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Physics. — “*Selective absorption and anomalous scattering of light in extensive masses of gas.*” By Prof. W. H. JULIUS.

(Communicated in the meeting of January 28, 1911).

§ 1. *Resonance and damping.* — The dispersion of light is at present generally explained by supposing the molecules of the medium to contain electrons, bound to positions of equilibrium by quasi-elastic forces. Such electrons have their own period of vibration T ; their motion will continually increase by resonance, if in the incident beam of light there are waves of the same period. As light, somewhere in the visible spectrum, may give impulses at the rate of 500 billion a second to a resonant electron, one conceives that in a very short time the amplitude of the electron will considerably increase, even though the impulses may be weak.

Waves of a slightly different period T' are also able to set the same electrons moving; they force the period T' upon them; but the resulting amplitude of the forced vibration is the smaller, the greater the value of $\pm(T - T')$. The absorption spectra of gases will appear, however, to be only slightly affected by this phenomenon. We may refer to a general result obtained by HELMHOLTZ¹⁾ as to how the intensity of the forced vibrations of a system is determined by the rate of extinction of the free vibrations, and the difference of the periods, $T - T'$. If the extinction is slow, sensible amplitudes are not reached unless T' differs *very* little from T . Now we know from the phenomena of interference with great differences of path, that the free vibrations of electrons are so little subject to damping, as to have more than $\frac{1}{10}$ of their energy left after 300.000 vibrations have been accomplished. Applying HELMHOLTZ's formula to the case, we find, that the energy of the forced vibrations of period T' will be smaller than $\frac{1}{10}$ of the energy of the exactly resonant vibrations of period T , if the wave-lengths corresponding to those two periods differ as little as 0,01 Å. So the width of the region of sensible co-vibrating scarcely exceeds some 0,02 Å.

The amplitude of a resonant electron cannot, of course, go on increasing infinitely, and no doubt the quasi-elastic force will cease from being proportional to the displacement when the latter has become very great. As soon as the amplitude surpasses a certain value, the electron will be strained out of the molecule and set to moving independently: then radiation has led to ionisation. But

¹⁾ HELMHOLTZ, Die Lehre von den Tonempfindungen (1877), S. 233 und 642.

absorption of light is not always accompanied by ionisation; in most cases the above-said critical value evidently is not exceeded, which means that some cause must exist by which the increase of the amplitude is limited. Such a cause is formally accounted for by introducing a "resistance", which opposes the motion of the electron and is supposed to be proportional to its velocity. The equations of motion of an electron, moving under the influence of a (periodically changing) electric force (XYZ), therefore have the form

$$m \frac{d^2 x}{dt^2} + h \frac{dx}{dt} + kx = eX, \text{ etc.} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where m means the mass, e the charge of the electron, while the magnitude of the quasi-elastic force is determined by k , that of the resistance by h .

We propose to study in this paper the nature of the damping parameter h , and to inquire into the influence which the damping forces exert on the intensity of light propagated through very extensive gaseous media, like the atmospheres of the sun and the stars.

§ 2. *Recalling some results of the dispersion theory.* — We know that the dispersion theory answers the question, how a given periodically changing electric force is propagated through a medium containing a large number of electrons, the motion of which is represented by a set of equations of the above form ¹⁾.

Let the medium contain \mathfrak{N} similar molecules per unit of volume, each of them furnished with a few differently connected electrons, so that there are a limited number of periods of free vibrations; then we only have to apply the general equations of the electromagnetic field to this charged medium, and, using the notation introduced by W. VOIGT ²⁾, which is well adapted to our purpose, we find the following solution:

$$n^2 = n^2(1 - i\kappa)^2 = 1 + \sum \frac{4\pi\mathfrak{N} \frac{e^2}{m}}{\frac{k}{m} + i \frac{h}{m} \nu - \nu^2} = 1 + \sum \frac{\rho}{\nu_0^2 + i\nu'\nu - \nu^2} \quad . \quad (2)$$

In this equation n represents the complex index of refraction,

¹⁾ The equations of motion of the electron with which LORENTZ starts (The Theory of Electrons, p. 139) contain two additional terms, and, therefore, are more general than the set (1). In the problem of which we are going to treat, we may omit those terms, because 1st we need not account for an exterior magnetic field, and 2nd we only wish to apply our results to media of low density.

