the locus of the point forming together with these a system of 9 double points of a curve of order six not degenerating into a cubic curve to be counted double we shall for the present represent by j_n, whilst by c_n an arbitrary curve of order n is meant.

§ 3. If D_9 is a point of j_n there exists a (non-degenerated) curve of order six possessing a double point in each of the points $D_1, D_2, \ldots D_9$; furthermore we can lay through these 9 points a cubic curve. If these

[1]) SALMON FIEDLER, Höhere ebene Kurven, p. 42.

two curves are represented respectively by the equations $w_6 = 0$ and $u_3 = 0$, then by $w_6 + \lambda u^2_3 = 0$ a pencil of curves of order six is determined, each of which possesses in each of the base points of the pencil a double point. Hence

If D_1, $D_2 \ldots D_9$ are double points of a curve of order six not degenerating into a cubic curve to be counted double, these points are the base points of a pencil of curves of order six with nine common double points.

For shortness'sake we shall in future call such a pencil a c_6 pencil with 9 double points; of course through each point passes *one* curve of this pencil.

§ 4. Out of the c_3 pencil determined by the base points $D_1, D_2 \ldots D_8$ we choose one u_3; we furthermore introduce a variable c_6, possessing double points in $D_1, D_2 \ldots D_8$, but being for the rest undetermined. These two curves intersect each other in 2 more points; the line connecting these two points intersects u_3 in another point T. This last point is according to the *Residual Theory* of SYLVESTER a fixed point, i. e. independent of the c_6 chosen by us. Point T is easy to determine; it is evident that if we take for c_6 a c_3 counting double, T is the tangential point of B_9, the ninth base point of the c_3 pencil through $D_1, D_2 \ldots D_8$.

If of a pencil c_6 the double points $D_1, D_2 \ldots D_8$ and a point P are determined, then according to the preceding another point P' of c_6 is determined; if namely we lay through $D_1, D_2 \ldots D_8$ and P a cubic u_3 and if we determine on it the tangential point T of B_9, then the third point of intersection of TP with u_3 is the point P' under discussion. Farthermore it is evident from this that only these points of u_3 having T as tangential point *can* be the ninth double points of a c_6, which has already double points in $D_1, D_2 \ldots D_8$. If however we impose the condition that this c_6 may not have degenerated into a c_3 counting double, then one of these points, viz. B_9 does not count; for, there is not a single non-degenerated c_6 which has double points in $D_1, D_2 \ldots D_8$ and B_9. For then we should be able to bring a c_3 through these points and a point chosen arbitrarily on the curve c_6 and the curves c_3 and c_6 would have nineteen points in common. If however J, J', and J'' are the other points of u_3 having T as tangential point, then each c_6, having double points in $D_1, D_2 \ldots D_8$ and passing through one of these points, will have there two points in common with u_3.

§ 5. We can determine a c_6 pencil by the double points $D_1, D_2 \ldots D_8$

and 2 points P and Q chosen arbitrarily; the two other base points of the pencil P' and Q' are then by this completely determined. From the preceding is evident, how we can determine those two last points and also, that *one* of these points e. g. P' is independent of Q and the other Q' of P.

We now again understand by u_3 an arbitrary cubic through $D_1, D_2 \ldots D_8$, whilst also J, J' and J'' have the same meaning as in § 4. We then further regard the c_6 pencil (β), having $D_1, D_2 \ldots D_8$ as double points and moreover J and an arbitrary point Q as single base points. An arbitrary curve out of (β) will touch u_3 in J whilst the last base point of (β) lies on the c_3 through $D_1, D_2 \ldots D_8$ and Q and is to be determined in the way indicated above; the line touching in J the u_3 as well as an arbitrary curve out of (β) we shall call j. If we then draw through J an arbitrary line l and if A is a point moving along l, then always through A passes *one* curve a_6 out of the pencil (β); if we allow A to coincide with J then the lines j and l will both have in J two points in common with a_6. From this ensues that J is now a double point of a_6 and lies therfore on the curve which we have indicated by j_n.

