Citation:

J.C. Kluyver, On the integrating of series term-by-term, in:
KNAW, Proceedings, 13 II, 1910-1911, Amsterdam, 1911, pp. 850-857

culture-drop can have no harmful consequences; after the above mentioned investigations of CLARK and DUGGAR caution is however advisable.

Taking everything into consideration among the changes which NIEUWENHUIS proposes in my method, there is only one with which (and then only partly) I am in agreement and which is also incorporated in my simplified apparatus (of which I had wished, for certain reasons, to postpone the publication): the two needle holders are replaced by one. When however he advises always the use of not more than one needle in isolating, we differ entirely. If he had carried out more extensive investigations with difficult material, for instance, with bacteria, instead of with the much larger fungus spores, he would not have suggested any alteration.

I should be glad if the fact that the needles can now be obtained commercially, should lead to a more extensive use of my method. I am convinced that it has a large sphere of usefulness. Those who from the detailed description in my last publication conclude that the method is too difficult, are mistaken : I have thought it preferable to go into a minute description in order to make it as easy as possible for those who use it, however tempting it was to give the impression of a too great simplicity by a cursory description.

For the rest, those who hesitate to isolate bacteria, can work with yeast or moulds.

The technique of the method will improve by extensive use, for I cannot imagine it to be perfect.

With regard to the alterations, proposed by NIEUWENHUIS, I felt obliged to say at once, that in my opinion, they are not improvements.

Finally one hint more: I advise those who can afford it to procure the apparatus with a moveable stand, on which the microscope is placed; although it seems to be more complicated it is the more convenient in use.

Mathematics. — *"On the integrating of series term-by-term".* Communicated by Mr. J. C. KLUYVER.

(Communicated in the meeting of January 28, 1911).

When the function $F(x)$ is developed in a series converging uniformly for $a \leq x \leq b$, we can integrate term-by-term, i.e., out of the equation

$$F(x) = \sum_{0}^{\infty} u_n(x)$$

follows

$$\int_a^b F(x)\,dx = \sum_0^\infty \int_a^b u_n(x)\,dx.$$

We may not conclude to this last equation as soon as one of the two limits of the domain of integration is just excluded from the interval of the uniform convergence.

Wellknown instances have shown, that in that case the rule of the integration term-by-term may hold or not.

Instances. The series $\sum_0^\infty u_n(x)$ converges uniformly for $0 \leqq x < 1$.

I. $\sum_0^\infty u_n(x)$ converges for $x = 1$.

$u_n(x) = (n+1)(1-x)\,x^n - (n+2)(1-x)\,x^{n+1}$.

$$F(x) = (1-x) \quad , \quad \int_0^1 F(x)\,dx = \frac{1}{2} \quad , \quad \sum_0^\infty \int_0^1 u_n(x)\,dx = \frac{1}{2}.$$

$u_n(x) = n(1-x)\,e^{-n(1-x)^2} - (n+1)(1-x)\,e^{-(n+1)(1-x)^2}$.

$$F(x) = 0 \quad , \quad \int_0^1 F(x)\,dx = 0 \quad , \quad \sum_0^\infty \int_0^1 u_n(x)\,dx = -\frac{1}{2}.$$

$$u_n(x) = \frac{(1-x)\,n^{\frac{3}{2}}}{1+(1-x)^4 n^2} - \frac{(1-x)(n+1)^{\frac{3}{2}}}{1+(1-x)^4 (n+1)^2}.$$

$$F(x) = 0, \quad \int_0^1 F(x)\,dx = 0 \quad , \quad \sum_0^\infty \int_0^1 u^n(x)\,dx \text{ diverges.}$$

II. $\sum_0^\infty u_n(x)$ diverges for $x = 1$.

$u_n(x) = (-1)^n x^n$.

$$F(x) = \frac{1}{1+x} \quad , \quad \int_0^1 F(x)\,dx = \log 2 \quad , \quad \sum_0^\infty \int_0^1 u_n(x)\,dx = \log 2.$$

