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these oscillations were remarkably smaller and had greater latency.

I have likewise tried to conduct the active current of the nervus acusticus of the frog. The nervus acusticus of this experimental animal can easily be reached without injuring the normal circulation or the brain. I have however not succeeded in observing an oscillation of the string-galvanometer. This may be partly attributed to the insufficient sensibility of the instrument, on the other hand the sensibility of the frog for sounds is exceedingly trifling. A frog poisoned with strychnine which showed symptoms of spasms when being blown, did still react with muscular spasms at a shot in the immediate vicinity, but did not do so when the shot was fired at some distance. Of the different tones it was those of a low vibrating figure that caused the greater reaction upon such like frogs, the high tones often had no influence at all. From the experiments of YERKES [1]) about the vigorating influence of the tone on the effect of a mechanical irritation it appears that vibrations of 50—10000 per second, in some way or other, cause an irritation to the nervous system of the frog.

I can moreover communicate that like PIPER [2]) I could show an electric current in a pike with the string-galvanometer when with a glass-rod the otolith was moved The unpolarisable electrodes were placed in such a way that one of them touched the nervus acusticus at the parietes of the emptied skull cavity, the other stood at some indifferent point of this parietes. Care was taken that neither the electrode nor the object could move from their places. I could not observe any electric action caused by a sound of whatever nature that was conveyed by the air to the head of the pike.

Mathematics. — *"On quartic curves of deficiency zero with a rhamphoid cusp and a node."* By Prof. GEORGE MAJCEN of Agram. (Communicated by Prof. JAN DE VRIES).

1. We shall here consider the quartic curve, which has as equation

$$(mx_3^2 + nx_1x_2)^2 - x_2x_3^2(a^2x_2 - bx_3) = 0 \quad . \quad . \quad . \quad . \quad (k_4)$$

It is easy to prove, that the represented curve has a rhamphoid cusp in the vertex A (1,0,0) of the triangle of reference, that the cuspidal tangent is the side $x_2 = 0$ of this triangle, and that the vertex B (0,1,0) is a node of the curve. The side $x_1 = 0$ is chosen

1) YERKES Journ of Comp. neurol and Psychol XV, p. 279.

2) PIPER. Zentralblatt f. Physiol 1906 Bd. I, p. 293.

as the harmonic conjugate of $x_3 = 0$ with respect to the two tangents of the double point B.

Indeed, the first polar curve of the point A with respect to k_4 breaks up into $x_2 = 0$ and the conic

$$m x_3^2 + n x_1 x_2 = 0,$$

which has in the vertex A five points in common with k_4. Evidently the only simple tangent of k_4, passing through the cusp A, is represented by

$$a^2 x_2 - b x_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (t)$$

The equation of a conic, which touches k_4 on $x_2 = 0$ in the cusp A and still in two other points, can be written in the well known form :

$$x_3^2 + 2\varepsilon (m x_3^2 + n x_1 x_2) + \varepsilon^2 x_2 (a^2 x_2 - b x_3) = 0.$$

If such a conic degenerates into two straight lines, one of which will be the tangent $x_2 = 0$, then the other must be the only double tangent belonging to k_4.

If we put $2m\varepsilon = -1$, it follows from the last equation

$$x_2 = 0, \quad 4mn x_1 - (a^2 x_2 - b x_3) = 0 \quad . \quad . \quad . \quad . \quad . \quad (d)$$

and we have the equation of the *double tangent d*.

From the form of this equation is evident, that the double tangent passes through the point of intersection of the lines

$$x_1 = 0 \text{ and } a^2 x_2 - b x_3 = 0.$$

We can now say: The line (BR) joining the double point (B) to the point (R) of intersection of the double tangent with the simple tangent (t), passing through the cusp (A), is the *harmonic conjugate* of the line AB with respect to the two tangents of the double point B.

2. A pencil of conics having the two common tangents $x_2 = 0$, $x_1 = 0$, with the points of contact A and B respectively, is indicated by the equation

$$x_3^2 + \mu x_1 x_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Each of these conics cuts the curve k_4 moreover in two points M, N; let us determine the equation of the right line MN.

