Crystallography. — *"On the orientation of microscopic crystal-sections."* By J. SCHMUTZER. (Communicated by Prof. C. E. A. WICHMANN).

(Communicated in the meeting of December 24, 1910).

If in a slide a crystal shows the traces of three non-parallel planes it is possible to determine the orientation of the section without making use of the extinction angle. If the optic constants of an anisotropic mineral are known, it is sufficient to know the extinction angle and the apparent angle between two planes (crystal, cleavage- or twinning-planes) as will be more distinctly demonstrated in a subsequent communication.

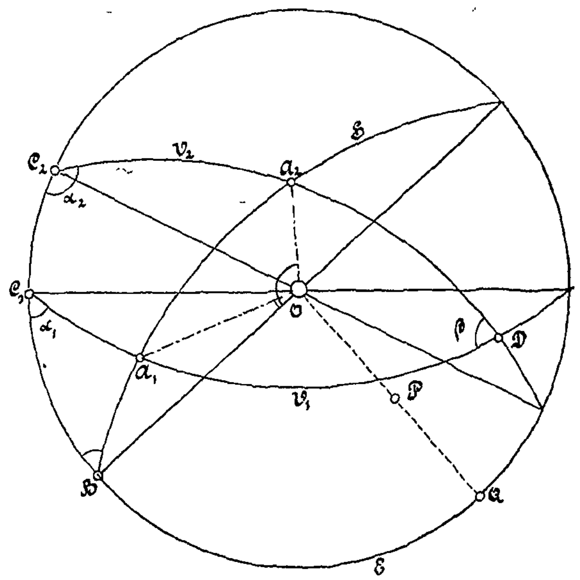

Fig 1.

In fig. 1 the crystal planes E, V_1, and V_2, the former of which is supposed to be the projection plane, are cut by the secant plane S, producing with it the secants OB, OA_1, and OA_2. In stead of the two angles α_1 and α_2 one measures in the secant plane, consequently in the slide, between the planes $V_1 : E$ and $V_2 : E$ the apparent angles $A_1OB = h_1$ and $A_2OB = h_2$.

Be the secant-plane S given by its pole P_1, of which the height $PQ = \sigma$ is measured > 0 above and < 0 below the equator plane, whilst the azimuth $C_1BQ = \varrho$ is > 0 if measured in opposite direction to the hands of a clock, then is in ΔBA_1C_1:

$$\cot A_1B = \frac{\sin A_1BC_1 \cot A_1C_1B + \cos A_1BC_1 \cos BC_1}{\sin BC_1}$$

$$-\cot h_1 = \frac{\sin(90-\sigma)\cot \alpha_1 + \cos(90-\sigma)\cos(\varrho-90)}{\sin(\varrho-90)}$$

or

$$\cot h_1 \cos \varrho = \cos \sigma \cot \alpha_1 + \sin \sigma \sin \varrho \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now is $\smile C_1C_2 = \gamma$ a constant that can be calculated with the help of the given angles α_1, α_2 and β from ΔC_1C_2D and that is measured from C_1 positively in a direction opposite to the hands of a clock. In ΔBA_2C_2 is then :

$$\cot A_2B = \frac{\sin A_2BC_2 \cot A_2C_2B + \cos A_2BC_2 \cos BC_2}{\sin BC_2}$$

$$-\cot h_2 = \frac{\sin(90-\sigma)\cot \alpha_2 + \cos(90-\sigma)\cos(\varrho-90-\gamma)}{\sin(\varrho-90-\gamma)}$$

$$-\cot h_2 = \frac{\cos \sigma \cot \alpha_2 + \sin \sigma \sin(\varrho-\gamma)}{-\cos(\varrho-\gamma)}$$

$$\cot h_2 (\cos \varrho \cos \gamma + \sin \varrho \sin \gamma) = \cos \sigma \cot \alpha_2 + \sin \sigma \sin(\varrho-\gamma)$$

$$\cot h_2 \cos \varrho \cos \gamma = \cos \sigma \cot \alpha_2 + \sin \sigma \sin(\varrho-\gamma) - \cot h_2 \sin \varrho \sin \gamma \quad . \quad . \quad (2)$$

If one divides (2) by (1), then becomes

$$\cot h_2 \cos \gamma (\cos \sigma \cot \alpha_1 + \sin \sigma \sin \varrho) =$$
$$= \cot h_1 \{\cos \sigma \cot \alpha_2 + \sin \sigma \sin(\varrho-\gamma) - \cot h_2 \sin \varrho \sin \gamma\}.$$

$$\cos \sigma (\cot \alpha_1 \cot h_2 \cos \gamma - \cot \alpha_2 \cot h_1) + \sin \sigma \{\sin \varrho (\cot h_2 - \cot h_1)\cos \gamma +$$
$$+ \cot h_1 \cos \varrho \sin \gamma\} + \cot h_1 \cot h_2 \sin \varrho \sin \gamma = 0.$$

If one supposes

$$\left.\begin{array}{l} \cot \alpha_1 \cot h_2 \cos \gamma - \cot \alpha_2 \cot h_1 = a \\ (\cot h_2 - \cot h_1)\cos \gamma = b \\ \cot h_1 \sin \gamma = c \\ \cot h_1 \cot h_2 \sin \gamma = d \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

then the formula changes into

$$a \cos \sigma + \sin \sigma (b \sin \varrho + c \cos \varrho) + d \sin \varrho = 0$$

$$\cos \sigma = -\frac{b \sin \varrho + c \cos \varrho}{a} \sin \sigma - \frac{d}{a} \sin \varrho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If one substitutes this value of $\cos \sigma$ in (1), one obtains:

$$\cot h_1 \cos \varrho = -\cot \alpha_1 \{(b \sin \varrho + c \cos \varrho)\sin \sigma + d \sin \varrho\} + a \sin \sigma \sin \varrho =$$
$$= \sin \sigma \{-(b \sin \varrho + c \cos \varrho)\cot \alpha_1 + a \sin \varrho\} - d \cot \alpha_1 \sin \varrho$$

from which follows:

