Mathematics. — "*The oscillations about a position of equilibrium where a simple linear relation exists between the frequencies of the vibrations*". (Third Part). By H. J. E. BETH. (Communicated by Prof. D. J. KORTEWEG).

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§ 1. In my dissertation [1]) were investigated the oscillations about a position of equilibrium of a mechanism with two degrees of freedom where a linear relation exists between the principal frequencies of vibration for which relation the sum of coefficients is $S \leq 4$. In what follows this investigation will be extended to a mechanism with an arbitrary number of degrees of freedom.

In the first place we shall trace the influence of a relation between two of the frequencies of vibration. Then the relations shall be discussed which are possible between 3 or 4 of the frequencies of vibration. Relations of more than 4 of the frequencies of vibration are outside our consideration, as we have always to keep in mind $S \leq 4$.

RELATIONS BETWEEN TWO OF THE FREQUENCIES OF VIBRATION.

§ 2. We imagine a mechanism with k degrees of freedom. Between the frequencies of vibration n_1 and n_2 of the principal coordinates q_1 and q_2 exists the relation

$$\gamma n_1 = n_2 + \varrho,$$

where $\gamma = 1$, 2 or 3. The remaining $k-2$ principal frequencies of vibration $n_3, n_4, \ldots . n_k$ do not appear in the relation ; we suppose moreover that between the k frequencies or between some of them no exact or approximate relation exists except the just mentioned one.

By the *disturbing terms of the first kind* in the equations of movement we shall understand terms which are always disturbing, also when no relation exists. When substituting the expressions for the coordinates by first approximation we find out of such a term a term having the same period and the same phase as the coordinate to which the equation, where the disturbing term appears, relates more in particular. These disturbing terms are of order h^3 or higher.

By *disturbing terms of the second kind* we understand such as owe their disturbing property to the existing relations. When substituting as above we find out of such a term a term corresponding to the coordinate in period but not in phase. These disturbing terms are of order h^{S-1} or higher.

[1]) Amsterdam, 1910; also These Proceedings, page 618—635 and page 735—750 (1910); Archives Néerlandaises, Séries II, Vol. XV, page 246—283 (1910).

§ 3. $S = 3\,(2n_1 = n_2 + \varrho)$. In general disturbing terms of the second kind of order h^2 appear. So the disturbing terms of the first kind can be left out, because they are at least of order h^3 and because in the equations we use the disturbing terms of the lowest order only. The disturbing terms of order h^2 appear only in the equations for q_1 and q_2, and they contain no other coordinates but q_1 and q_2. On account of this these two equations, in as far as they must be considered to give the first approximation, get the same form as we have formerly found for an arbitrary mechanism with two degrees of freedom only. The coordinates q_1 and q_2 bear themselves as if they were the only coordinates. As in the equations for $q_3, q_4, \dots q_k$ no disturbing terms of order h^2 appear, these coordinates bear themselves for first approximation as if no relation existed.

§ 4. $S = 4\,(3n_1 = n_2 + \varrho)$. Both kinds of disturbing terms are at least of order h^3. Disturbing terms of the second kind of order h^3 appear in the equations for q_1 and q_2 only and they contain no other coordinates than q_1 and q_2; so they are the same terms as those which appear in case $S = 4$ for a mechanism with two degrees of freedom as disturbing terms of the second kind.

However it is clear that disturbing terms of the first kind of order h^3 appear in all equations and that they will contain the coordinates $q^3, \dots q_k$ as well as q_1 and q_2. We reduce them in the following manner.

In the first equation disturbing terms of the first kind of order h^3 are those with: $q_1^3,\ q_1^2\ddot{q}_1,\ q_1\dot{q}_1^2,\ q_1q_2^2,\ q_1q_3^2 \dots q_1q_k^2,\ \ddot{q}_1q_2^2, \ddot{q}_1q_3^2 \dots \ddot{q}_1q_k^2,\ q_1\dot{q}_2^2,$ $q_1\dot{q}_3 \dots q_1\dot{q}_k^2,\ q_1q_2\ddot{q}_2,\ q_1q_3\ddot{q}_3 \dots q_1q_k\ddot{q}_k$.

If we take as solution at first approximation

$$q_r = \frac{\sqrt{\alpha_r}}{n_r}\cos\,(n_rt + 2n_r\beta_r) \quad (r = 1, 2, 3 \dots k)$$

then in the terms of higher order we may substitute $-n_r^2q_r$ for $\ddot{q}_r$ and $\alpha_r - n_r^2q_r^2$ for $\dot{q}_r^2$. We then retain as disturbing terms of the first kind in the first equation only those with

$$\alpha_rq_1,\ q_1^3,\ q_1q_2^2,\ q_1q_3^2 \dots q_1q_k^2.$$

We then substitute $\frac{\alpha_2}{2n_2^2}q_1$ etc. for $q_1q_2^2$.

If we deduce in the same way the disturbing terms of the first kind also in the remaining equations, we shall find that these terms are in the different equations the derivatives resp. according to $q_1, q_2 \dots q_k$ of

$$\sum_{r=1}^{r=k} A_r q_r^4 + \sum_{r=1}^{r=k} \sum_{s=1}^{s=k} B_{rs} \alpha_r q_s^2,$$

where A_r and B_{rs} represent constants.

So we have shown that there is a function of which the disturbing terms in the equations are the derivatives. If we write it as function of the α's, β's and t and afterwards leave out the terms containing t explicitly, we find:

$$-R = \chi_2 (\alpha_1, \alpha_2, \ldots \alpha_k) + \varrho' h^2 \alpha_2 + m_1 \alpha_1 \sqrt{\alpha_2} \cos 6n_1 (\beta_1 - \beta_2).$$

Here χ_2 is a homogeneous quadratic function of the α's; the term $\varrho' h^2 \alpha_2$ is inserted in order to take in the wellknown way the residue of relation into account.

The α's and β's must now be determined as functions of t with the aid of the following system of equations:

$$\dot{\alpha}_r = \frac{\partial R}{\partial \beta_r}, \quad \dot{\beta}_r = -\frac{\partial R}{\partial \alpha_r}. \quad (r = 1, 2, 3, \ldots k).$$

We immediately notice that now also

$$\dot{\alpha}_1 = -\dot{\alpha}_2.$$

Hence

$$\alpha_1 + \alpha_2 = \text{constant}.$$

From the absence of $\beta_3, \beta_4 \ldots \beta_k$ in R it is evident that:

$$\dot{\alpha}_3 = \dot{\alpha}_4 = \ldots = \dot{\alpha}_k = 0.$$

Hence

$$\alpha_3 = \text{constant}, \quad \alpha_4 = \text{constant}, \ldots \alpha_k = \text{constant}.$$

Here $\dot{\alpha}_1$ and $\dot{\alpha}_2$ have the same form as for the mechanism with two degrees of freedom.

The expressions for $\dot{\beta}_1$ and $\dot{\beta}_2$ contain both, besides the terms which they have for the mechanism with two degrees of freedom, one more linear function of $\alpha_3, \alpha_4, \ldots \alpha_k$. On account of what was just found these functions can be reduced to constants of order h^2. Let $m_1 h^2$ be this constant term in the second member of the equation for $\dot{\beta}_1$, $m_2 h^2$ the term in the second member of the equation for $\dot{\beta}_2$. This is then the influence of these terms that the frequency of vibration n_1 must be increased by $m_1 h^2$ and the residue of relation by $6n_1 (m_2 - m_1) h^2$.