²⁾ W. VOIGT, Magneto- und Electro-optik, (1908) p. 107.

n the real index of refraction, z the index of extinction, while ν means the frequency (i.e. the number of vibrations in the time 2π) of the kind of light we are considering. The meaning of q , ν_0 and ν' is clear from the formula. That $\sqrt{\frac{h}{m}} = \nu_0$ is the frequency of the "free vibration" of the electron becomes apparent, if in equation (1) we put $h = 0$ and $X = 0$, which means that no damping and no external electric force are supposed to exist. Each kind of electrons contributes a term in the sum.

Separating the real from the imaginary terms in (2) leads to the relations

$$\left. \begin{aligned} n^2(1 - z^2) &= 1 + \sum \frac{q(\nu_0^2 - \nu^2)}{(\nu_0^2 - \nu^2)^2 + \nu'^2\nu^2} \\ 2n^2z &= \sum \frac{q\nu\nu'}{(\nu_0^2 - \nu^2)^2 + \nu'^2\nu^2} \end{aligned} \right\} \dots \dots (3)$$

from which n and z have to be solved. This problem becomes much simpler, and yet scarcely less general, if we confine our attention to the surroundings of each of the characteristic frequencies ν_0 separately. In equation (2) we therefore put apart the term relating to the selected ν_0 , and designate the other terms of the sum by a variable index h :

$$n^2 = 1 + \sum \frac{q_h}{\nu_h^2 + i\nu'_h\nu - \nu^2} + \frac{q}{\nu_0^2 + i\nu'\nu - \nu^2} \dots \dots (4)$$

Because we only consider such values of ν as differ little from ν_0 , we are allowed to replace ν by ν_0 in the terms of the summation Σ , and then to neglect $i\nu'_h\nu_0$ relatively to $\nu_h^2 - \nu_0^2$ (the damping connected with ν_h being imperceptible near ν_0). Writing $\nu - \nu_0 = \mu$, and therefore (μ being small) $\nu^2 - \nu_0^2 = 2\nu_0\mu$, we may put, instead of (4):

$$n^2 = 1 + \sum \frac{q_h}{\nu_h^2 - \nu_0^2} - \frac{q}{\nu_0(2\mu - i\nu')}$$

or

$$n^2 = n_0^2 - \frac{q}{\nu_0(2\mu - i\nu')}, \dots \dots (5)$$

where n_0 represents the value which, in the small spectral region we are considering, the index of refraction would show if there were no electrons having the proper frequency ν_0 .

One might proceed to the separation of the real from the imaginary part of this simplified equation, and then solve n and z ; but as the result would not yet be very simple, we shall first consider the special case that the modulus of the complex second term is small

compared with n_0^2 , for all values of μ lying within the region we are concerned with. Then $n^2 - n_0^2$ may be replaced by $2n_0(n - n_0)$, and separating the real from the imaginary part, we find:

$$n = n_0 - \frac{Q\mu}{n_0 v_0 (4\mu^2 + v'^2)} \dots \dots \dots (6)$$

$$n\kappa = \frac{Qv'}{2n_0 v_0 (4\mu^2 + v'^2)} \dots \dots \dots (6a)$$

which formulae easily show the symmetry of the curves representing n and $n\kappa$ as functions of μ .

As to n , it is clear indeed that $n = n_0$ for $\mu = 0$, and that for some positive value of μ (i.e. on the violet side of v_0) n is smaller than n_0 by a certain amount, while for an equal negative value of μ (on the red side) n is larger than n_0 by the same amount. For $\mu = -\frac{1}{2}v'$, n reaches a maximum:

$$n = n_0 + \frac{Q}{4 n_0 v_0 v'}$$

and for $\mu = +\frac{1}{2}v'$ a minimum:

$$n = n_0 - \frac{Q}{4 n_0 v_0 v'}$$

The attenuation coefficient $n\kappa$ has its greatest value $\frac{Q}{2 n_0 v_0 v'}$ at the point $\mu = 0$; passes the value $\frac{Q}{4 n_0 v_0 v'}$, which is half that of the maximum, at $\mu = \pm \frac{1}{2}v'$, i.e. exactly there where the maximum

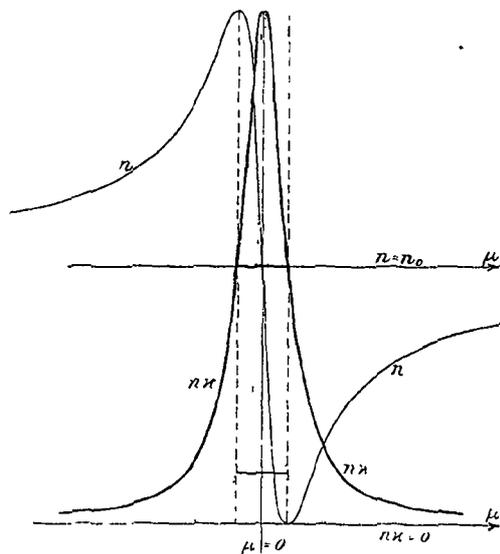


Fig. 1.