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$$\sin\sigma = \frac{a \cot h_1 \cos\varrho + d \cot \alpha_1 \sin\varrho}{(a - b\cot\alpha_1)\sin\varrho - c\cot\alpha_1 \sin\varrho}.$$

When worked out this formula furnishes:

$$\sin\sigma = -\frac{a\cos\varrho + \cot\alpha_1 \cot h_2 \sin\gamma \sin\varrho}{\cot\alpha_1 \sin\gamma\cos\varrho + (\cot\alpha_2 - \cot\alpha_1\cos\gamma)\cos\varrho}.$$

If one supposes herein again

$$\left.\begin{array}{l} \cot\alpha_1 \cot h_2 \sin\gamma = e \\ \cot\alpha_2 - \cot\alpha_1 \cos\gamma = f \\ \cot\alpha_1 \sin\gamma = g \end{array}\right\} \quad \ldots\ldots\ldots \quad (5)$$

then becomes

$$\sin\sigma = -\frac{a\cos\varrho + e\sin\varrho}{g\cos\varrho + f\sin\varrho} \quad \ldots\ldots \quad (6)$$

We found above

$$\cos\sigma = -\frac{b\sin\varrho + c\cos\varrho}{a}\sin\sigma - \frac{d}{a}\sin\varrho$$

$$\cos\sigma = \left(\frac{b\sin\varrho + c\cos\varrho}{a}\right)\left(\frac{a\cos\varrho + e\sin\varrho}{g\cos\varrho + f\sin\varrho}\right) - \frac{d}{a}\sin\varrho =$$

$$= \frac{(b\sin\varrho + c\cos\varrho)(a\cos\varrho + e\sin\varrho) - d(g\cos\varrho + f\sin\varrho)\sin\varrho}{a(g\cos\varrho + f\sin\varrho)} =$$

$$= \frac{\sin^2\varrho\,(eb - fd) + \sin\varrho\cos\varrho\,(ab + ec - gd) + ac\cos^2\varrho}{a(g\cos\varrho + f\sin\varrho)} =$$

$$= \frac{h\sin^2\varrho + b\sin\varrho\cos\varrho + c\cos^2\varrho}{g\cos\varrho + f\sin\varrho} \quad \ldots\ldots\ldots\ldots \quad (7)$$

in which

$$h = \cot h_2 \sin\gamma \quad \ldots\ldots\ldots\ldots \quad (8)$$

The variables ϱ and σ are consequently separated; the ratio

$$\sin^2\sigma + \cos^2\sigma = 1$$

furnishes

$$(a\cos\varrho + e\sin\varrho)^2 + (h\sin^2\varrho + b\sin\varrho\cos\varrho + c\cos^2\varrho)^2 = (g\cos\varrho + f\sin\varrho)^2$$

$$\cos^2\varrho\,(a^2 - g^2) + 2\sin\varrho\cos\varrho\,(ae - fg) + \sin^2\varrho\,(e^2 - f^2) +$$
$$+ (h\sin^2\varrho + b\sin\varrho\cos\varrho + c\cos^2\varrho)^2 = 0.$$

If one introduces:

$$\cos^2\varrho = \frac{1 + \cos 2\varrho}{2}; \quad \sin^2\varrho = \frac{1 - \cos 2\varrho}{2}; \quad 2\sin\varrho\cos\varrho = \sin 2\varrho,$$

the latter ratio changes into:

$$\cos^2 2\varrho\,\{(c-h)^2 - b^2\} + 2\cos 2\varrho\,(a^2 + c^2 + f^2 - e^2 - g^2 - h^2) +$$
$$+ \{b^2 + (h + c)^2 + 2(a^2 + e^2 - g^2 - f^2)\} +$$
$$+ 2\sin 2\varrho\,\{2(ae - fg) + b(h + c) + b(c - h)\cos 2\varrho\} = 0$$

If one supposes

$$\left.\begin{array}{l} \dfrac{2(a^2+c^2+f^2-e^2-g^2-h^2)}{(c-h)^2-b^2}=p \\[2ex] \dfrac{b^2+(h+c)^2+2(a^2+e^2-g^2-f^2)}{(c-h)^2-b^2}=q \\[2ex] \dfrac{4(ae-fg)+2b(h+c)}{(c-h)^2-b^2}=r \\[2ex] \dfrac{2b(c-h)}{(c-h)^2-b^2}=s \end{array}\right\} \quad . \quad . \quad . \quad . \quad (9)$$

then becomes

$$\cos^2 2\varrho + p\cos 2\varrho + q = -(r + s\cos 2\varrho)\sin 2\varrho \quad . \quad . \quad . \quad (10)$$

and

$$\cos^4 2\varrho + \frac{2(p+rs)}{1+s^2}\cos^3 2\varrho + \frac{p^2+2q+r^2-s^2}{1+s^2}\cos^2 2\varrho + $$
$$+ \frac{2(pq-rs)}{1+s^2}\cos 2\varrho + \frac{q^2-r^2}{1+s^2} = 0 \quad . \quad . \quad . \quad (11)$$

This equation can now be solved; the value found for ϱ, introduced into (6) furnishes σ.