If inversely it is given that J is a point of j_n and if we bring a c_3 through $D_1, D_2 \ldots D_8$ and J, then J must possess the same tangential point on c_3 as B_9. We have then proved:

If we generate a c_6 pencil with double points in the points D_1, $D_2 \ldots D_8$ chosen arbitrarily and single base points in a point J of the curve j_n and in a point Q chosen arbitrarily, then the curves of this pencil have in J a common tangent. In this pencil is included a curve, having in J a ninth double point.

§ 6. We have seen, that on an arbitrary curve u_3 out of the c_3 pencil having $D_1, D_2 \ldots D_8$ and B_9 as base points lie three points of j_n; these points have on u_3 the same tangential point T as B_9. We now regard first the locus of T when u_3 describes the c_3 pencil which we shall now call (β'). Each line l through B_9 determines *one* curve out of (β'), touching it; so l intersects the indicated locus besides in B_9 in *one* point. Farthermore this locus has in B_9 a triple point, three curves out of (β') possessing in B_9 a point of inflexion. The point T describes therefore a quartic curve t_4 possessing in B_9 a triple point; the points $D_1, D_2 \ldots D_8$ lie also on t_4, as each of the lines B_9D is touched by *one* curve.

Let T' be the tangential point of D_1 on u_3, then if again u_3 describes the pencil (β'), T' describes a quartic curve t'_4; t_4 and t'_4 have

41*

besides the base points of (β') three more points in common, which points to the fact that three times one and the same point is at the same time tangential point of B_9 and D_1. If, however, B_9 and D_1 have on a curve out of (β') the same tangential point, then D_1 will lie on the curve j_n. This last will be cut by u_3 in each of the points $D_1, D_2 \ldots D_8$ three times and once in three other points; so it is of order nine.

If $D_1, D_2 \ldots D_8$ are points chosen arbitrarily, then the locus of the point which can be the ninth double point of a curve of order six already possessing double points in $D_1, D_2 \ldots D_8$ is a curve j_n of order nine with triple points in $D_1, D_2 \ldots D_8$.

Moreover we have found the following generation of j_n:

If we determine on a curve u_3 out of the c_3-pencil (β') with the base points $D_1, D_2 \ldots, D_8$ the points having the same tangential points, as the ninth base point B_9, then if u_3 describes the pencil (β') these points will describe the curve j_n.

§ 7. We shall now show analytically that the curve j_n is of order nine and possesses triple points in $D_1, D_2, \ldots D_8$.

To this end we regard the net of curves $v \equiv w_6 + \lambda u^2_3 + \mu v^2_3 = 0$, where $w_6 = 0$ represents a curve of order six with double points in $D_1, D_2 \ldots D_8$, whilst $u_3 = 0$ and $v_3 = 0$ are the equations of two cubic curves through those eight points. The curves of the net passing through an arbitrary point form a pencil; we choose the pencil of curves passing through an arbitrary point J of j_n. In § 5 we have seen that in this pencil appears *one* curve possessing in J a ninth double point; therefore:

Each point of j_n is the ninth double point of one of the curves contained in the net (v).

§ 8. We take an arbitrary triangle $O_1 O_2 O_3$ as triangle of coordinates; the locus of the double points of the net $v \equiv w_6 + \lambda u^2_3 + + \mu v^2_3 = 0$ is then found by elimination of λ and μ out of the equations $\frac{dv}{dx_1} = 0$, $\frac{dv}{dx_2} = 0$ and $\frac{dv}{dx_3} = 0$.

As equation of that locus we then find:

$$uv \left\{ \frac{dw}{dx_1}\left(\frac{du}{dx_2}\frac{dv}{dx_3} - \frac{du}{dx_3}\frac{dv}{dx_2}\right) + \frac{dw}{dx_2}\left(\frac{du}{dx_3}\frac{dv}{dx_1} - \frac{du}{dx_1}\frac{dv}{dx_3}\right) + \right.$$
$$\left. + \frac{dw}{dx_3}\left(\frac{du}{dx_1}\frac{dv}{dx_2} - \frac{du}{dx_2}\frac{dv}{dx_1}\right)\right\} = 0.$$