and the minimum of n are found; and, with increasing μ , approaches zero on both sides. Fig. 1 (taken from VOIGT l.c. p. 115) shows the curves representing the two functions. We may consider v' as measuring the width of the dark line appearing in the spectrum. Indeed, the slope of the intensity curve is much steeper than the slope of nz , because the strength of the transmitted light (provided we are not dealing with the radiation through a thick atmosphere,

cf. § 5) is given by $I = I_0 e^{-\frac{4\pi}{\lambda} n' z}$, z being the length of the path through the gas. The dotted curve represents the reciprocal value of the intensity of the transmitted light on the supposition that for $\mu = 0$ we have $I = \frac{1}{100} I_0$, so that at $\mu = \pm \frac{1}{2} v'$, where nz has half its maximum value, we find $I = \frac{1}{10} I_0$. Almost the whole of the dark line thus lies between $\mu = -\frac{1}{2} v'$ and $\mu = +\frac{1}{2} v'$).

§ 3. *Cases in which the anomaly of the dispersion curve is greater.* Our object being to apply the results of the theory to the interpretation of the solar spectrum, we must allow for the possibility that perhaps not in all cases the modulus of $\frac{e}{v_0(2\mu - i v')}$ may be taken to be small as compared with n_0^2 (e.g. when we are concerned with very strong lines, like the calcium lines *H* and *K*), we therefore return to equation (5). Separating the real from the imaginary part, we obtain

$$n^2 - n^2 z^2 - n_0^2 = \frac{-2Q\mu}{v_0(4\mu^2 + v'^2)} \quad \text{and} \quad n^2 z = \frac{\frac{1}{2} Q v'}{v_0(4\mu^2 + v'^2)}.$$

The substitution of the second equation into the first one leads to

$$n^2 - n_0^2 = \frac{-2Q\mu + \frac{1}{2} Q v' z}{v_0(4\mu^2 + v'^2)} \quad \text{or} \quad n - n_0 = \frac{-Q\mu + \frac{1}{4} Q v' z}{\frac{1}{2}(n + n_0)v_0(4\mu^2 + v'^2)} \quad (7)$$

from which we deduce

$$n = n_0 + \frac{\frac{1}{4} Q v' z}{\frac{1}{2}(n + n_0)v_0(4\mu^2 + v'^2)} - \frac{Q\mu}{\frac{1}{2}(n + n_0)v_0(4\mu^2 + v'^2)}. \quad (8)$$

A similar position as, according to (6), the curve n takes with respect to the straight line n_0 , it assumes according to (8) with respect to the curve $n_0 + \sigma$ (if σ represents the second term, which is variable with n and z). By that term σ the character of the curve n is, however, scarcely influenced, because $\frac{1}{4} v' z$ is small in com-

¹⁾ We shall see later on that in the light which has traversed the *solar atmosphere*, the apparent width of the real absorption lines must be even less, because part of the attenuation depends on scattering, and this part follows a law different from the exponential one.

parison with μ . Even within the region of strong absorption, the term σ is of little consequence. Let us for instance consider the frequency where $\mu = \frac{1}{2} v'$. There the ratio of the second to the third term is

$$\frac{1}{4} v' z : \mu = \frac{1}{4} v' z : \frac{1}{2} v' = \frac{1}{2} z.$$

There are no experimental data at my disposal from which the values of z might be deduced for calcium vapour; but for sodium vapour the maximum value of nz was found to be 10^{-3} ¹⁾, which, at the indicated spot of the spectrum, makes $nz = 5 \times 10^{-4}$, and, consequently, $\frac{1}{2} z = \frac{1}{n} \times 2,5 \times 10^{-4}$. For greater values of $\pm \mu$ the said ratio decreases rapidly. So the second term of the second member of (8) may be neglected.