If however in the equation (6)

$$a\cos\varrho + e\sin\varrho = g\cos\varrho + f\sin\varrho = 0,$$

then $\sin\sigma$ becomes indefinite. In that case is

$$tg\,\varrho = -\frac{a}{e} = -\frac{g}{f}$$

$$af = eg$$

consequently

$$(\cot\alpha_1 \cot h_2 \cos\gamma - \cot\alpha_2 \cot h_1)(\cot\alpha_2 - \cot\alpha_1 \cos\gamma) =$$
$$= \cot^2\alpha_1 \cot h_2 \sin^2\gamma = \cot^2\alpha_1 \cot h_2 (1-\cos^2\gamma)$$
$$\cot\alpha_1 \cot h_2 (\cot\alpha_2 \cos\gamma - \cot\alpha_1) + \cot\alpha_2 \cot h_1 (\cot\alpha_1 \cos\gamma - \cot\alpha_2) = 0$$
$$\frac{\cot\alpha_1 \cot h_2}{\cot\alpha_2 \cot h_1} = \frac{\cot\alpha_1 \cos\gamma - \cot\alpha_2}{\cot\alpha_1 - \cot\alpha_2 \cos\gamma} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

σ is then found by introducing the value of ϱ in (1).

What is said here finds an application to the determination of the plane that cuts the octahedron planes $(1\bar{1}1)$, (111) and $(\bar{1}11)$ in such a way that the traces of the planes $(1\bar{1}1):(111)$ and $(\bar{1}11):(111)$ enclose right angles. If one supposes (111) to be the equator plane, then becomes

$\alpha_1 = 180° - 70°31'43''$, $\{V_1 = (1\bar{1}1)\}$; $\alpha_2 = 70°31'43''$, $\{V_2 = (\bar{1}11)\}$; $\gamma = -60°$; $h_1 = h_2 = 90°$; $\varrho = -30°$.

The equation (12) changes into $0 = 0$; in (6) not only the numerator becomes $= 0$ because $a = e = 0$; but also the denominator because

$$g \cos \varrho + f \sin \varrho =$$
$$= \cot \alpha_1 \sin \gamma \cos \varrho + (\cot \alpha_2 - \cot \alpha_1 \cos \gamma) \sin \varrho =$$
$$= \cot \alpha_1 \{\sin \gamma \cos \varrho - (1 + \cos \gamma) \sin \varrho\} =$$
$$= \cot \alpha_1 \left\{ - \sin^2 60^\circ + \left(1 + \frac{1}{2}\right) \sin 30^\circ \right\} =$$
$$= \cot \alpha_1 \left(- \left(\frac{1}{2}\sqrt{3}\right)^2 + \frac{3}{2} \cdot \frac{1}{2} \right) = 0$$

From (1) one finds on the contrary

$$\cot h_1 \cos \varrho = 0 = \cos \sigma \cot \alpha_1 + \sin \sigma \sin \varrho$$
$$tg\, \sigma = - \frac{\cot \alpha_1}{\sin \varrho} = - \frac{\cot 70^\circ 31' 43''}{\sin 30^\circ}$$

from which $\sigma = -35^\circ 15' 53''$, the angle between $(001):(111) - \frac{\pi}{2}$.

The secant plane is consequently (001).

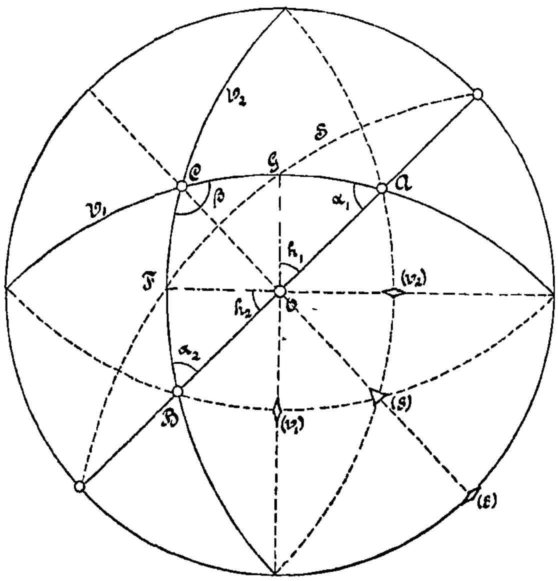

Fig. 2.

A single example may suffice to demonstrate the applicability of the formulas deducted above; I choose for it a problem in which the result obtained can easily be controlled.

The three rhombododecahedron planes (101), (110) and (011) are cut by a plane in such a way, that their traces include angles of 60°. What is the orientation of the secant-plane?

Take E (110) as equator plane; $V_1 : E = \alpha_1 = -60°$; $V_2 \cdot E = \alpha_2 = 60°$; $h_1 = \angle GOA = 60°$; $h_2 = \angle FOB = -60°$;

$$\gamma = \frown AB = (\bar{1}11) : (\bar{1}11) = -109°28'17''.$$

$$\cot \alpha_1 = -\cot \alpha_2 = -\cot h_1 = \cot h_2 .$$

So according to (3), (5) and (8):

$$a = \cot^2 \alpha_1 (\cos \gamma - 1) \qquad e = \cot^2 \alpha_1 \sin \gamma$$
$$b = 2 \cot \alpha_1 \cos \gamma \qquad f = -\cot \alpha_1 (1 + \cos \gamma)$$
$$c = -\cot \alpha_1 \sin \gamma \qquad g = \cot \alpha_1 \sin \gamma$$
$$d = -\cot^2 \alpha_1 \sin \gamma \qquad h = \cot \alpha_1 \sin \gamma .$$