$u_n(x) = (n+1)\,x^n - (n+2)\,x^{n+1}$.

$$F(x) = 1 \quad , \quad \int_0^1 F(x)\,dx = 1 \quad , \quad \sum_0^\infty \int_0^1 u_n(x)\,dx = 0.$$

$u_n(x) = (-1)^n (n+1)\,x^n$.

$$F(x) = \frac{1}{(1+x)^2} \quad , \quad \int_0^1 F(x)\,dx = \frac{1}{2} \quad , \quad \sum_0^\infty \int_0^1 u_n(x)\,dx \text{ diverges.}$$

When the upper limit b of the range of integration belongs no more to the interval of the uniform convergence, we can prove still pretty simply in some suppositions, that the integrating term-by-term gives a correct result.

We are sure of this, when the series $\sum_0^\infty u_n(x)$ converges uniformly for $a \leqq x < b$, when the series of integrals converges absolutely and when moreover each term $u_n(x)$ has a constant sign in the whole domain of integration.

For, now we find in the first place for $a \leqq t < b$

$$\int_a^t F(x)\,dx = \sum_0^\infty \int_a^t u_n(x)\,dx.$$

Of this last series the absolute value of each term is smaller than the corresponding term of the convergent series

$$\sum_0^\infty \left| \int_a^b u_n(x)\,dx \right| ,$$

from which ensues that with respect to t the series

$$\sum_0^\infty \int_a^t u_n(x)\,dx$$

converges uniformly in the domain $a \leqq t \leqq b$.

The principal property of the uniformly convergent series furnishes then immediately

$$\int_a^b F(x)\,dx = \underset{t=b}{Lim} \sum_0^\infty \int_a^t u_n(x)\,dx = \sum_0^\infty \int_a^b u_n(x)\,dx.$$

Very often this theorem proves sufficient. Thus we find that for $0 \leqq x < 1$ the equation holds:

$$F(x) = \frac{\left(log\,\frac{1}{x}\right)^{s-1}}{1+x} = \sum_0^\infty (-1)^n \left(log\,\frac{1}{x}\right)^{s-1} x^n,$$

and the development at the right-hand side is in this interval uniformly convergent.

The series of the integrals

$$\Gamma(s) \sum_1^\infty \frac{(-1)^{n-1}}{n^s}$$

converges absolutely, if only $s > 1$ and under this condition therefore the equation

$$\int_0^1 \frac{\left(\log \frac{1}{x}\right)^{s-1}}{1+x}\, dx = \int_0^\infty \frac{y^{s-1}}{e^y+1}\, dy = \Gamma(s) \sum_1^\infty \frac{(-1)^{n-1}}{n^s}$$

will hold.

However the theorem under discussion does not serve to show that the above equation remains correct for $0 < s \leqq 1$. Here as well as in other cases this theorem needs amplifying and as such the following theorem can sometimes serve.

When $F(x)$ is developed in a series of continuous functions we shall be able to deduce

$$\int_a^b F(x)\, dx = \sum_0^\infty \int_a^b u_n(x)\, dx,$$

out of

$$F(x) = \sum_0^\infty u_n(x)$$

as soon as is given:

1st. $\sum_0^\infty u_n(x)$ is convergent for $a \leqq x < b$.

2nd. $\sum_0^\infty \int_a^b u_n(x)\, dx$ converges.

3rd. The function $u_n(x)$ does not change its sign in the interval $a \leqq x \leqq b$.

4th. $\left|\frac{u_{n+1}(x)}{u_n(x)}\right|$ is monotonic with respect to x and that for all values of the index n in the same sense.