But in the denominator of the third term we meet with the variable factor $\frac{1}{2}(n + n_0)$, where in (6) the constant factor n_0 occurs. It follows from this circumstance that now the dispersion curve does not show the perfect symmetry of the one which represents equation (6). The character of the deviation becomes apparent from equation (7) if, omitting the term $\frac{1}{2} \nu v' z$, we write:

$$n^2 = n_0^2 - \frac{2q\mu}{v_0(4\mu^2 + v'^2)}.$$

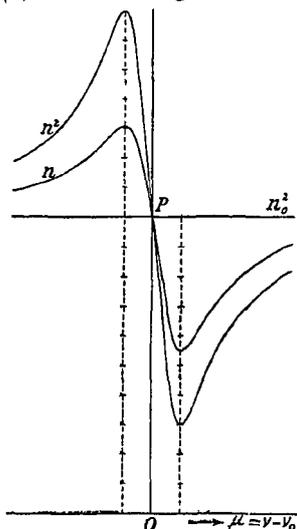


Fig. 2.

With respect to the point of intersection P of the horizontal line n_0^2 (fig. 2) with the vertical line $\mu = 0$ the curve representing n^2 is symmetrical. Let us now suppose $n_0 = 1$, then the line n_0 coincides with the line n_0^2 . Constructing, in the same figure, the curve whose ordinates are the square roots out of the ordinates of the first curve, we immediately see, that the "anomaly" of the index of refraction n is greater on the violet than on the red side of the line. It is questionable whether absorbing vapours will present cases in which this difference is great enough to show itself in the observations.

§ 4. *On the nature of the damping parameter.* — A permanently acting cause why the vibrations of an electron die out, is the fact that it radiates waves in all directions, thus "scattering" its kinetic energy. An electron, moving with the variable velocity v , experiences a force due to its own field, opposite to its acceleration, and, in

¹⁾ W. Voigt, l. c. p. 142.

first approximation, proportional to $\frac{dv}{dt}$; this accounts for the inertia.

But the force also contains a term, proportional to $\frac{d^2v}{dt^2}$, viz: ¹⁾

$$r = \frac{2}{3} \frac{e^2}{c^3} \cdot \frac{d^2v}{dt^2}.$$

If now the motion is periodical: $v = b \cos \nu t$, we shall have $\frac{d^2v}{dt^2} = -\nu^2 v$, so that we may put

$$r = -\frac{2}{3} \frac{\nu^2 e^2}{c^3} \cdot \frac{dv}{dt}$$

where x means the elongation at the time t .

This term of the force may therefore be considered to express a "resistance", being proportional to the velocity and having the opposite direction. The numerical value of the coefficient is small, and that the resulting attenuation of the vibrations really is insignificant, appears from the known phenomena of interference with great differences of path, which show that after some 100000 vibrations the amplitude of an electron has scarcely diminished. On the basis of this cause of damping we are not able to account for the *absorption* of the incident light, i.e. for a transformation of the radiant energy into heat or other forms of energy. The scattered light remains radiant energy of the vibration periods occurring in the original beam of light.

In order to explain absorption, LORENTZ assumes, that the vibrations of an electron excited by incident waves of light, go on undisturbed only during a certain interval of time τ , and that then, for instance in consequence of the collisions of the molecules, their energy is transformed and distributed among other systems. ²⁾ This idea may be expressed mathematically by giving the damping parameter h the value $\frac{2m}{\tau}$. It is not necessary, however, to identify τ with the mean length of time elapsing between two successive collisions of a molecule; indeed, after a much shorter interval τ the amplitude of a resonant (or almost resonant) electron might already have increased to such a value, that also the other components of the molecule to which it belongs have been thoroughly shaken, and have assumed

¹⁾ LORENTZ, The theory of electrons, p. 49; Encyklopädie der math. Wiss. V. 2, 188; ABRAHAM, Theorie der Electricität II, S. 72, 123. In the above formula e is expressed in the C. G. S. unit reposing on COULOMB'S law.

²⁾ LORENTZ, The theory of electrons, p. 141.

part of its energy. In that case we are already witnessing an "absorption" process; the further transformation of the energy into heat, etc. ensues by collisions.

According to this conception, *absorption* of radiant energy only takes place, when by resonance certain electrons are set vibrating to such a degree, that energy is imparted by them in an irreversible way to other parts of the systems to which they belong. As to the particular conditions of this process, we can only guess at them. LORENTZ has shown¹⁾ that the damping influence of *collisions* may approximately be expressed by introducing into the equations of motion a term, proportional to the velocity of the electron. But if the absorption process already begins within the molecule, before a new encounter takes place (as LORENTZ thinks probable, l. c. p. 142), it is quite conceivable that the amplitude, and, therefore, the velocity of the resonant electron must have increased beyond a certain limit, before a continual transfer of energy to other parts of the same molecule can result²⁾. If the connections really are of that kind, then waves which on account of imperfect resonance impart only *small* mean velocities to the electron, will suffer *no absorption at all*: the part of the damping parameter that is due to absorption, will sink to zero at a certain small distance on both sides of the centre of the absorption line. This, of course, is an hypothesis to which, in the absence of a deeper knowledge of the internal structure of the molecules, we are unable to give a solid foundation, but which may be put to the test by scrutinizing the deductions following from it.