Then $\alpha_1, \alpha_2, \beta_1$ and β_2 are determined out of the same equations as in the case of the mechanism with two degrees of freedom. The coordinates of q_1 and q_2 behave here too as if they were the only ones. The influence of the $k-2$ remaining degrees of freedom consists in a modification of n_1 and n_2, which modification is of order h^2 and dependent on the amplitudes of the remaining $k-2$ vibrations.

Then $\dot{\beta}_3, \dot{\beta}_4, \ldots\ldots \beta_4$ are, as the form of R tells us, linear functions of $\alpha_1, \alpha_2 \ldots\ldots \alpha_k$. As $\alpha_3, \alpha_4 \ldots\ldots \alpha_k$ are constant and as α_2 can be expressed in α_1 we can write $\dot{\beta}_3, \dot{\beta}_4, \ldots\ldots \dot{\beta}_k$ as linear functions of α_1. So the coordinates $q_3, q_4 \ldots\ldots q_k$ feel the influence of the relation, however only in the phase, not in the amplitude. As α_1 (just as α_2, β_1, and β_2) was determined before already as function of t, we can also determine $\beta_3, \beta_4, \ldots\ldots \beta_k$. So the problem has been reduced to quadratures.

§ 5. $S = 2\ (n_1 = n_2 + \varrho)$. All disturbing terms which we have to regard are again of order h^3. For this case the peculiarity appears that all disturbing terms of the second kind must be regarded at the same time as disturbing of the first kind. So a term $q_1 q_2^2$ in the first equation gives as disturbing terms a term with $\cos\,(nt + 2n\beta_2)$ and one with $\cos\,(nt + 4n\beta_2 - 2n\beta_1)$.

Just as was done above for the case $S = 4$ we can prove easily also for this case that, apart from a modification of their frequency of vibration, the coordinates q_1 and q_2 behave as if we had to do with a mechanism with two degrees of freedom, whilst the remaining coordinates feel the influence of relation in their phase, but not in their amplitude.

Relation between three of the frequencies of vibration for which $S = 3$.

§ 6. The only relation which for $S = 3$ remains to be discussed runs:

$$n_1 + n_2 - n_3 = \varrho.$$

Just as was done in § 3 for the case of a relation between two of the frequencies of vibration for which $S = 3$ we can also show here, that at first approximation only the coordinates q_1, q_2, and q_3 feel the influence of the relation and that q_1, q_2, and q_3 behave as if they were the only coordinates. So we can restrict ourselves to a mechanism with but three degrees of freedom; q_1, q_2, and q_3 are the principal coordinates.

As in the equations of motion terms of order h^2 appear already among the disturbing terms we need not take into account any terms of a higher order than h^3 in the expressions for the kinetic energy and the potential function. Hence

$$T = \tfrac{1}{2} \sum_{r=1}^{r=3} \dot{q}_r^2 + \tfrac{1}{2} \sum_{r=1}^{r=3} \sum_{s=1}^{s=3} (P_{rr} \dot{q}_r^2 + 2\, P_{rs} \dot{q}_r \dot{q}_s),$$

$$U = \tfrac{1}{2} \sum_{r=1}^{r=3} n_r^2 q_r^2 + H_3 (q_1, q_2, q_3),$$

where H_3 represents a homogeneous function of order three, and

$$P_{rs} \equiv a_{rs} q_1 + b_{rs} q_2 + c_{rs} q_3.$$

The equation of LAGRANGE for the coordinate q_1 runs:

$$\ddot{q}_1 + n_1^2 q_1 = -\sum_{r=1}^{r=3} P_{1r} \ddot{q}_r - \sum_{r=1}^{r=3} (b_{1r} \dot{q}_2 + c_{1r} \dot{q}_3) \dot{q}_r + \tfrac{1}{2} \sum_{r=1}^{r=3} a_{rr} \dot{q}_r^2 + a_{23} \dot{q}_2 \dot{q}_3 - \frac{\partial H_3}{\partial q_1}.$$

When in the terms of the second member $\ddot{q}_r$ is replaced by $-n_r^2 q_r$, then in the case of the supposed relation we have to regard as disturbing in this equation the terms with $\dot{q}_2 \dot{q}_3$ and those with $q_2 q_3$. Omitting all the remaining terms of a higher order the equation becomes:

$$\ddot{q}_1 + n_1^2 q_1 = (c_{12} n_2^2 + b_{13} n_3^2 - p) q_2 q_3 - (-a_{23} + b_{13} + c_{12}) \dot{q}_2 \dot{q}_3.$$

(p being the coefficient of the term $q_1 q_2 q_3$ in H_3).

We may now replace $\dot{q}_2 \dot{q}_3$ by $n_2 n_3 q_2 q_3$, because these two products furnish the same disturbing term when we substitute for q_2 and q_3 the expressions to be taken at first approximation. We then find:

$$\ddot{q}_1 + n_1^2 q_1 = \{a_{23} n_2 n_3 + b_{13} n_3 (n_3 - n_2) + c_{12} n_2 (n_2 - n_3) - p\} q_2 q_3.$$

Putting in the second member $n_1 + n_2 - n_3 = 0$ we finally find:

$$\ddot{q}_1 + n_1^2 q_1 = (a_{23} n_2 n_3 + b_{13} n_1 n_3 - c_{12} n_1 n_2 - p) q_2 q_3.$$

In this way we can also simplify the two other equations; we must then bear in mind that in the second equation $\dot{q}_1 \dot{q}_3$ must be replaced by $n_1 n_3 q_1 q_3$, in the third equation however $\dot{q}_1 \dot{q}_2$ by $-n_1 n_2 q_1 q_2$.

The result is that the equations of motion are to be written as follows:

$$\ddot{q}_r + n_r^2 q_r - \frac{\partial R}{\partial q_r} = 0 \qquad (r = 1, 2, 3). \quad . \quad . \quad . \quad . \quad (1)$$

where

$$-R = \{p - (a_{23} n_2 n_3 + b_{13} n_1 n_3 - c_{12} n_1 n_2)\} q_1 q_2 q_3.$$

If, however, we take as abridged 3rd equation

$$\ddot{q}_3 + (n_1 + n_2)^2 q_3 = 0,$$

then we find:

$$-R = \{p - (a_{23} n_2 n_3 + b_{13} n_1 n_3 - c_{12} n_1 n_2)\} q_1 q_2 q_3 - \\ - \varrho (n_1 + n_2) q_3^2 \equiv p' q_1 q_2 q_3 - \varrho (n_1 + n_2) q_3^2$$

Pure relation $n_1 + n_2 - n_3 = 0$.