Here is

$$\cot \alpha_1 = -\frac{1}{3}\sqrt{3} \quad ; \cot^2 \alpha_1 = \frac{1}{3}$$

$$\cos \gamma = -\frac{1}{3} \quad ; \sin \gamma = -\frac{2}{3}\sqrt{2}.$$

so that (9):

$$p = \frac{2(a^2+c^2+f^2-e^2-g^2-h^2)}{(c-h)^2-b^2} = -\frac{2}{21}$$

$$q = \frac{b^2 + (h+c)^2 + 2(a^2+e^2-g^2-f^2)}{(c-h)^2 - b^2} = -\frac{1}{7}$$

$$r = 2\,\frac{2(ae-fg) + b(h+c)}{(c-h)^2 - b^2} = -\frac{4}{21}\sqrt{2}$$

$$s = \frac{2b\,(c-h)}{(c-h)^2-b^2} = -\frac{4}{7}\sqrt{2}$$

further is in equation (11):

$$l = \frac{2\,(p+rs)}{1+s^2} = \frac{4}{27} = \frac{4}{3^3}$$

$$m = \frac{p^2+2q+r^2-s^2}{1+s^2} = -\frac{14}{27} = -\frac{14}{3^3}$$

$$n = \frac{2\,(pq-rs)}{1+s^2} = -\frac{20}{81} = -\frac{20}{3^4}$$

$$o = \frac{q^2-r^2}{1+s^2} = -\frac{23}{729} = -\frac{23}{3^6}.$$

Consequently we have to solve the equation:

$$\cos^4 2\varrho + \frac{4}{3^3}\cos^3 2\varrho - \frac{14}{3^3}\cos^2 2\varrho - \frac{20}{3^4}\cos 2\varrho - \frac{23}{3^6} = 0 \quad . \quad (13)$$

Suppose $\cos 2\varrho = x - \frac{1}{4}\,l = x - \frac{1}{3^3}$, then (13) can be changed into:

$$x^4 + \lambda x^2 + \mu x + \nu = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (14)$$

in which

$$\lambda = m - \frac{3}{8} l^2 = -\frac{2^7}{3^5}$$

$$\mu = n - \frac{1}{2} lm + \frac{1}{8} l^3 = -\frac{2^{12}}{3^9}$$

$$\nu = o - \frac{1}{4} ln + \frac{1}{16} l^2 m - \frac{3}{256} l^4 = -\frac{2^{12}}{3^{11}}$$

If one composes now the equation

$$y^3 + a'y^2 + b'y + c' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

in which

$$a' = \frac{\lambda}{2} = -\frac{2^6}{3^5}$$

$$b' = \frac{\lambda^2 - 4\nu}{16} = \frac{2^{12}}{3^{11}}$$

$$c' = -\frac{\mu^2}{64} = -\frac{2^{18}}{3^{18}}$$

then the roots of (14) are given by

$$x_{1.2.3.4} = \pm \sqrt{y_1} \pm \sqrt{y_2} \pm \sqrt{y_3},$$

in which

$$\sqrt{y_1}\sqrt{y_2}\sqrt{y_3} = -\frac{\mu}{8} = \frac{2^9}{3^9} > 0.$$

In order to solve (15) one supposes $y = z - \frac{1}{3} a'$, by which the equation changes into:

$$z^3 + vz + w = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

In this is

$$v = b' - \frac{1}{3} a'^2 = \frac{2^{12}}{3^{11}} - \frac{1}{3}\left(\frac{2^6}{3^5}\right)^2 = 0$$

$$w = c' - \frac{1}{3} a'b' + \frac{2}{27} a'^3 = -\frac{2^{18}}{3^{18}} + \frac{1}{3} \cdot \frac{2^6 . 2^{12}}{3^5 . 3^{11}} - \frac{2}{3^3}\,\frac{2.^{18}}{3^{15}} =$$

$$= \frac{-2^{18} + 3.2^{18} - 2.2^{18}}{3^{18}} = 0.$$

(16) furnishes consequently 3 equal roots $z = 0$.

(15) furnishes 3 equal roots $y = z - \frac{1}{3} a' = \frac{2^6}{3^6}$.

To (14) correspond the values

$$x_1 = +\sqrt{y} + \sqrt{y} + \sqrt{y} = 3\sqrt{y} = \frac{2^3}{3^2}$$

$$x_2 = +\sqrt{y} - \sqrt{y} - \sqrt{y} = -\sqrt{y} = -\frac{2^3}{3^3}.$$

The roots of (13) are consequently represented by:

$$\cos 2\varrho_1 = x_1 - \frac{1}{3^3} = \frac{2^3}{3^2} - \frac{1}{3^3} = \frac{23}{3^3}$$

$$2\varrho_1 = \pm 31°35'8'' ; \quad \varrho_1 = \pm 15°47'34''$$

$$\cos 2\varrho_2 = x_2 - \frac{1}{3^3} = -\frac{2^3}{3^3} - \frac{1}{3^3} = -\frac{1}{3}$$

$$2\varrho_2 = \pm 109°28'16'' ; \quad \varrho_2 = \pm 54°44'8''.$$

The corresponding values of σ are calculated from (6)·

$$\sin \sigma = -\frac{a \cos \varrho + e \sin \varrho}{g \cos \varrho + f \sin \varrho} = -\frac{\cot^2 \alpha_1 (\cos \gamma - 1) \cos \varrho + \cot^2 \alpha_1 \sin \gamma \sin \varrho}{\cot \alpha_1 \sin \gamma \cos \varrho - \cot \alpha_1 (1 + \cos \gamma) \sin \varrho} =$$

$$= -\cot \alpha_1 \frac{(\cos \gamma - 1) \cos \varrho + \sin \gamma \sin \varrho}{\sin \gamma \cos \varrho - (1 + \cos \gamma) \sin \varrho} = -\cot \alpha_1 \frac{-\frac{4}{3} \cos \varrho - \frac{2}{3} \sqrt{2} \sin \varrho}{-\frac{2}{3} \sqrt{2} \cos \varrho - \frac{2}{3} \sin \varrho} =$$

$$= -\cot \alpha_1 . \frac{2 \cos \varrho + \sin \varrho . \sqrt{2}}{\sqrt{2} . \cos \varrho + \sin \varrho} = -\cot \alpha_1 . \sqrt{2} . = \frac{1}{3} \sqrt{6} \quad . \quad . \quad . \quad (17)$$

in which however the term $\sin \varrho + \sqrt{2} . \cos \varrho$ disappears in the numerator as well as in the denominator.