The factor uv in the first member of this equation means simply that each point of the curves u_3 and v_3 counting double can be regarded as a double point; as locus of the point J we find:

$$\frac{dw}{dx_1}\left(\frac{du}{dx_2}\frac{dv}{dx_3}-\frac{dv}{dx_2}\frac{du}{dx_3}\right)+\frac{dw}{dx_2}\left(\frac{du}{dx_3}\frac{dv}{dx_1}-\frac{du}{dx_1}\frac{dv}{dx_3}\right)+$$
$$+\frac{dw}{dx_3}\left(\frac{du}{dx_1}\frac{dv}{dx_2}-\frac{du}{dx_2}\frac{dv}{dx_1}\right)=0.$$

This equation of order nine represents the curve j_n which for the future we shall call j_9.

We now allow the vertex O_3 of the triangle of coordinates to coincide with D_1; the equations of the curves w_6, u_3, and v_3 are ranged according to the descending powers of x_3, written thus:

$$w_6 \equiv x^4_3 (ax^2_1 + 2bx_1 x_2 + cx^4_2) + \ldots = 0$$
$$u_3 \equiv x^2_3 (a'x_1 + b'x_2) + \ldots = 0$$
$$v_3 \equiv x^2_3 (a''x_1 + b''x_2) + \ldots = 0$$

The first member of the equation representing the curve j_9 evidently possesses now no term in which x_3 appears to a lower power than the sixth; so D_1 is a triple point of j_9 and D_2, $D_3 \ldots D_8$ likewise.

§ 9. To the curves of order six possessing in $D_1, D_2 \ldots D_8$ double points belongs one degenerated into the line D_1D_2 and a curve of order five having in $D_3, D_4 \ldots D_8$ double points and passing moreover through D_1 and D_2. The latter is cut by D_1D_2 in three points more which must lie on j_9; thus on each of the lines connecting $D_i D_k$ three points of j_9 can be indicated.

Let us suppose a conic $D_1, D_2 \ldots D_5$ and a quartic possessing in D_6, D_7 and D_8 double points and passing also through $D_1, D_2 \ldots D_5$; then these form together also a c_6 with double points in $D_1, D_2 \ldots D_8$; the remaining three points of intersection of the two curves lie on j_9. Thus on each of the 56 conics $D_i D_k D_l D_m D_n$ three points of j_9 are determined.

Each c_3 through $D_1, D_2 \ldots D_8$ cuts j_9 besides in these points in three points more; we have already seen how these points can be determined. We have also seen that B_9, the ninth base point of the c_3 pencil, does not lie on j_9; by allowing the vertex O_3 of the triangle of coordinates to coincide with B_9 we can easily deduce this out of the equation of j_9.

§ 10. Let w_6 be a curve of order six possessing in $D_1, D_2 \ldots D_9$ double points, whilst u_3 is the cubic through those points; then by

$\beta \equiv w_6 + \lambda u_3^2 = 0$ is represented a pencil (β) of curves of order six with nine double points. To a pencil of curves of order nine belong in general $3(n-1)^2$ curves possessing a double point; this amount must however be diminished by seven for each common double point which the curves possess in the base points. So we can expect that there will be twelve points, which can appear as tenth double point of a curve out of the pencil (β). It seems however desirable to prove that in this case too where one of the curves is a c_3 counting double the number of these points is twelve.

The points indicated are found by elimination of λ out of the equations:

$$\frac{d\beta}{dx_1} = 0\,,\ \frac{d\beta}{dx_2} = 0 \text{ and } \frac{d\beta}{dx_3} = 0, \text{ or}$$

$$\frac{dw}{dx_1} + \lambda \frac{du}{dx_1} = 0\,,\ \frac{dw}{dx_2} + \lambda \frac{du}{dx_2} = 0\,,\ \frac{dw}{dx_3} + \lambda \frac{du}{dx_3} = 0.$$