§ 7. As first approximation of system (1) we take:

$$q_1 = \frac{\sqrt{\alpha_1}}{n_1} \cos (n_1 t + 2 n_1 \beta_1),$$

$$q_2 = \frac{\sqrt{\alpha_2}}{n_2} \cos (n_2 t + 2 n_2 \beta_2),$$

$$q_3 = \frac{\sqrt{\alpha_3}}{n_1 + n_2} \cos \{(n_1 + n_2) t + 2 (n_1 + n_2) \beta_3\}.$$

The α's must necessarily be positive quantities and in general do not become zero during the motion. We now write $-R \equiv p' q_1 q_2 q_3$ as a function of the α's, β's, and t, and we then omit the terms, containing t explicitly. Then we find:

$$-R = p'' \sqrt{\alpha_1 \alpha_2 \alpha_3} \cos \varphi,$$

where

$$p'' = \frac{p'}{4 n_1 n_2 (n_1 + n_2)},$$

$$\varphi = 2 \{n_1 \beta_1 + n_2 \beta_2 - (n_1 + n_2) \beta_3\}.$$

The system of equations, determining the variability of the α's and β's, runs:

$$\left.\begin{array}{ll} \dot{\alpha}_1 = 2 n_1 p'' \sqrt{\alpha_1 \alpha_2 \alpha_3} \sin \varphi, & \dot{\beta}_1 = \dfrac{p''}{2} \dfrac{\sqrt{\alpha_1 \alpha_2 \alpha_3}}{\alpha_1} \cos \varphi, \\ \dot{\alpha}_2 = 2 n_2 p'' \sqrt{\alpha_1 \alpha_2 \alpha_3} \sin \varphi, & \dot{\beta}_2 = \dfrac{p''}{2} \dfrac{\sqrt{\alpha_1 \alpha_2 \alpha_3}}{\alpha_2} \cos \varphi, \\ \dot{\alpha}_3 = -2 (n_1 + n_2) p'' \sqrt{\alpha_1 \alpha_2 \alpha_3} \sin \varphi, & \dot{\beta}_3 = \dfrac{p''}{2} \dfrac{\sqrt{\alpha_1 \alpha_2 \alpha_3}}{\alpha_3} \cos \varphi, \end{array}\right\} \quad \ldots \quad (2)$$

An integral of this system is:

$$\sqrt{\alpha_1 \alpha_2 \alpha_3} \cos \varphi = constant \quad \ldots \ldots \ldots \quad (3)$$

Furthermore we notice that:

$$\frac{\dot{\alpha}_1}{n_1} = \frac{\dot{\alpha}_2}{n_2} = - \frac{\dot{\alpha}_3}{n_1 + n_2},$$

$$\dot{\alpha}_1 + \dot{\alpha}_2 + \dot{\alpha}_3 = 0.$$

Therefore:

$$\frac{\alpha_1}{n_1} - \frac{\alpha_2}{n_2} = C_2 h^2, \quad \frac{\alpha_1}{n_1} + \frac{\alpha_3}{n_1 + n_2} = C_3 h^2, \quad \frac{\alpha_2}{n_2} + \frac{\alpha_3}{n_1 + n_2} = (C_3 - C_2) h^2 \quad (4)$$

$$\alpha_1 + \alpha_2 + \alpha_3 = constant. \quad \ldots \ldots \ldots \quad (5)$$

Here C_2 and C_3 are constants; C_3 is positive and $C_3 > C_2$. We suppose C_2 to be positive too, which does not imply a restriction; for if C_3 were negative, we should have but to exchange the coordinates q_1 and q_2.

If we put

$$\frac{\alpha_1}{n_1} = \zeta h^2,$$

then

$$\frac{\alpha_2}{n_2} = (\zeta - C_2)h^2, \quad \frac{\alpha_3}{n_1 + n_2} = (C_3 - \zeta)h^2.$$

This gives to (3) the form:

$$\sqrt{\zeta(\zeta - C_2)(C_3 - \zeta)} \cos \varphi = k, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where k represents a constant.

The first equation of (2) becomes by introduction of ζ:

$$\dot{\zeta} = 2p'' \sqrt{n_1 n_2 (n_1 + n_2)}\, h \sqrt{\zeta(\zeta - C_2)(C_3 - \zeta)} \sin \varphi \quad . \quad . \quad (7)$$

Elimination of φ between (6) and (7) furnishes

$$\frac{d\zeta}{\pm \sqrt{\zeta(\zeta - C_2)(C_3 - \zeta) - k^2}} = 2p'' \sqrt{n_1 n_2 (n_1 + n_2)}\, h dt.$$

On account of this ζ can be determined as function of t, which makes α_1, α_2, and α_3 to be known; then φ can be found out of (3); finally β_1, β_2, and β_3 out of (2).

§. Let us suppose relation (6):

$$\sqrt{\zeta(\zeta - C_2)(C_3 - \zeta)} \cos \varphi = k$$

to represent a curve on polar coordinates; we take ζ as radius vector, φ as polar angle.

As α_1, α_2, and α_3 are positive, ζ, $\zeta - C_2$ and $C_3 - \zeta$ are positive, ζ remains between C_2 and C_3. So we have to regard only curves situated between the circles $\zeta = C_2$ and $\zeta = C_3$.

The curves remain on the right or on the left of O according to k being positive or negative. In fig. 1 the curves have been drawn for definite values of C_2 and C_3, for some values of k.

The distances of the points of intersection of a curve with the axis of the angles is found as the positive roots of the equation:

$$\zeta(\zeta - C_2)(C_3 - \zeta) - k^2 = 0.$$

For a given value of C_2 and C_3 there is a maximal value of k^2, for which this equation has two equal roots and below which it has 3 real ones. For this value the curve has contracted to an isolated point. This concerns a special case of motion.

Another special case we have for $k^2 = 0$. Degeneration takes place to the point $\zeta = 0$, the circles $\zeta = C_2$ and $\zeta = C_3$, and the right line $\cos \varphi = 0$.

Further more there are special cases for special values of C_2 and

C_3. If $C_2 = C_3$, then of necessity $k = 0$; so this belongs to the second special case. If $C_2 = 0$, then by putting $\zeta = C_3 \zeta'$ the relation (6) passes into the one which we had with the mechanism with two degrees of freedom, for which $n_2 = n_1$.

The special cases will be discussed in § 14.

§ 9. *Osculating curves.* In order to illustrate the motion of the mechanism somewhat better, we use an image point.

To this end we choose the point whose rectangular coordinates x, y, and z are at an arbitrary moment equal to the values of the principal coordinates q_1, q_2, and q_3 at that moment. The motion of this point is then given by

$$x = \frac{\sqrt{\alpha_1}}{n_1} \cos (n_1 t + 2 n_1 \beta_1),$$

$$y = \frac{\sqrt{\alpha_2}}{n_2} \cos (n_2 t + 2 n_2 \beta_2),$$

$$z = \frac{\sqrt{\alpha_3}}{n_1 + n_2} \cos \{(n_1 + n_2)\, t + 2\, (n_1 + n_2)\, \beta_3\}.$$

By eliminating t between these equations two by two and by ascribing to the α's and β's, the values at a definite moment, we find the projections of the osculating curves on the planes of coordinates.

These projections are LISSAJOUS curves; the osculating curves themselves we can call LISSAJOUS twisted curves.

§ 10. Such a twisted curve remains enclosed inside a rectangular parallelopiped bounded by the planes:

$$x = \pm \frac{\sqrt{\alpha_1}}{n_1}, \quad y = \pm \frac{\sqrt{\alpha_2}}{n_2}, \quad z = \pm \frac{\sqrt{\alpha_3}}{n_1 + n_2}.$$

In consequence of the variability of the α's this enclosed parallelopiped varies continually. The vertices move along a twisted curve, which according to (3) projects itself on the XY-plane as a hyperbola, on the XZ and on the YZ-plane as an ellipse. Out of (5) follows that this curve is situated on an ellipsoid, whose axes lying on the axes of coordinates are in the ratio:

$$\frac{1}{n_1} : \frac{1}{n_2} : \frac{1}{n_1 + n_2}.$$

As the α's change periodically between definite limits, the vertices will move to and fro along the above mentioned twisted curve between two extreme positions. (fig. 2).