From

$$\sin \varrho + \sqrt{2} . \cos \varrho = 0$$

one finds

$$tg\, \varrho = - \sqrt{2},$$

to which corresponds

$$\varrho = - 54°44'8'' \text{ (or } 180° - 54°44'8'')$$

and this is one of the two values ϱ_2. Whilst from (17) results the value $\sigma = + 54°44'8''$ (or $180° - 54°44'8''$) the value of σ corresponding to $\varrho = - 54°44'8''$ must be calculated from (1).

$$\cot h_1 = \frac{\cos \sigma \cot \alpha_1 + \sin \sigma \sin \varrho}{\cos \varrho}$$

$$\frac{1}{3}\sqrt{3} = \frac{-\frac{1}{3} \sqrt{3} \cos \sigma - \frac{1}{3} \sqrt{6} \sin \sigma}{\frac{1}{3} \sqrt{3}}$$

$$\frac{1}{3} \sqrt{3} = - \cos \sigma - \sqrt{2} \sin \sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$\cos^2 \sigma + \frac{2}{3} \sqrt{3} \cos \sigma + \frac{1}{3} = 2 (1 - \cos^2 \sigma)$$

$$\cos^2 \sigma + \frac{2}{9} \sqrt{3} \cos \sigma - \frac{5}{9} = 0$$

$$cos\,\sigma = -\frac{1}{9}\sqrt{3} \pm \sqrt{\frac{3+45}{81}} = -\frac{1}{9}\sqrt{3} \pm \frac{4}{9}\sqrt{3} = +\frac{1}{3}\sqrt{3} \text{ or } -\frac{5}{9}\sqrt{3}.$$

To $cos\,\sigma = \frac{1}{3}\sqrt{3}$ corresponds

$$\varrho = \pm\, 54°44'8''$$

of these two values only $\sigma = -\,54°44'8''$ cooresponds to (18). A further investigation shows that the pole of the secant plane S is represented by

$$\varrho = -\,54°44'8'', \quad \sigma = -\,54°44'8''.$$

whereas the other values of ϱ and σ do not correspond. The secant plane is the octahedronplane (111) as appears clearly from fig. 2; σ corresponds to the angle between (001) : (111) the cosinus of which is represented by

$$cos\,\theta = \frac{h_1h_2 + k_1k_2 + l_1l_2}{\sqrt{h_1^2+k_1^2+l_1^2}\sqrt{h_2^2+k_2^2+l_2^2}} = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}.$$

If the planes E, V_1 and V_2 (vide fig. 1) lie in one zone the formulas (10) and (6) obtain a much less complicate form. Because $\gamma = 0$ the coefficients (3), (5) and (8) change into

$$\left.\begin{aligned} a &= cot\,\alpha_1\,cot\,h_2 - cot\,\alpha_2\,cot\,h_1 \\ b &= cot\,h_2 - cot\,h_1 \\ f &= cot\,\alpha_2 - cot\,\alpha_1 \\ c &= d = e = g = h = 0' \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (19)$$

so that:

$$cos^2\,2\varrho - \frac{2(a^2+f^2)}{b^2}\,cos\,2\varrho - \frac{b^2+2(a^2-f^2)}{b^2} = 0 \; . \; . \; . \quad (20)$$

$$sin\,\sigma = -\frac{a}{f}\,cot\,\varrho \quad . \; . \; . \; . \; . \; . \; . \; . \quad (21)$$

In an amphibole crystal the planes $(1\bar{1}0)$, 110 and (010) are lying in one zone; (110) may serve as equator plane. Whilst the real angles between $(\bar{1}10)$: (110) and (010) : (110) are resp. $\alpha_1 = 55°50'$ and $\alpha_2 = -\,62°5'$ the apparent angles, measured in a rock slide, amounted to $h_1 = 43°$ and $h_2 = -\,79°$.

(19) gives:

$$a = 0{,}43625 \quad ; \quad a^2 = 0{,}19032$$
$$b = -\,1{,}26674 \quad ; \quad b^2 = 1{,}60454$$
$$f = -\,1{,}20858 \quad ; \quad f^2 = 1{,}46070$$

from which:

$$\frac{a^2+f^2}{b^2} = 1.02875 \quad ; \quad \frac{b^2+2(a^2-f^2)}{b^2} = -\,0{,}58074$$

consequently :

$$\cos^2 2\varrho - 2(1{,}02875)\cos 2\varrho + 0{,}58074 = 0$$

$$\cos 2\varrho = 1{,}02875 \pm \sqrt{(1{,}02875)^2 - 0{,}58074} =$$

$$= 1{,}02875 \pm 0{,}68948 = 1{,}71823 \text{ or } 0.33927.$$

Only the second value corresponds, so that:

$$2\varrho = 70°10' \text{ or } - 70°10'$$

$$\varrho = 35°5' \text{ or } - 35°5' .$$

$$\sin\sigma = -\frac{a}{f}\cot\varrho = -\frac{0{,}43625}{-1{,}20858}\cot\varrho = 0{,}36096 \cot\varrho$$

from which

$$\varrho_1 = 35°5' , \sigma_1 = 30°55'$$

$$\varrho_2 = - 35°5' , \sigma_2 = - 30°55'.$$

As will appear afterwards the optic extinction offers an expedient to decide in a given case whether one has to do with the secant plane $(\varrho_1\sigma_1)$ or with $(\varrho_2\sigma_2)$.