By eliminating x_3^2 out of the equations (k_4) and (1), we find

$$x_1 = 0, \quad x_2^2 = 0, \text{ and } MN \equiv (n - \mu m)^2 x_1 + \mu (a^2 x_2 - b x_3) = 0 \quad (2)$$

so all these lines MN, passing through the point R, determine a pencil $[R]$. We put

$$\frac{\mu}{(n - \mu m)^2} = \varrho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and from this ensues the equation

$$m^2\varrho \cdot \mu^2 - (2\varrho mn + 1)\mu + n^2\varrho = 0,$$

giving the correspondence between ϱ and μ. Each value for ϱ furnishes *two* values for μ, and for each value of μ we find *one* value for ϱ.

The curve k_4 can therefore be determined *by means of an involutory pencil of conics* (1) *and a projective pencil of rays* (2). It is easy to see, that these two pencils have $x_1 = 0$ as a corresponding common element and that consequently the generated curve of order five breaks up into $x_1 = 0$ and the curve k_4. All those conics of the pencil (1) have two tangents $x_1 = 0$, $x_2 = 0$ in common, and the vertex (R) of the pencil of rays is situated on the first of these tangents.

3. The points of contact D_1 and D_2 on the double tangent d, are projected out of the cusp A by two right lines, the equation of which will be obtained by eliminating x_1 out of (k_4) and (d); so from

$$[4m^2x_3^2 + x_2(a^2x_2 - bx_3)]^2 - 16m^2x_2x_3(a^2x_2 - bx_3) = 0$$

we find

$$AD_1, AD_2 \equiv 4m^2x_3^2 - x_2(a^2x_2 - bx_3) = 0.$$

By eliminating $x_2(a^2x_2 - bx_3)$ out of the latter equation and (k_4), we have

$$(mx_3^2 + nx_1x_2)^2 - 4m^2x_3^4 = 0$$

or

$$mx_3^2 + nx_1x_2 \pm 2mx_3^2 = 0,$$

therefore

$$nx_1x_2 - mx_3^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$nx_1x_2 + 3mx_3^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

On these conics lie the points of intersection of k_4 with the pair of lines AD_1, AD_2. The first conic (4) gives by combination with the equation (k_4) again the double tangent (d), and the second conic (5) furnishes by eliminating x_3^2 out of (k_4) and (5) the equation

$$3m(2nx_1x_2)^2 + 9nx_1x_2^2(a^2x_2 - bx_3) = 0$$

or

$$4mnx_1 + 3(a^2x_2 - bx_3) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

On this line lie the two points D'_1, D'_2 of intersection of k_4 with the projecting rays AD_1, AD_2.

The line $D'_1D'_2$ cuts the curve k_4 again in two other points E_1, E_2 and bears four projecting rays out of the cusp A. This quadruple of rays will be obtained by eliminating x_1 out of (6) and (k_4), namely

$$16m^4x_3^4 - 40m^2x_2x_3^2(a^2x_2 - bx_3) + 9x_2^2(a^2x_2 - bx_3)^2 = 0,$$

and consequently the expression to the left must be divisible by the left side of the equation for AD_1, AD_2, i. e. by

$$4m^2x_3^2 - x_2(a^2x_2 - bx_3).$$

The division gives the equation of the pair

$$AE_1, AE_2 \equiv 4m^2x_3^2 - 9x_2(a^2x_2 - bx_3) = 0.$$

By eliminating $x_2(a^2x_2 - bx_3)$ out of the latter equation and (k_4), we obtain

$$(mx_3^2 + nx_1x_2)^2 - 4m^2x_3^4 = 0,$$

therefore

$$3nx_1x_2 + mx_3^2 = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

$$3nx_1x_2 + 5mx_3^2 = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

On the conic (7) are situated the points E_1, E_2, and on the conic 8) the points E'_1, E'_2 as the fourth intersections of k_4 with the pair of lines AE_1, AE_2.