§ 11. Besides on the α's the form of an osculating curve depends moreover on the β's. However for an osculating curve described in a definite parallelopiped it depends not on 3, but only on 2 quantities, as is evident when we change the origin of time. We can get:

$$x = \frac{\sqrt{\alpha_1}}{n_1} \cos \{n_1 t + 2(n_1\beta_1 - n_1\beta_3)\},$$

$$y = \frac{\sqrt{\alpha_2}}{n_2} \cos \{n_2 t + 2(n_2\beta_2 - n_2\beta_3)\},$$

$$z = \frac{\sqrt{\alpha_3}}{n_1+n_2} \cos (n_1+n_2)\, t.$$

The form of the osculating curve evidently depends on the quantities $\beta_1-\beta_3$ and $\beta_2-\beta_3$. So if we put:

$$\frac{\sqrt{\alpha_1}}{n_1} = A\,,\ \frac{\sqrt{\alpha_2}}{n_2} = B\,,\ \frac{\sqrt{\alpha_3}}{n_1+n_2} = C,\ 2(n_1\beta_1 - n_1\beta_3) = a,\ 2(n_2\beta_2 - n_2\beta_3) = b,$$

then we find

$$\left.\begin{aligned} x &= A \cos (n_1 t + a), \\ y &= B \cos (n_2 t + b), \\ z &= C \cos (n_1+n_2)\, t. \end{aligned}\right\} \quad . \ . \ . \ . \ . \ . \ . \ . \quad (8)$$

It is evident out of (7) that in the extreme parallelopipeds curves are described for which $\sin \varphi = 0$. So for these curves

$$2(n_1\beta_1 + n_2\beta_2 - n_1\beta_3 - n_2\beta_3) = l\pi,$$

where l is an integer,

$$a + b = l\pi$$

$$b = l\pi - a.$$

So the curves described in the extreme parallelopipeds are given by:

$$\left.\begin{aligned} x &= A \cos (n_1 t + a), \\ y &= B \cos (n_2 t - a + l\pi), \\ z &= C \cos (n_1 + n_2)\, t. \end{aligned}\right\} \quad . \ . \ . \ . \ . \ . \ . \quad (9)$$

§ 12. The literature concerning the LISSAJOUS twisted curves seems to restrict itself to a paper of A. RIGHI (Il Nuovo Cimento, vol IX and X, 1873). RIGHI discusses only the case that the periods of the three mutually perpendicular vibrations have a common measure and he investigates which properties of symmetry these curves can have.

Let us put in (8) $t = +\tau$ and $t = -\tau$ and let us call the values of x, y, and z belonging to these values of t, respect. x_1, y_1, z_1 and x_2, y_2, z_2, then we find

$\frac{1}{2}(x_1+x_2) = A \cos a \cos n_1\tau$, $\frac{1}{2}(y_1+y_2) = B \cos b \cos n_2\tau$, $z_1 = z_2$.

The curve represented by (8) has therefore with respect to directions of chords parallel to the XY-plane as diameter a curve represented by the equations:

$$\left.\begin{aligned} x_3 &= A' \cos n_1 t, \\ y_3 &= B' \cos n_2 t, \\ z_3 &= C \cos (n_1 + n_2)\, t \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

where

$$A' = A \cos a \ , \ B' = B \cos b.$$

To investigate the curves represented by (8) we can start from the simple curves represented by (10). In fig. 3 such a curve is given perspectively, in fig. 4 (continuous lines) by projections for the case that n_1 and n_2 are commensurable and that we have $n_1 = 2n_2$; the twisted curve begins and ends in two vertices of the circumscribed parallelopiped and is described backwards and forwards.

When a curve (10) is constructed we must bear in mind that

$$x = x_3 - \delta_1 \sin n_1 t \ , \ y = y_3 - \delta_2 \sin n_2 t \ , \ z = z_3,$$

where

$$\delta_1 = A \sin a \ , \ \delta_2 = B \sin b$$

So we can think the curve (8) as described by a point moving along the curve (10) and vibrating at the same time according to the X- and Y-direction.

From this we can see how the osculating curve changes for increasing values of a and b, and we can make out when it shows double points. In fig. 4 the projections are represented (dotted lines) of an osculating curve for $n_1 = 2n_2$ and small values of a and b.

§ 13. For the curves represented by (9) exists a simple method to construct the ZX- and ZY-projections, when the XY-projection is given. We can imagine l as even; the curves for odd values of l are the mirror image of the curves for even values of l with respect to the XZ-plane. The XY-projection is an entirely arbitrary LISSAJOUS curve; for $t = 0$ the projection of the point is on a diagonal of the circumscribed rectangle.

Now however follows out of (9)

$$\cos^{-1} \frac{z}{C} = \cos^{-1} \frac{x}{A} + \cos^{-1} \frac{y}{B} .$$

In fig. 5 is given how for every point on the XY-projection z is to be constructed. It is easy to show, that the points of intersection of (9) with the XY-plane lie on the ellipse:

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$$\frac{x^2}{A^2}+\frac{y^2}{B^2}=1.$$

The points where the curve touches the planes $z=\pm C$ are projected in the right lines

$$\frac{x}{A}=\pm\frac{y}{B}.$$

§ 14. *Special cases.* At the conclusion of § 8 the special cases were named which may appear. They are:

A. *For given values of C_2 and C_3 we find $k=0$.* The relation

$$\sqrt{a_1 a_2 a_3}\cos\varphi=0$$

allows of various possibilities.

1. One of the a's is continually zero. Not one of these forms of motion, however, proves to be possible on substitution into (1).

2. $\cos\varphi=0$; φ is continually $\frac{\pi}{2}$ or $\frac{3\pi}{2}$. The form of motion changes periodically between those for which $\zeta=C_2$ ($a_2=0$) and those for which $\zeta=C_3$ ($a_3=0$).

B. *For given C_2 and C_3 we find k^2 maximum.* Here ζ is constant, so the a's are also constant; the circumscribed parallelopiped does not change. The β's increase uniformly with the time; the osculating curve changes its form; $\sin\varphi=0$ remains however. The osculating curve is thus represented by (9); a increases uniformly with the time.

C. *C_2 is equal to zero.* Then $\frac{a_1}{n_1}=\frac{a_2}{n_2}$. The movement of the vertices of the circumscribed parallelopiped takes place along a plane curve; the plane passes through the Z-axis. When $C_2=0$ and at the same time $k=0$, then the form of the movement approaches asymptotically the Z vibration.

D. $C_2=C_3$. An X-vibration continually takes place.

Approximated relation $n_1+n_2-n_3=\varrho$.

§ 15. We must imagine ϱ to be of order h. Now

$$-R=p' q_1 q_2 q_3-(n_1+n_2)\varrho q_3^2.$$

As first approximation we take for q_1, q_2, and q_3 the same expressions as in the case of the pure relation. We find for R as function of the a's and β's

$$-R=p''\sqrt{a_1 a_2 a_3}\cos\varphi-\frac{\varrho}{2(n_1+n_2)}a_3.$$

We can again write down the system of equations which indicates the variability of the α's and β's. This system has again as integrals:

$$\frac{\alpha_1}{n_1}-\frac{\alpha_2}{n_2}=constant,\quad \frac{\alpha_1}{n_1}+\frac{\alpha_3}{n_1+n_2}=constant,\quad \frac{\alpha_2}{n_2}+\frac{\alpha_3}{n_1+n_2}=constant\,,$$

$$\alpha_1+\alpha_2+\alpha_3=constant.$$

Lét us again put

$$\frac{\alpha_1}{n_1}=\zeta h^2,\quad \frac{\alpha_2}{n_2}=(\zeta-C_2)\,h^2,\quad \frac{\alpha_3}{n_1+n_2}=(C_3-\zeta)\,h^2,$$

then the integral $R=constant$ takes the form:

$$\sqrt{\zeta\,(\zeta-C_2)\,(C_3-\zeta)}\cos\varphi=\varrho'\,(k-\zeta),$$

where k represents a constant and

$$\varrho'=\frac{\varrho}{2p''\sqrt{n_1n_2(n_1+n_2)}h}.$$

We now see easily in what way the coordinates are to be found as functions of time.