In a graphical way the problem of the orientation of crystal-sections can be solved in a considerably simpler way. To do so one can make use of diagrams, that give for any discretional angle between two planes, the apparent angle h as a function of ϱ and σ.

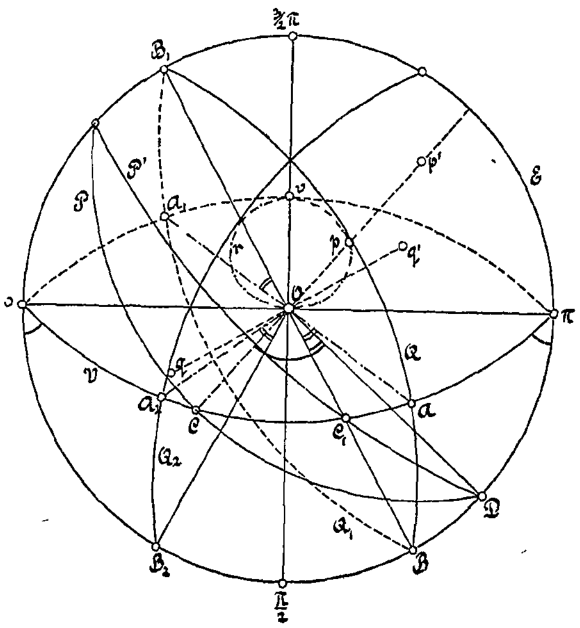

Fig. 3.

In fig. 3 be E again the projection plane, V the crystalplane, that with E includes an angle α, (0) $O(\pi)$ the secant line between E and V; the azimuth of a secant plane (P,Q) be measured from (0) positively opposite to the direction of the hands of a clock. If one indicates the globe octants (0) $\left(\frac{\pi}{2}\right), \left(\frac{\pi}{2}\right)(\pi), (\pi)\left(\frac{3}{2}\pi\right)$ and $\left(\frac{3}{2}\pi\right)$ (0) respectively by the numbers 1, 2, 3, and 4, in so far as they lie above the equatorplane, by 5, 6, 7, and 8 in so far as they lie below it, then one can deduct from the figure with regard to the sign of h in the various octants what follows.

If $0 < \alpha < \frac{\pi}{2}$, and if the pole q of the secant plane Q lies in the first octant, h becomes $= \angle AOB > 0$. In the opposite 7th octant one finds for the secant plane (q', Q') a negative apparent angle; if Q and Q' coincide, so that the distance of the poles is $q : q' = \pi$, then h_1 becomes $= -h$.

If the pole of the secantplane lies in the 2nd octant, then h becomes < 0; if one applies Q_2 with regard to the plane $O\left(\frac{\pi}{2}\right)$ symmetrically with Q then h_2 becomes again $= \angle A_2OB_2 = -h\,(\angle AOB)$. If at last the pole lies in the plane $O\left(\frac{\pi}{2}\right)$, then $h = 0$ independently of the value of σ.

In the octants 3 and 4 the pole of the secant plane moves exclusively within the obtuse angle $E:V$. If with a constant value of ϱ, we substitute for the pole (p) of the plane (P) successively all the values of σ between 0 and $\frac{\pi}{2}$, it appears that for $\sigma = 0$, $h = 0$; if σ becomes > 0, as in the plane $(p'P')$, h becomes $= \angle C_1OD < 0$; this negative angle becomes larger if σ increases till the pole lies in p, and h has become $= \angle COD = -\frac{\pi}{2}$. Now the negative angle surpasses the value $-\frac{\pi}{2}$ and is consequently measured positively as far as $O\left(h < \frac{\pi}{2}\right)$.

By construction the point p can easily be found, because $\frown Cp = \frac{\pi}{2}$.

From (1) follows, that here

$$\cot h \cos \varrho = 0 = \cos \sigma \cot \alpha + \sin \sigma \sin \varrho$$

$$tg\,\sigma = -\frac{cot\,\alpha}{sin\,\varrho}\,.$$

From this ratio the course of the curve vpO can be deducted, which is the locus of all poles of secant planes that cut the angle between the planes $V:E$ in a way so as to produce the apparent angle $h=\frac{\pi}{2}$. For $\varrho=\pi$ one finds $\sigma=\frac{\pi}{2}$; $\varrho=\frac{3}{2}\pi$ produces $tg\,\sigma=cot\,\alpha$, or $\sigma=\frac{\pi}{2}-\alpha$; whilst for $\varrho=2\pi$, σ becomes again $\frac{\pi}{2}$.