In order to prove this theorem we must show in the first place, that the series to be integrated converges uniformly for $a \leqq x < b$. In the main this follows out of the fourth datum, which states that for all values of n the inequality

$$\left| \frac{u_{n+1}(x)}{u_n(x)} \right| \leqq \left| \frac{u_{n+1}(y)}{u_n(y)} \right|$$

will exist when $x < y$, or that for all values of n that inequality will hold when $x > y$.

I first suppose that the inequality holds for $x < y$. On the ground of the third datum we find

$$\alpha_n = \frac{u_n(x)}{u_n(y)} \text{ and } \alpha_{n+1} = \frac{u_{n+1}(x)}{u_{n+1}(y)}$$

to be positive numbers and the inequality expresses that $\alpha_{n+1} \leqq \alpha_n$.

In the sequence

$$\alpha_p, \quad \alpha_{p+1}, \quad \alpha_{p+2}, \ldots$$

the numbers are therefore not ascending and as according to the first datum the series $\sum_0^\infty u_n(y)$ converges, we conclude from the well-known lemma of ABEL, that

$$\sum_p^\infty u_n(y)\alpha_n = \sum_p^\infty u_n(x)$$

is situated between $G\alpha_p$, and $K\alpha_p$, where G and K denote successively the upper limit and the lower one of the sums:

$$u_p(y), \quad u_p(y) + u_{p+1}(y), \quad \ldots, \; u_p(y) + u_{p+1}(y) + u_{p+2}(y) + \ldots$$

If we take p large enough $|G|$ and $|K|$ remain below an arbitrary small quantity ε, so that we have for p large enough:

$$\left|\sum_p^\infty u_n(x)\right| < \varepsilon\alpha_p .$$

In the supposition under discussion here concerning the fourth datum follows out of the convergence of $\sum_0^\infty u_n(y)$ the uniform convergence of the series $\sum_0^\infty u_n(x)$ for all x, satisfying $a \leqq x < y$, and as we can make y tend to b, the uniform convergence for $a \leqq x < b$ has been proved.

In the same way we might have concluded out of

$$\left|\frac{u_{n+1}(x)}{u_n(x)}\right| \leqq \left|\frac{u_{n+1}(y)}{u_n(y)}\right|$$

$$(x > y)$$

the uniform convergence of $\sum_0^\infty u_n(x)$ in the range $y < x \leqq b$, and as this series converges for $x = a$, we should have uniform convergence in the whole interval of integration $a \leqq x \leqq b$, from which would immediately follow what is to be proved. So we have only to investigate further the supposition

$$\left|\frac{u_{n+1}(x)}{u_n(x)}\right| \leqq \left|\frac{u_{n+1}(y)}{u_n(y)}\right|,$$

$$(x < y)$$

where the series converges uniformly in the domain $a \leqq x < b$ and where divergence for $x = b$ remains possible.

When we put

$$\beta_n = \frac{\int_a^t u_n(x)dx}{\int_a^b u_n(y)dy},$$

then this number, because $u_n(x)$ has a definite sign in the range of integration $u_n(x)$, is positive and smaller than unity.

The numbers

$$\beta_p, \quad \beta_{p+1}, \quad \beta_{p+2}, \ldots$$

are not ascending, for we have

$$\beta_n - \beta_{n+1} = \frac{\int_a^t |u_n(x)|dx}{\int_a^b |u\ (y)|dy} - \frac{\int_a^t |u_{n+1}(x)|dx}{\int_a^b |u_{n+1}(y)|dy},$$

or

$$\beta_n - \beta_{n+1} = \frac{\int_a^t dx \int_t^b dy\ |u_n(x)|\ |u_n(v)| \left\{ \left|\frac{u_{n+1}(y)}{u_n(y)}\right| - \left|\frac{u_{n+1}(x)}{u_n(x)}\right| \right\}}{\int_a^b |u_n(y)|dy \ . \int_a^b |u_{n+1}(y)|dy},$$

and as in the double integral y always remains larger than x, the double integral can in connection with the supposition just made concerning the fourth datum never be negative.