§ 16. A survey of the general and special cases which can appear, as well as an insight into the manner in which the transition takes place on one hand to the case of the pure relation, on the other hand to the general case, where no relation exists, is to be obtained by representing the relation between ζ and φ on polar coordinates, ζ being the radius vector, φ the polar angle. In

$$\sqrt{\zeta\,(\zeta-C_2)\,(C_3-\zeta)}\cos\varphi=\varrho'\,(k-\zeta)$$

we may represent ϱ' as positive; for the curves for negative values of ϱ' are the mirror images with respect to the right line $\varphi=\frac{\pi}{2}$ of the curves for positive values of ϱ'.

We give to C_2 and C_3 constant values and we find for a certain value of ϱ' the forms of the curves satisfying the different possible values of k. We then see how this system of curves varies when ϱ' passes through all values from very little to very large.

For every value of ϱ three cases can be distinguished:

1. $k>C_3$. As ζ remains smaller than C_3, the second member, so also $\cos\varphi$, remains positive. Curves on the right of O.

2. $k<C_2$. As ζ remains larger than C_2, the second member, so also $\cos\varphi$, remains negative. Curves on the left of O.

3. $C_2<k<C_3$. The second member, therefore also $\cos\varphi$, becomes zero for $\zeta=k$. Curves which surround O.

The curves represented by the above relation lie therefore either entirely on one side of O, or they surround O.

The domains of the plane occupied by these different kinds of curves are bounded by the curves which correspond to $k = C_2$ and $k = C_3$.

For these values of k a degeneration takes place.

For $k = C_2$ in:

$$\zeta = C_2 \text{ and } \sqrt{\xi(C_3-\xi)} \cos\varphi = -\varrho' \sqrt{\xi - C_2}.$$

The latter curve lies on the left of O, it begins and ends in the points: $\xi = C_2$, $\varphi = \pm \frac{\pi}{2}$.

For $k = C_3$ in:

$$\xi = C_3 \text{ and } \sqrt{\xi(\xi - C_2)} \cos\varphi = \varrho' \sqrt{C_3 - \xi}.$$

The latter curve lies on the right of O; it commences and ends on the points: $\xi = C_3$, $\varphi = \pm \frac{\pi}{2}$.

To investigate how the system of curves varies when ϱ' is changed, it is sufficient to investigate the variation of the degenerated curves. The result is, that the domain of the curves surrounding O is very small for small values of ϱ' and it extends according as ϱ' increases, so that those curves are most important for great values of ϱ'.

So we have for small values of ϱ' by preference the case, that φ moves to and fro between two extreme opposite values, for great values of ϱ' by preference the case that assumes all values.

Furthermore we notice that according as ϱ' increases the curves surrounding O as centre, i. o. w. ξ and on account of this the α's vary but little. We thus approach the general case where ϱ' has become so great, that we can no longer speak of a relation.

Here too we get for each value of ϱ' for the maximal and minimal value of k an isolated point on the axis of the angles.

Fig. 6 gives some curves for a rather small value of ϱ', fig. 7 for a rather great value of ϱ'; the —·— lines indicate the degenerated curves.

Relations between 3 of the frequencies of vibration for which $S = 4$.

§ 17. Two of these relations have to be discussed, namely:

$$(A) \qquad n_1 + 2n_2 - n_3 = \varrho,$$
$$(B) \quad -n_1 + 2n_2 - n_3 = \varrho.$$

We commence with the determination of the disturbing terms of the second kind in the equations of motion. These contain no other coordinates than q_1, q_2, and q_3, and they appear only in the equations of motion which refer more in particular to these coordinates. So it is clear that to determine the disturbing terms of the second kind we can restrict ourselves to a mechanism with three degrees of freedom.

In the equations there are no disturbing terms among the terms of order h^2; terms of a higher order than h^3 are not inserted. Hence we can write the potential function and the kinetic energy as follows:

$$U = \tfrac{1}{2} \sum_{r=1}^{r=3} n_r^2 q_r^2 + H_4 (q_1, q_2, q_3),$$

$$T = \tfrac{1}{2} \sum_{r=1}^{r=3} \dot{q}_r^2 + \tfrac{1}{2} \sum_{r=1}^{r=3} (P_{rr} \dot{q}_r^2 + 2P_{rs} \dot{q}_r \dot{q}_s),$$

where H_4 represents a homogeneous function of degree 4, and

$$P_{rs} = \tfrac{1}{2} a_{rs} q_1^2 + \tfrac{1}{2} b_{rs} q_2^2 + \tfrac{1}{2} c_{rs} q_3^2 + e_{rs} q_1 q_2 + f_{rs} q_1 q_3 + h_{rs} q_2 q_3 .$$

If e. g. we write the equation of motion for q_1, then for the relations (A) and (B) the following terms

$$q_2^2 q_3 ,\ q_2 \ddot{q}_2 q_3 ,\ q_2^2 \ddot{q}_3 ,\ \dot{q}_2^2 q_3 \text{ and } \dot{q}_2 q_2 \dot{q}_3 ,$$

are to be regarded as disturbing.

Let us replace in these terms $\ddot{q}_3$ by $-n_3^2 q_3$, $\ddot{q}_2$ by $-n_2^2 q_2$, $\dot{q}_2^2$ by $-n_2 q_2^2$, and $\dot{q}_2 \dot{q}_3$ by $n_2 n_3 q_2 q_3$.

Let us omit all non-disturbing terms of order h^3 and let us make use in the disturbing terms of the relation $n_1 \pm 2n_2 \mp n_3 = 0$ (which is permissible, as ϱ is of order h^2); we then find that the first equation can be written as follows:

$$\ddot{q}_1 + n_1^2 q_1 = (\mp n_1 n_2 h_{12} \pm \tfrac{1}{2} n_1 n_3 b_{13} - \tfrac{1}{2} n_2^2 f_{22} + n_2 n_3 e_{23} - p)\, q_2^2 q_3 +$$

disturbing terms of the first kind (p being the coefficient of a term $q_1 q_2^2 q_3$ in U).

Of the $\mp$ and $\pm$ signs the top one must be taken in the case of the relation (A), the bottom one in the case of the relation (B).

When determining the disturbing terms of the second kind in the equations q_2 and q_3, and when reducing these terms according to the method given just now, we find as result that the disturbing terms are the derivatives of one and the same function, namely of

$$p' q_1 q_2^2 q_3 ,$$

where

$$p' = p \pm n_1 n_2 h_{12} \mp n_1 n_3 b_{13} + \tfrac{1}{2} n_2^2 f_{22} - n_2 n_3 e_{23} .$$

This part of the function of disturbance can be again expressed in the same manner in the α's and the β's.

As disturbing terms of the first kind we have but to take the terms, which we have determined in § 4. For those terms are independent of the relation.

So we find that the disturbing function expressed in the α's and the β's takes the following form:

$$-R = \chi_2(\alpha_1, \alpha_2, \ldots \alpha_k) + \varrho' h^2 \alpha_3 + m_1 \alpha_2 \sqrt{\alpha_1 \alpha_3} \cos \varphi,$$

where again χ_2 represents a homogeneous, quadratic function of the α's.