The equation of $E'_1E'_2$ will be acquired by combination of (8) with (k_4); if we eliminate x_3^2, we obtain:

$$E'_1E'_2 \equiv 4mnx_1 + 15(a^2x_2 - bx_3) = 0 \quad . \; . \; . \; . \quad (9)$$

In pursuing these projections in this manner we can show that the general equation of all these lines $D'_1D'_2, E'_1E'_2, F'_1F'_2$, and so on, will be

$$4mnx_1 + [(2k)^2 - 1](a^2x_2 - bx_3) = 0 \quad . \; . \; . \; . \quad (10)$$

k being any entire positive number or zero. All these projections are also elements of the pencil $[R]$.

The parameters in the equation (10) belonging to the mentioned projections are of the form

$$\varrho_i = \frac{(2k)^2 - 1}{4mn}, (i = k = 0, 1, 2, 3, \ldots\ldots).$$

We conclude from this that

> *the cross ratio of any four projections, determined by the equation of the form* (10) *is independent of the coefficients in the equation of* k_4, *or, this cross ratio for the same four values of* k *is unaltered for all curves of the considered form.*

The double tangent d, having the equation

$$4\,mnx_1 - (a^2x_2 - bx_3) = 0, \quad . \; . \; . \; . \; . \; . \; . \quad (d)$$

belongs also to the projections (10); indeed, the equation (10) furnishes the equation (d) for $k = 0$.

Retaining the three lines

$$x_1 = 0$$

$$a^2 x_2 - b x_3 = 0$$

and

$$4\,mn x_1 - (a^2 x_2 - b x_3) = 0,$$

we can change the fourth ray, the equation of which is of the form (10). The cross ratio of these four lines will be:

$$\Delta = 1 - (2k)^2 \qquad (k = 1, 2, 3, 4, \ldots\ldots),$$

therefore the value of Δ *is independent of the curve* k_4, *and is a function of* k *alone.*

4. We have seen, that in the projective generation of k_4 to any ray of pencil $[R]$ correspond two conics of the involutory pencil. The values of the parameters μ for these conics, which correspond to the right lines, indicated by (10), will be determined out of the equation, with respect to (3):

$$\frac{(2k)^2 - 1}{4mn} = \frac{\mu}{(n - m\mu)^2}$$

This quadratic equation furnishes two pairs of values for μ, namely

$$\mu_{1,2} = \frac{n\,[2k \pm 1]}{m\,[2k \mp 1]}.$$

We can now determine any number of discrete points of k_4 as follows; putting

$$p \equiv a^2 x_2 - b x_3 = 0$$

we can write

$$\left.\begin{aligned} 4\,mn x_1 + [2k+1]\,[2k-1]\,p &= 0 \\ m\,[2k \mp 1]\,x_3^2 + n\,[2k \pm 1]\,x_1 x_2 &= 0 \end{aligned}\right\} \quad (k = 0, 1, 2, 3, \ldots\ldots),$$

m, n being whatever constant numbers and p any right line passing through A[1]. If we eliminate x_1 out of the equations of the latter system, we shall obtain two pairs of pencils with non-consecutive rays in a correspondence (1, 2) i. e. $[R]$, $[A]$, having the equations

$$\left.\begin{aligned} 4\,mn x_1 + [2k+1]\,[2k-1]\,p &= 0 \\ 4\,m^2 x_3^2 - [2k \pm 1]^2\,p x_2 &= 0 \end{aligned}\right\} \quad (k = 0\ 1, 2, 3, \ldots),$$

where the coefficients have an interesting form.

[1] In my paper: *Ein Satz über die ebene Kurve 4. Ordnung mit einer Spitze 2. Art*, Sitzungsberichte der K. Akademie in Wien, IIa, CXIX, 1910, I have considered a few similar relations for this curve of deficiency *one*. Next time I shall treat the same relations for a quartic curve with a spinode and a rhamphoid cusp (deficiency zero).