By this elimination we find:

$$\frac{\dfrac{dw}{dx_1}}{\dfrac{du}{dx_1}} = \frac{\dfrac{dw}{dx_2}}{\dfrac{du}{dx_2}} = \frac{\dfrac{dw}{dx_3}}{\dfrac{du}{dx_3}},$$

which equations represent three curves, whose common points of intersection — if only differing from the base points of (β) — are the demanded double points. (The factor $u = 0$, which we have omitted means that each point of u_3 can be regarded as a double point,

We write them in this form:

$$\frac{dw}{dx_1}\frac{du}{dx_2} - \frac{dw}{dx_2}\frac{du}{dx_1} = 0 \quad . \ . \ . \ . \ . \ . \ . \ . \quad (1)$$

$$\frac{dw}{dx_1}\frac{du}{dx_3} - \frac{dw}{dx_3}\frac{du}{dx_1} = 0 \quad . \ . \ . \ . \ . \ . \ . \ . \quad (2)$$

$$\frac{dw}{dx_2}\frac{du}{dx_3} - \frac{dw}{dx_3}\frac{du}{dx_2} = 0 \quad . \ . \ . \ . \ . \ . \ . \ . \quad (3)$$

The curves represented by (1) and (2) have forty-nine points of intersection; among these there are however ten which do not lie on (3), viz. the points which satisfy $\dfrac{dw}{dx_1} = 0$ and $\dfrac{du}{dx_1} = 0$. The remaining thirty-nine points must still be diminished by the points of intersection lying in $D_1, D_2 \ldots D_9$. If again we allow the vertex O_3 of the triangle of coordinates to coincide with D_1 and if we note down the equations of w_6 and u_3 ranged according to the descending powers of x_3 (see § 8), it is evident, that (2) and (3) possess in D_1 a double point, whilst (1) has in D_1 a single point; farthermore they have all three in D_1 a common tangent with the equation

$$x_1\,(ab'-a'b) + x_2\,(bb'-ac) = 0.$$

So in each of the base points of the pencil (β) lie three common points of intersection of (1), (2) and (3); besides $D_1, D_2 \ldots D_9$ the curves (1), (2) and (3) have $39 - 3 \times 9 = 12$ more common points So there are really 12 curves, possessing besides $D_1, D_2 \ldots D_9$ still a tenth double point.

We can directly indicate those twelve points. Each of those points must lie on the curve j_9, determined by $D_1, D_2 \ldots D_8$ and likewise on the curve j'_9, which is determined in the same way by $D_2, D_3 \ldots D_9$. These two curves have 81 points of intersection of which, however, nine lie in each of the points $D_2, D_3 \ldots D_8$ and three in each of the points D_1 and D_9. The remaining points are those indicated.

To a c_6-pencil with nine double points belong twelve curves possessing still a tenth double point.

§ 11. I wish to draw attention to another property of these points. If $P(x'_1, x'_2, x'_3)$ is an arbitrary point then the polar lines of P with respect to the curves out of the pencil (β) are represented by

$$x_1\left(\frac{dw}{dx_1} + \lambda u\,\frac{du}{dx_1}\right)_P + x_2\left(\frac{dw}{dx_2} + \lambda u\,\frac{du}{dx_2}\right)_P + x_3\left(\frac{dw}{dx_3} + \lambda u\,\frac{du}{dx_3}\right)_P = 0.$$

We shall now put the question whether it is possible to give P such a position that the polar line of P with respect to each curve out of the pencil is the same. Evidently for that it is necessary that the coordinates of P satisfy the equation $u = 0$, or the equations,

$$\frac{\dfrac{dw}{dx_1}}{\dfrac{du}{dx_1}} = \frac{\dfrac{dw}{dx_2}}{\dfrac{du}{dx_2}} = \frac{\dfrac{dw}{dx_3}}{\dfrac{du}{dx_3}} \quad \ldots\ldots\ldots \quad (I)$$

So P must be on u_3 or — as the system of equations (I) is the same as the system which we came across in § 10 — P must be one of the 12 points found there. Hence:

If $D_1, D_2 \ldots D_{10}$ are the double points of a rational curve of order six, then the polar line of one of these points with respect to the curves out of the c_6-pencil possessing the other nine as double points, is a fixed line.