The second term is inserted because we take as first approximation:

$$q_1 = \frac{\sqrt{\alpha_1}}{n_1} \cos(n_1 t + 2n_1\beta_1),$$

$$q_2 = \frac{\sqrt{\alpha_2}}{n_2} \cos(n_2 t + 2n_2\beta_2),$$

$$q_3 = \frac{\sqrt{\alpha_3}}{2n_2 \pm n_1} \cos\{(2n_2 \pm n_1)t + 2(2n_2 \pm n_1)\beta_3\}; \text{ etc.}$$

Furthermore we find

$$\varphi = \pm 2n_1\beta_1 + 4n_2\beta_2 - 2(2n_2 \pm n_1)\beta_3 \equiv \pm 2n_1(\beta_1 - \beta_3) + 4n_2(\beta_2 - \beta_3).$$

§ 18. We can again suppose the differential equations written down, determining the variability of the α's and β's. We then immediately find:

$$\alpha_4 = \textit{constant}, \quad \alpha_5 = \textit{constant}, \ldots \ldots \ldots \alpha_k = \textit{constant}.$$

However, $\beta_4, \beta_5 \ldots \beta_k$ are variable. The coordinates $q_4, q_5 \ldots q_k$ experience the influence of the relation in their phase, but not in their amplitude.

Let us regard in particular the equations for $\dot{\alpha}_1$, $\dot{\alpha}_2$ and $\dot{\alpha}_3$:

$$\dot{\alpha}_1 = \pm 2m_1 n_1 \alpha_2 \sqrt{\alpha_1\alpha_3} \sin \varphi,$$

$$\dot{\alpha}_2 = + 4m_1 n_2 \alpha_2 \sqrt{\alpha_1\alpha_3} \sin \varphi,$$

$$\dot{\alpha}_3 = - 2(2n_2 \pm n_1) m_1 \alpha_2 \sqrt{\alpha_1\alpha_3} \sin \varphi.$$

We deduce from this:

$$\mp \frac{\dot{\alpha}_1}{n_1} = -\frac{\dot{\alpha}_2}{2n_2} = \frac{\dot{\alpha}_3}{2n_2 \pm n_1},$$

$$\dot{\alpha}_1 + \dot{\alpha}_2 + \dot{\alpha}_3 = 0.$$

§ 19. *Case A.*

$$\frac{\alpha_1}{n_1} - \frac{\alpha_2}{2n_2} = (C_1 - C_2)h^2, \quad \frac{\alpha_2}{2n_2} + \frac{\alpha_3}{2n_2 + n_1} = C_2 h^2, \quad \frac{\alpha_1}{n_1} + \frac{\alpha_3}{2n_2 + n_1} = C_1 h^2,$$

where C_1 and C_2 are positive constants.

If we put

$$\alpha_3 = (2n_2 + n_1) h^2 \zeta,$$

we then find

$$\alpha_1 = n_1 (C_1 - \zeta) h^2, \quad \alpha_2 = 2 n_2 (C_2 - \zeta) h^2.$$

The integral $R =$ constant then takes the form:

$$(C_2 - \zeta) \sqrt{\zeta (C_1 - \zeta)} \cos \varphi = p\zeta^2 + q\zeta + r.$$

Case B.

$$\frac{\alpha_1}{n_1} + \frac{\alpha_2}{2n_2} = C_1 h^2, \quad \frac{\alpha_2}{2n_2} + \frac{\alpha_3}{2n_2 - n_1} = C_3 h^2, \quad \frac{\alpha_3}{2n_2 - n_1} - \frac{\alpha_1}{n_1} = (C_3 - C_1) h^2,$$

where C_1 and C_3 are positive constants.

If we put

$$\alpha_2 = 2 n_2 h^2 \zeta.$$

then we find

$$\alpha_1 = n_1 (C_1 - \zeta) h^2, \quad \alpha_3 = (2n_2 - n_1)(C_3 - \zeta) h^2.$$

The integral $R =$ constant gets the form:

$$\zeta \sqrt{(C_1 - \zeta)(C_3 - \zeta)} \cos \varphi = p\zeta^2 + q\zeta + r.$$

§ 20. In case A we find that ζ lies continually between 0 and C_1 or between 0 and C_2 according as C_1 or C_2 is the smaller. In Case B we find that ζ lies continually between 0 and C_1 or between 0 and C_3 according as C_1 or C_3 is the smaller.

When again we represent the relations between ζ and φ as polar coordinates, we find curves of quite the same kind as in the case of a mechanism with two degrees of freedom for which $n_2 = 3n_1$.

So there are curves which do not enclose O and which therefore relate to forms of motion, where φ runs to and fro between two limits; and curves which do enclose O and which therefore relate to forms of motion, where φ takes all values. The transition is formed by a curve, having a double point on the axis of the angles; this points to a particular case, where the form of motion tends asymptotically to a movement where $\sin \varphi = 0$ and ζ is constant. To another special case the isolated point refers situated on the axis of the angles; it points to a form of motion, where $\sin \varphi = 0$ and where ζ is constant.

§ 21. *Osculating curves*. The osculating curves of the image point are again Lissajous twisted curves. The vertices of the circumscribed parallelopiped move along a twisted curve lying on an ellipsoid, whose axes lying along the coordinate axes have lengths proportional to $\frac{1}{n_1} : \frac{1}{n_2} : \frac{1}{2n_2 \pm n_1}$; the twisted curve projects itself in case A on the XY-plane as a hyperbola, on the YZ- and the ZY-plane as an ellipse, in case B on the XZ-plane as a hyperbola on the XY- and the YZ-plane as an ellipse.

The osculating curve described in a definite parallelopiped can be given in the equations

$$x = A\cos(n_1 t + a),$$
$$y = B\cos(n_2 t + b),$$
$$z = C\cos(2n_2 \pm n_1)\,t,$$

where a and b represent the momentary values resp. of $2n_1(\beta_1 - \beta_3)$ and of $2n_2(\beta_2 - \beta_3)$.

In the extreme parallelopiped curves are described for which $\sin\varphi = 0$, so for which

$$2b \pm a = l\pi,$$

where l is an integer.

For this case we have, if we suppose l to be even,

$$\cos^{-1}\frac{z}{C} = 2\cos^{-1}\frac{y}{B} \pm \cos^{-1}\frac{x}{A}.$$

Relations between 4 of the frequencies of vibration for which $S = 4$.

§ 22. There are two of these relations to be discussed, namely:

$$(A)\quad n_1 + n_2 + n_3 - n_4 = \varrho,$$
$$(B)\quad n_1 + n_2 - n_3 - n_4 = \varrho.$$

After the preceding it will be clear that we have to take.

$$U = \tfrac{1}{2}\sum_{r=1}^{r=4} n_r^2 q_r^2 + H_4(q_1, q_2, q_3, q_4),$$

$$T = \tfrac{1}{2}\sum_{r=1}^{r=4} \dot{q}_r^2 + \tfrac{1}{2}\sum_{r=1}^{r=4}\sum_{s=1}^{s=4}(P_{rr}\dot{q}_r^2 + 2P_{rs}\dot{q}_r\dot{q}_s),$$

where

$$P_{rs} = \tfrac{1}{2}a_{rs}q_1^2 + \tfrac{1}{2}b_{rs}q_2^2 + \tfrac{1}{2}c_{rs}q_3^2 + \tfrac{1}{2}d_{rs}q_4^2 + e_{rs}q_1 q_2 + f_{rs}q_1 q_3 + \\ + g_{rs}q_1 q_4 + h_{rs}q_2 q_3 + k_{rs}q_2 q_4 + l_{rs}q_3 q_4.$$

We can again point out which terms in the different equations of motion are to be regarded as disturbing, and we can reduce them according to the method indicated in § 17.