5. The line $RB \equiv x_1 = 0$ cuts k_4 still in two points P_1, P_2; projecting these points out of the cusp A, we obtain two lines having as equation

$$m^2 x_3^2 - x_2 (a^2 x_2 - b x_3) = 0. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Eliminating $x_2 (a^2 x_2 - b x_3)$ or $x_2 \cdot p$ out of k_4 and the latter equation, we have

$$(m x_3^2 + n x_1 x_2) \pm m x_3^2 = 0$$

therefore

$$x_1 = 0 \quad , \quad x_2 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

$$2 m x_3^2 + n x_1 x_2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

The equation (b) defines a conic, passing through the two points $P'_1 P'_2$ in which the curve k_4 is cut still by the pair of lines (11). By eliminating x_3^2 out of (k_4) and (b) we obtain the equation of $P'_1 P'_2$ in the form:

$$P'_1 P'_2 \equiv m n x_1 + 2p = 0.$$

On the line $P'_1 P'_2$ lie two other points Q_1, Q_2 common to k_4 and $P'_1 P'_2$; so we can now project the points Q_1, Q_2 out of A by two lines cutting k_4 still in the fourth intersections Q'_1, Q'_2, and so on.

There is no difficulty to show, that the general equation of all these projections $P'_1 P'_2$, $Q'_1 Q'_2$, $S'_1 S'_2, \ldots$, will be

$$m n x_1 + k(k+1) p = 0 \quad , \quad (k = 0, 1, 2, 3, \ldots), \quad . \quad . \quad (12)$$

and we see, that all these projections are again elements of the pencil $[R]$.

By means of the involutory pencil of conics (1) we find with respect to (3) and (12):

$$\frac{k(k+1)}{mn} = \frac{\mu}{(n - m\mu)^2}.$$

From this equation follows:

$$\mu_1 = \frac{n(k+1)}{mk} \quad , \quad \mu_2 = \frac{nk}{m(k+1)},$$

therefore any line having the form (12) cuts k_4 on the two conics:

$$\left.\begin{array}{l} m k x_3^2 + n(k+1) x_1 x_2 = 0 \\ m(k+1) x_3^2 + nk \; x_1 x_2 = 0 \end{array}\right\} \quad (k = 0, 1, 2, 3, \ldots) \quad . \quad (13)$$

By eliminating x_1 out of (12) and (13) we obtain two pairs of pencils with non-consecutive rays in correspondence (1, 2), by means of which any number of discrete points of k_4 can be determined; thus

$$\left.\begin{array}{r} mnx_1 + k(k+1)p = 0 \\ m^2x_3{}^2 - (k+1)^2 x_2 p = 0 \\ m^2 x_2{}^2 - k^2 x_2 p = 0 \end{array}\right\},$$

k being any entire positive number or zero.

Let us observe, that any four lines having an equation of the form (12), give a cross ratio which is independent of the coefficients in the equation of k_4, or, what is the same, that the *cross ratio* for the same four values of $k = 0, 1, 2, 3, \ldots.,$ is *unaltered for all the curves of the considered form.*

If we retain the three fixed rays

$$x_1 = 0 \quad , \quad p = 0,$$
$$4\, mnx_1 - p = 0,$$

any line of the form (12) gives with these three rays an absolute constant cross ratio for all the curves of the species k_4, where k is a constant number:

$$\Delta' = -\frac{1}{4k(k+1)},$$

also *a function of the chosen value of k for the same curve k_4.*

6. We have already indicated the two systems of projections, the *first* of which is acquired by projecting the two points of contact on the double tangent out of the cusp A, and the *second* by projecting the two common points to k_4 and $x_1 = 0$ out of the same centre of projection. We take now two of those projections, belonging to *various systems* for the same value of k, having the equations

$$4mnx_1 + [(2k)^2 - 1]\, p = 0$$
$$mnx_1 + k(k+1)\, p = 0.$$

By the term "same value of k" for the two systems is meant that the *same number of projections* was made in both systems. These rays of the pencil $[R]$ bear with the fixed pair of rays

$$4mnx_1 - p = 0 \quad \ldots\ldots\ldots \quad (d)$$
$$p = 0$$

a cross ratio Δ''. By means of the parameters

$$\frac{mn}{k(k+1)},\ \frac{4mn}{(2k)^2-1},\ 0,\ -4mn,$$

we shall obtain

$$\Delta'' = \left(\frac{2k}{2k+1}\right)^2, \ (k = 0, 1, 2, 3, \ldots.),$$

therefore *a value invariable for all the curves k_4 with a rhamphoid cusp and a node,* if the *same* value of k for all such curves has been chosen.