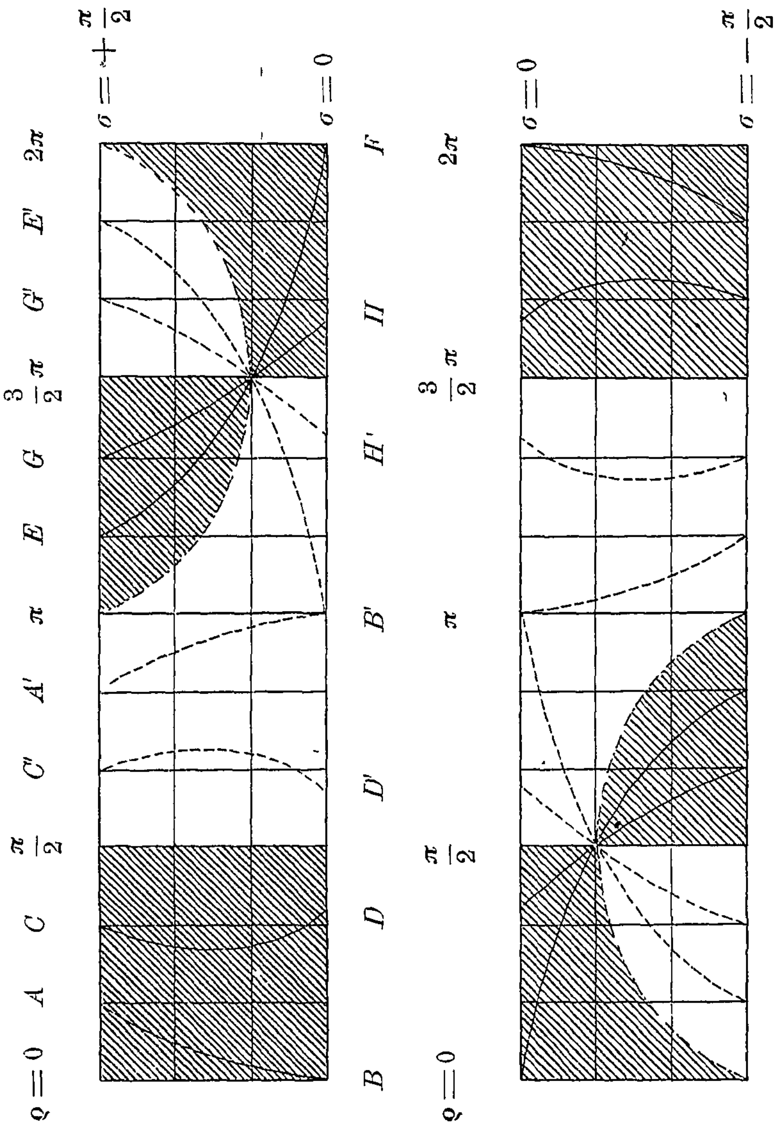

Fig. 4 and 5.

It is easy to see that the areas $O(\pi)\left(\frac{3}{2}\pi\right)vpO$ and Ovr furnish a negative, $O(0)\left(\frac{3}{2}\pi\right)vrO$ and Ovp on the contrary furnish a positive value for h. For the octants 5 and 6 we find the same as for 3 and 4, only h takes here the opposite sign.

In the figures 4 and 5 one finds the scheme of the h-diagram for $\alpha = 60°$ with the area-division discussed here; the secant planes the poles of which lie in the shadowed areas, form a positive angle h, the white areas form a negative angle. The squares into which the figures are divided, are 30° by 30°; the meridiancircles, on which σ is measured out, are drawn as parallel lines, so that a line equal in length to the equator takes the place of the pole Q. In fig. 4 the octants 1—4, in fig. 5 the octants 5—8 are represented.

In the first octant the curve AB indicates the locus of the poles of all the secant planes that produce $h = 60°$; the curve CD gives $h = 30°$; symmetrically with regard to the line $\left(\varrho = \frac{\pi}{2}\right)\left(\sigma = \frac{\pi}{2}\right) \cdot \left(\varrho = \frac{\pi}{2}\right)\left(\sigma = 0\right)$, to which corresponds $h = 0$ lie in the 2nd octant the curves $C'D'$ with $h = -30°$ and $A'B'$ with $h = -60°$.

If for $\varrho = \pi$, σ varies from 0° to $\frac{\pi}{2}$, then h takes successively all values from $-60°$ to $-90°$; here the curve Opv of fig. 3 begins going over the point $\left(\varrho = \frac{3}{2}\pi,\ \sigma = \frac{\pi}{2} - \alpha = 30°\right)$ to $\left(\varrho = 2\pi,\ \sigma = \frac{\pi}{2}\right)$. The curves EF and GH produce here again values $h = 60°$ and $h = 30°$, the curves $G'H'$ and $E'B'$ respectively $-30°$ and $-60°$.

The diagram for $-\frac{\pi}{2} < \sigma < 0°$ needs no further elucidation; it plainly expresses the above-mentioned identity of the angle h for planes with poles $+\varrho, \pm\sigma$ and $-\varrho, \mp\sigma$. Consequently the octants 1 : 2, 3 : 4, 5 : 6, 7 : 8 are but for the sign symmetrical with regard to the plane $\left(\frac{\pi}{2}\right) : \left(\frac{3}{2}\pi\right)$, cf. fig. 3, whilst the octantpairs (1,2) : (7,8) and (5,6) : (3,4) are symmetrical, but for the sign of σ with regard to the plane $(0) : (\pi)$.

As appears further from fig. 3 the diagram for $\alpha = 60°$ is the same as for $\alpha = -60°$, if one substitutes for ϱ the value $\varrho \pm \pi$, and consequently changes the octant pairs (1,2) and (7,8) resp. for (3,4) and (5,6).

The diagrams of which I have served myself for the graphical solution of the problems discussed above, and which will be published elsewhere, give for angles α varying between 0 and $\pm \frac{\pi}{2}$ and progressing with 10°, the values of h for secant planes, of which azimuth and height of poles, likewise progressing with 10°, vary between $\varrho = 0$ and 2π, $\sigma = 0$ and $\pm \frac{\pi}{2}$. By interpolation the value of h for any indifferent secant plane can be found from it for every value of α with sufficient accuracy.