The result of the reduction is that the disturbing terms of the second kind in the equations for q_1, q_2, q_3, and q_4 are the derivatives resp. to q_1, q_2, q_3, and q_4 of:

$(-n_1 n_2 l_{12} \mp n_1 n_3 k_{13} + n_1 n_4 h_{14} \mp n_2 n_3 g_{23} + n_2 n_4 f_{24} \pm n_3 n_4 e_{34} - p)\, q_2 q_3 q_4$
(p being the coefficient of a term $q_1 q_2 q_3 q_4$ in H_1).

Of the $\mp$ and $\pm$ signs we must take the top one in the case of the relation (A) and the bottom one in the case of the relation (B).

We take as first approximation:

$$q_1 = \frac{\sqrt{\alpha_1}}{n_1} \cos (n_1 t + 2n_1 \beta_1),$$

$$q_2 = \frac{\sqrt{\alpha_2}}{n_2} \cos (n_2 t + 2n_2 \beta_2),$$

$$q_3 = \frac{\sqrt{\alpha_3}}{n_3} \cos (n_3 t + 2n_3 \beta_3),$$

$$q_4 = \frac{\sqrt{\alpha_4}}{n_1 + n_2 \pm n_3} \cos \{(n_1 + n_2 \pm n_3) t + 2(n_1 + n_2 \pm n_3)\beta_4\};\ \text{etc.}$$

We have then to take as function R:

$$-R = \chi_2 (\alpha_1, \alpha_2, \ldots \alpha_k) + \varrho' h^2 \alpha_4 + m_1 \sqrt{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \cos \varphi,$$

in which

$$\varphi = 2n_1 \beta_1 + 2n_2 \beta_2 \pm 2n_3 \beta_3 - 2(n_1 + n_2 \pm n_3)\beta_4 \equiv$$
$$\equiv 2n_1 (\beta_1 - \beta_4) + 2n_2 (\beta_2 - \beta_4) \pm 2n_3 (\beta_3 - \beta_4).$$

§ 23. We can write down the equations which show the variability of the α's and β's with time and we find, as always in the case $S = 4$, that the coordinates $q_5, q_6 \ldots q_k$ feel the influence of the relation in their phase, but not in their amplitude.

We therefore occupy ourselves particularly with q_1, q_2, q_3, and q_4. The equations for $\dot{\alpha}_1$, $\dot{\alpha}_2$, $\dot{\alpha}_3$, and $\dot{\alpha}_4$ run:

$$\dot{\alpha}_1 = + 2n_1 m_1 \sqrt{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \sin \varphi,$$
$$\dot{\alpha}_2 = + 2n_2 m_1 \sqrt{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \sin \varphi,$$
$$\dot{\alpha}_3 = \pm 2n_3 m_1 \sqrt{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \sin \varphi,$$
$$\dot{\alpha}_4 = - 2(n_1 + n_2 \pm n_3) m_1 \sqrt{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \sin \varphi.$$

We deduce from this:

$$\frac{\dot{\alpha}_1}{n_1} = \frac{\dot{\alpha}_2}{n_2} = \pm \frac{\dot{\alpha}_3}{n_3} = - \frac{\dot{\alpha}_4}{n_1 + n_2 \pm n_3},$$
$$\dot{\alpha}_1 + \dot{\alpha}_2 + \dot{\alpha}_3 + \dot{\alpha}_4 = 0.$$

§ 24. *Case A.*

$$\frac{\alpha_1}{n_1} + \frac{\alpha_4}{n_1 + n_2 + n_3} = C_1 h^2, \frac{\alpha_2}{n_2} + \frac{\alpha_4}{n_1 + n_2 + n_3} = C_2 h^2, \frac{\alpha_3}{n_3} + \frac{\alpha_4}{n_1 + n_2 + n_3} = C_3 h^2,$$

$$\frac{a_1}{n_1}-\frac{a_2}{n_2}=(C_1-C_2)h^2,\frac{a_1}{n_1}-\frac{a_3}{n_3}=(C_1-C_3)h^2,\frac{a_2}{n_2}-\frac{a_3}{n_3}=(C_2-C_3)h^2,$$

where C_1, C_2, C_3, represent positive constants.

If we put:

$$a_4=(n_1+n_2+n_3)\,\zeta h^2,$$

we find

$$a_1=n_1\,(C_1-\zeta)\,h^2,\ a_2=n_2\,(C_2-\zeta)\,h^2,\ a_3=n_3\,(C_3-\zeta)\,h^2.$$

The integral R = constant takes the form:

$$\sqrt{\zeta\,(C_1-\zeta)\,(C_2-\zeta)\,(C_3-\zeta)}\cos\varphi=p\zeta^2+q\zeta+r.$$

Case B.

$$\frac{a_1}{n_1}+\frac{a_4}{n_1+n_2-n_3}=C_1h^2,\frac{a_2}{n_2}+\frac{a_4}{n_1+n_2-n_3}=C_2h^2,\frac{a_3}{n_3}-\frac{a_4}{n_1+n_2-n_3}=C_3h^2,$$

$$\frac{a_1}{n_1}-\frac{a_2}{n_2}=(C_1-C_2)h^2,\frac{a_1}{n_1}+\frac{a_3}{n_3}=(C_1+C_3)h^2,\frac{a_2}{n_2}+\frac{a_3}{n_3}=(C_2+C_3)h^2,$$

where C_1, C_2, and C_3 are positive constants.

If we put:

$$a_4=(n_1+n_2-n_3)\,\zeta h^2,$$

we find

$$a_1=n_1\,(C_1-\zeta)h^2,\ a_2=n_2\,(C_2-\zeta)h^2,\ a_3=n_3\,(C_3+\zeta)h^2.$$

The integral R = *constant* takes the form:

$$\sqrt{\zeta\,(C_1-\zeta)\,(C_2-\zeta)\,(C_3+\zeta)}\cos\varphi=p\zeta^2+q\zeta+r.$$

It is clear that the problem is again reduced to quadratures and that the coordinates with the help of elliptic functions can be expressed in the time.

§ 25. The radius vector ζ varies periodically between two limits, lying in case A between zero and the smaller of the three quantities C_1, C_2, and C_3, in case B between zero and the smaller of C_1 and C_2.

The curves representing the relation between ζ and φ have here again the same form as for the case of the relation $n_2 = 3n_1$.

Thus as general forms of motion we have those where φ takes all values and those where φ moves backward and forward between two opposite values; the first we have by preference for great values of the residue of relation.

Furthermore there is again a special case where the amplitudes are constant, and $\sin\varphi$ remains 0; and another special case where such a form of movement is asymptotically approached.

H. J. E. BETH. "The oscillations about a position of equilibrium where a simple linear relation exists between the frequencies of the vibrations." (Third part).

Fig. 1.

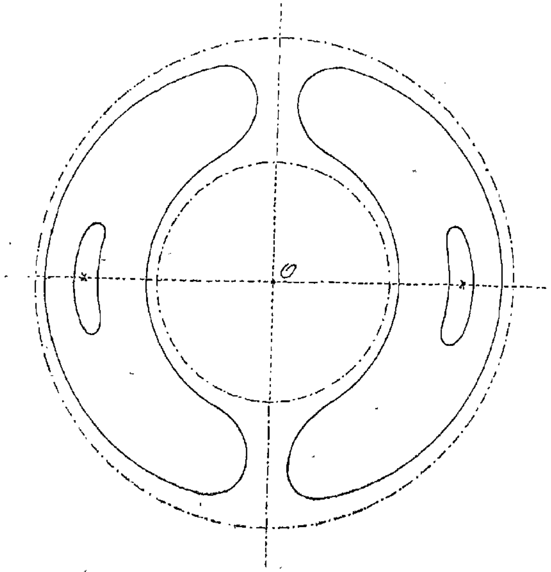


Fig. 2.