Again we can apply the lemma of ABEL, from which ensues that

$$\sum_p^\infty \beta_n \int_a^b u_n(y)dy = \sum_p^\infty \int_a^t u_n(x)dx$$

is lying between $\beta_p G'$ and $\beta_p K'$, where G' and K' denote successively the upper and lower limit of the sums

$$\int_a^b u_p(y)dy, \quad \int_a^b u_p(y)dy + \int_a^b u_{p+1}(y)dy, \ \ldots\ldots, \quad .$$

$$\int_a^b u_p(y)dy + \int_a^b u_{p+1}(y)dy + \int_a^b u_{p+2}(y)dy + \ldots .$$

By taking p large enough we can make $|G'|$ and $|K'|$ according to the second datum smaller than an arbitrary small quantity ε and for p large enough we have thus at the same time

$$\left| \sum_p^\infty \int_a^b u^n(y)dy \right| < \varepsilon,$$

$$\left| \sum_p^\infty \int_a^t u_n(x)dx \right| < \beta_p \, \varepsilon < \varepsilon.$$

By these inequalities is expressed that the series

$$\sum_0^\infty \int_a^t u_n(x)dx$$

converges with respect to t uniformly in the domain $a \leqq t \leqq b$. We have already proved that in the domain $a \leqq x < b$ the series $\sum_0^\infty u_n(x)$ converges uniformly, so that the equation

$$\int_a^t F(x)dx = \sum_0^\infty \int_a^t u_n(x)dx$$

certainly holds, and when we then finally apply the principal property of the uniformly convergent series we find when t tends to b

$$\int_a^b F(x)dx = \underset{t=b}{Lim} \sum_0^\infty \int_a^t u_n(x)dx = \sum_0^\infty \int_a^b u_n(x)dx.$$

With this we have given the proof of the enunciated theorem and it is clear that the proof holds if the third and fourth data only hold for all numbers n surpassing a definite number.

For the evaluation of the integral

$$\int_0^\infty \frac{y^{s-1}}{e^y+1} = \int_0^1 \frac{\left(log \frac{1}{x}\right)^{s-1}}{1+x} dx$$

(where $s > 0$) the theorem can be applied.

We have

$$\frac{\left(log \frac{1}{x}\right)^{s-1}}{1+x} - \left(log \frac{1}{x}\right)^{s-1} = \sum_1^\infty (-1)^n \left(log \frac{1}{x}\right)^{s-1} x^n,$$

and the series converges for $0 < x < 1$. The terms do not change signs for $0 < x < 1$, the quotient

$$\left|\frac{u_{n+1}(x)}{u_n(x)}\right| = x$$

increases with x. The series of the integrals

$$\Gamma(s) \sum_2^\infty \frac{(-1)^{n-1}}{n^s}$$

converges for $s > 0$. The theorem therefore holds and we find for all positive values of s

$$\int_0^\infty \frac{y^{s-1}}{e^y+1}\, dy = \Gamma(s) \sum_1^\infty \frac{(-1)^{n-1}}{n^s}.$$

In general we shall often be able to use this theorem when evaluating an integral of the form

$$\int_0^1 f(x)\, g(x)\, dx.$$

Suppose it possible to replace $f(x)$ by a power series $\sum_0^\infty a_n x^n$ such that the series $\sum_0^\infty a_n x^n g(x)$ diverges for $x = 1$, but that the series of the integrals

$$\sum_0^\infty a_n \int_0^1 x^n g(x)\, dx$$

is still convergent. Then this series will certainly be equal to the integral, if only $g(x)$ does not change its sign in the domain of integration, because then all conditions under which the theorem holds are satisfied.

We shall likewise, if the development

$$f(x) = \sum_0^\infty a_n x^n$$

holds for all finite values of x, be allowed to conclude by means of the theorem to the equation

$$\int_0^\infty f(x) e^{-x}\, dx = \sum_0^\infty n!\, a_n,$$

if this last development converges.