We see yet also, that the four points of intersection of the curve k_4 with each line h, passing through R, this point R, and the common point to h and $x_2 = 0$ are three pairs of *the same involution.* Then the pair $x_1 = 0$, $x_2 = 0$ is a degenerated conic of the pencil of conics (1) which bears k_4 with the projective pencil $[R]$.

7. A line passing through the point (M) of intersection of the double tangent (d) and the cuspidal tangent ($x_2 = 0$) has an equation of the form

$$\mu^2 x_2 + (4mnx_1 - a^2 x_2 + bx_3) = 0 \quad . \quad . \quad . \quad . \quad . \quad (14)$$

If we eliminate d out of (14) and (k_4), writing the equation of k_4 in the form

$$(mx_3^2 - nx_1 x_2)^2 + x_2 x_3^2 (4mnx_1 - a^2 x_2 + bx_3) = 0 \quad . \quad . \quad . \quad (k_4)$$

we shall obtain

$$\mu^2 x_2^2 x_3^2 - (mx_3^2 - nx_1 x_2)^2 = 0$$

therefore

$$\mu x_2 x_3 \pm (mx_3^2 - nx_1 x_2) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

To any ray of the pencil $[M]$ corresponds a pair of conics (15), which form an involutory system for all values of μ. The two conics of the conjugate pair have in the vertex B a pair of tangents

$$\mu x_3 \pm nx_1 = 0,$$

which is divided harmonically by the two lines $x_1 = 0$, $x_3 = 0$. All the conics of the involutory system osculate one another in the cusp A on $x_2 = 0$.

From this follows an other generation of k_4.

> *Let be given an involutory pencil of conics, which osculate each other in a point (A) on the common tangent ($x_2 = 0$), and a pencil of rays $[M]$ having its vertex (M) on the tangent $x_2 = 0$, then we can arrange a correspondence between these pencils in this manner, that the parameter of a ray in the pencil $[M]$ is the square of the parameter belonging to the corresponding conjugate pair of conics in the involutory pencil.*

The two pencils generate a curve of order five, which breaks up into the common corresponding right line $x_2 = 0$ and the curve k_4 of the considered species.

If we choose under all these conjugate pairs of conics that, for which the two tangents in the point B are identical with the tangents

$$nx_1 \pm ax_3 = 0$$

of the curve k_4 in the same point, then this pair will meet k_4 in two points U_1, U_2; we obtain the joining line of these points out of the equation of the considered pair of conics, i. e. out of

$$ax_2x_3 \pm (mx_3^2 - nx_1x_2) = 0;$$

so we have by the latter definition

$$U_1U_2 \equiv a^2x_2 + d = 0$$

or

$$U_1U_2 \equiv 4\,mnx_1 + bx_3 = 0,$$

and this is the line passing through the vertex B and the common point (M) to the double tangent and the cuspidal tangent. Therefore the four points U_1, U_2, B, and M lie on a straight line.

On each line passing through M we have obtained four points of k_4 as intersections of this line with two conics belonging to a pencil, which has three consecutive base-points in A, and the fourth base-point in B. To this pencil of conics belongs also the pair of lines $x_3 = 0$, $x_2 = 0$ as a degenerated conic. We can now say, that on each line s, passing through M, the two pairs of intersections with k_4, the point M and the common point to s and $x_3 = 0$ are *three pairs of elements of the same involution.*

All the relations considered here, remain unaltered, if the double point B is a "conjugate point" (acnode).

Anatomy. — "*On the development of the Hypophysis of Primates especially of Tarsius*". By Prof. L. Bolk.

(Communicated in the Meeting of November 26, 1910).

When studying an Embryo of Tarsius spectrum belonging to the embryological Institute of the Utrecht University (Catalogued as Tarsius n° 666), my attention was drawn by the peculiar shape of the pharyngeal part of the Hypophysis. In this Primate a form is developed more complicated than is known to us in other mammals. In most cases we know, as follows from the description of various authors, that the Hypophysis-vesicle unstrings itself from the roof-epithelium of the stomadeum, places itself against the anterior surface of the infundibularstem; and is then, when the nervous part of the Hypophysis begins to develop, invaginated by the latter. The pharyngeal — or more correctly expressed — the oral part of the Hypophysis