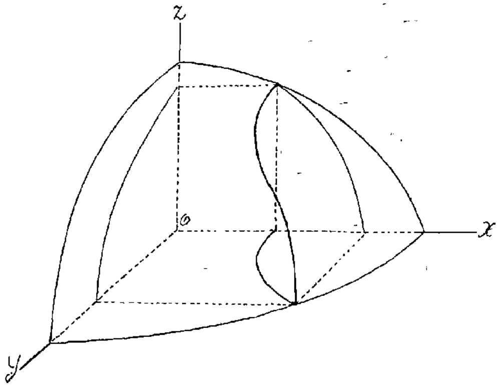


Fig. 3

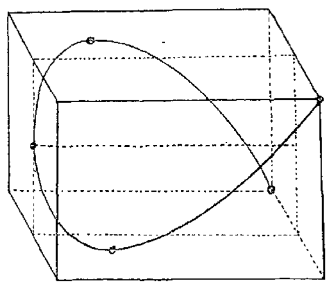


Fig. 4

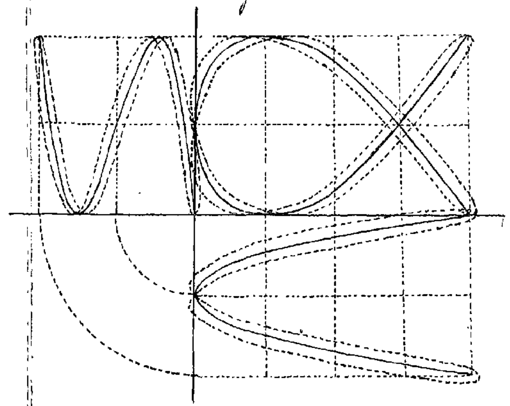


Fig. 5.

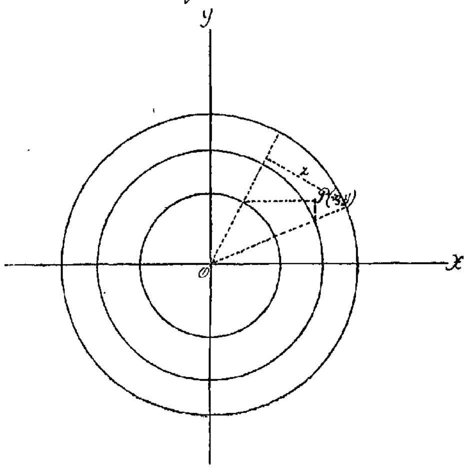


Fig. 6.

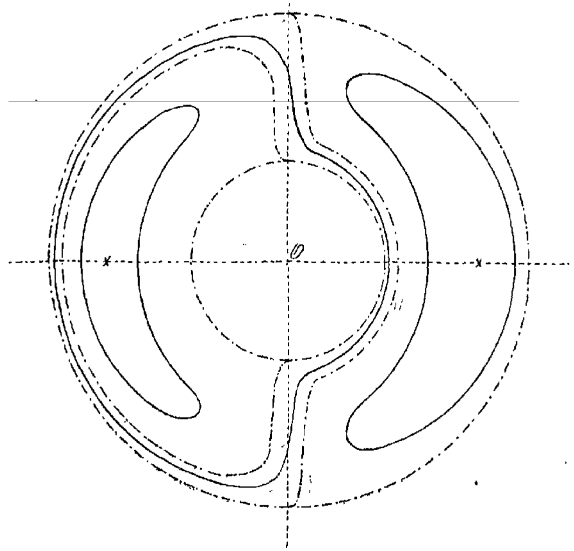
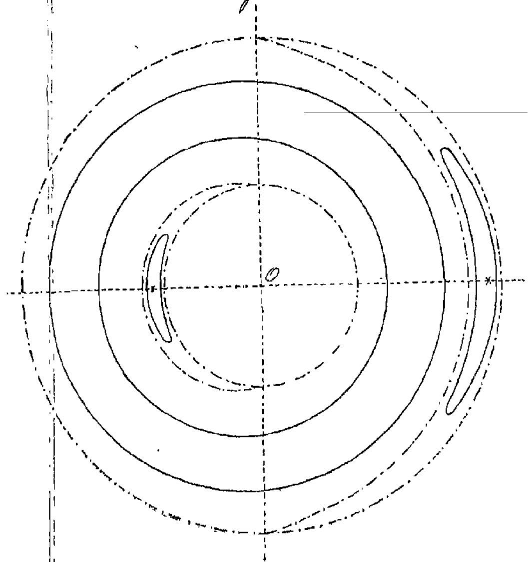


Fig. 7.



§ 26 *Osculating curves.* The image point which is to represent the movement of the mechanism with 4 degrees of freedom, moves in a space R_4. The coordinates x, y, z, and u of the image point on a rectangular system of coordinates are at every moment equal to q_1, q_2, q_3, q_4. Its movement is then determined by:

$$x = \frac{\sqrt{\alpha_1}}{n_1} \cos (n_1 t + 2n_1 \beta_1),$$

$$y = \frac{\sqrt{\alpha_2}}{n_2} \cos (n_2 t + 2n_2 \beta_2),$$

$$z = \frac{\sqrt{\alpha_3}}{n_3} \cos (n_3 t + 2n_3 \beta_3),$$

$$u = \frac{\sqrt{\alpha_4}}{n_1 + n_2 \pm n_3} \cos \{(n_1 + n_2 \pm n_3)\, t + 2\,(n_1 + n_2 \pm n_3)\, \beta_4\}.$$

If we ascribe to the α's and β's their momentary values, then these equations represent the osculating curves for the indicated moment. The osculating curve we can call a LISSAJOUS curve.

The curve remains enclosed inside a fourdimensional parallelotope bounded by the spaces:

$$x = \pm \frac{\sqrt{\alpha_1}}{n_1},\ y = \pm \frac{\sqrt{\alpha_2}}{n_2},\ z = \pm \frac{\sqrt{\alpha_3}}{n_3},\ u = \pm \frac{\sqrt{\alpha_4}}{n_1 + n_2 \pm n_3}.$$

By the variability of the α's the circumscribed parallelotope also changes; the vertices move backward and forward between two extreme positions along a wrung curve; this curve lies on a hyperellipsoid, whose axes lying along the axes of coordinates are proportional to $\frac{1}{n_1} : \frac{1}{n_2} : \frac{1}{n_3} : \frac{1}{n_1 + n_2 \pm n_3}$.

The form of the wrung LISSAJOUS curve in a definite parallelotope depends, as is found by a change of the origin of time, on the quantities

$$2n_1 (\beta_1 - \beta_4),\quad 2n_2 (\beta_2 - \beta_4),\quad 2n_3 (\beta_3 - \beta_4).$$

The osculating curves described in the extreme parallelotopes have the property that

$$2n_1 (\beta_1 - \beta_4) + 2n_2 (\beta_2 - \beta_4) \pm 2n_3 (\beta_3 - \beta_4) = l\pi.$$

For these curves the relation holds:

$$\cos^{-1} \frac{x}{A} + \cos^{-1} \frac{y}{B} \pm \cos^{-1} \frac{z}{C} = \cos^{-1} \frac{u}{D},$$

when A, B, C, and D are written for the amplitudes and l is supposed even.