The way in which the problem must be solved may be explained to two of the problems treated analytically above. If one considers fig. I, then it is clear, that by a graphical treatment one can find $\angle A_1OB = h_1$ in the diagram for $\alpha = \alpha_1$ i. e. with $\varrho = \frown C_1BQ_1$, $\sigma = PQ$; in the same way $\angle A_2OB = h_2$ in the diagram for $\alpha = \alpha_2$ with $\varrho_1 = \frown C_2BQ = \varrho - \gamma$ and $\sigma = PQ$.

If on the contrary one wishes to determine from the given angles α_1, h_1, and α_2, h_2 the locus of the secant plane $P(\varrho,\sigma)$, then the diagrams for α_2 and α_1 must be laid on each other, the latter with regard to the former turned over an angle $-\gamma$, and one must

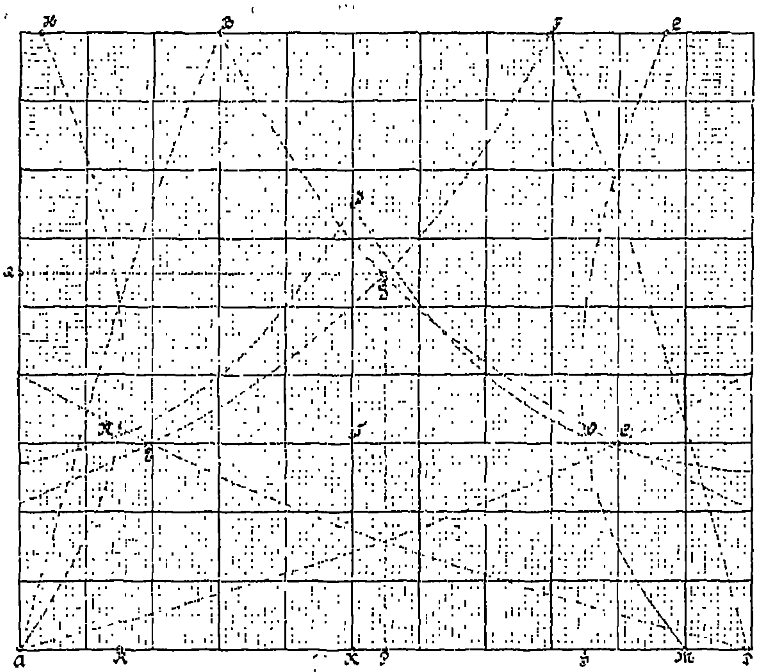

Fig. 6.

see where the curves $h_1(\alpha_1)$ and h_2 (α_2) meet. The meeting point indicates the pole $P(\varrho,\sigma)$ of the required secant plane.

In fig. 6 the graphical solution of two of the above discussed problems is represented. If one takes (cf. fig. 2) the rhombododecahedronplane (110) as equatorplane, then the planes (101) and (011) form angles with it equal to $\alpha_1 = -60°$ and $\alpha_2 = 60$. The secant plane S gives $h_1 = 60°$, $h_2 = -60°$; $\gamma = -109°28'17''$. If the diagram for α_1 is removed $109°^1/_2$ with regard to the diagram for α_2, and both are placed on each other, then it appears, that the curves h_1 (60°) and h_2 (-- 60°) meet only in the 5th octant. In the figure the curves for $\sigma > 0$ and $\sigma < 0$ are drawn side by side. If the azimuth of A is $= 0$, the curve $AB\,(\alpha_2 = 60°)$ indicates the poles for $h_2 = 60°$ with $\sigma > 0$; BC the poles for $h_2 = -60°$ with $\sigma < 0$, AC those for $h_2 = 60°$ with $\sigma < 0$. The azimuth of $D = 109°^1/_2$; FD gives the poles for $h_1 = -60°$ with $\sigma > 0$; FE and DE give those for $h_1 = 60°$ resp. $h_1 = -60°$ with $\sigma < 0$. The meeting point of the curves BC and FE gives the pole of the required secant plane with $\varrho = 54°^3/_4$ with regard to A, or $\varrho = -54°^3/_4$ with regard to D, and $\sigma = -54°^3/_4$.

The second problem refers to the amphibole-crystal spoken of on page 732, which is cut by the section plane in such a way that the apparent angles between the planes $(\bar{1}10):(110)$ and $(010):(110)$ amount respectively to $h_1 = 43°$ and $h_2 = -79°$.

In fig. 6 the zone-axis $(\varrho = 0)$ is indicated by K; the curves LOM and HNA indicate the locus of the poles for $h_1 = 43°$ with $\sigma > 0$ (1st octant) and $\sigma < 0$ (8th octant); the curves IO and IN indicate the locus of the poles for $h_2 = -79°$ with $\sigma > 0$ (1st octant) and $\sigma < 0$ (8th octant). The meeting point of the curves $LM : IO$ and $HA : IN$ gives the points $O\,(\varrho = 35°,\ \sigma = 31°)$ and $N\,(\varrho = -35°,\ \sigma = -31°)$ which values likewise correspond entirely to those found above in the analytical way.

Mathematics. — *"On the Integral equation of* FREDHOLM." By Prof. W. KAPTEYN.

1. Let

$$\varphi(x) = f(x) + \lambda \int^{b} K(xs)\,\varphi(s)\,ds \quad . \quad . \quad . \quad . \quad . \quad (1)$$

be the integral equation of FREDHOLM, in which the constants a, b, λ, and the functions $f(x)$ and $K(xy)$ are known, and $\varphi(x)$ is the function to be determined.