Suppose a beam of white light to pass through a rarefied gas having rather sharply defined periods of free vibrations³⁾, then, according to § 1, the spectral region in which effectual co-vibrating occurs, is only little wider — for each kind of electrons — than

¹⁾ LORENTZ, l. c. Note 57.

²⁾ This conception agrees very well with a new radiation hypothesis, recently proposed by PLANCK (Verhandl. d. Deutschen physikalischen Gesellschaft 13, p. 138), according to which *accumulating* radiant energy by resonance is a continuous process, whereas *emitting* radiant energy only takes place by definite "light-quanta". — Now, let the forced vibrations of an imperfectly resonant electron attain an intensity sufficient for it to emit light-quanta by itself, but not sufficient to shake the other electrons, belonging to the same molecule (and having frequencies of their own), to such a degree as would be necessary for them also to emit quanta: then part of the incident radiation is *scattered* by the electron first affected, but there is no *absorption*, no transformation of the accumulated energy into energy of some other kind.

³⁾ The proper periods are never defined with *perfect* sharpness, owing to the disturbing influence of collisions and to the DOPPLER-effect.

the region of their proper periods; but, strictly speaking, co-vibrating takes place in some degree throughout the spectrum, though with decreasing intensity as one recedes from the line corresponding to the free vibration. Now, according to our hypothesis, the same cannot be said with regard to *absorption*, this process being confined to a narrow part of the region of resonance.

Let us consider the share which *scattering* has in the act of damping. Within the region of strongest absorption it keeps in the back-ground; indeed, for sodium light the factor $\frac{2\nu^2 e^2}{3c^3}$ amounts to only 2×10^{-20} , while the value of the entire damping parameter in sodium vapour was found to be: $h = m\nu' = 7 \times 10^{-28} \times 11.2 \times 10^{10} = 7.8 \times 10^{-17}$ 1). But on the other hand, we have no ground to doubt, as in the case of absorption, the proportionality of this damping effect to the velocity, however small it may be 2). Scattering extends all over the spectrum, wherever the proper frequencies of the electrons may be found.

Indeed, apart from the theory of absorption and dispersion RAYLEIGH has proved that a beam of light of intensity J_0 and wave-length λ , after having travelled a distance x through a mass of gas whose index of refraction is n , and which contains N scattering molecules per unit volume, will have sunk to the intensity

$$J = J_0 e^{-\frac{32 \pi^3 (n-1)^2}{3 \lambda^4 N} x} \dots \dots \dots (9)$$

The quantity $s = \frac{32 \pi^3 (n-1)^2}{3 \lambda^4 N}$ is called the *coefficient of scattering*.

Let Δ be the density of the gas, then $\frac{n-1}{\Delta} = R$ may be considered constant for any given wave-length. As we may put Δ proportional to N , say $\Delta = f \cdot N$, the expression $\frac{n-1}{N} = f \cdot R$ is also a constant for any definite kind of light. Introducing it into the coefficient of scattering, we find

$$s = \frac{32 \pi^3 N f^2 R^2}{3 \lambda^4} = \frac{32 \pi^3 \Delta \cdot f \cdot R^2}{3 \lambda^4} \dots \dots \dots (10)$$

where R represents the *refraction constant* of the medium.

The coefficient of scattering is thus inversely proportional to the

1) From observations by HALLO, concerning the magnetic rotation of the plane of polarisation in sodium vapour, VOIGT calculated: $\nu' = 11,2 \times 10^{10}$. Cf. VOIGT, l. c. p. 142.

2) If we admit PLANCK's new radiation hypothesis, this statement will have to be corrected.

fourth power of the wave-length, direct proportional to the density, to the average mass contained in the medium per scattering particle and to the square of the refraction constant. As the latter varies strongly in the neighbourhood of a proper frequency, also if the damping parameter is small or even approaches zero (Cf. VORST, l.c. p. 118), the coefficient of scattering will assume widely different, and relatively great values in such a region; it only vanishes for waves travelling in the medium with the same velocity as in the ether, i.e. in places of the spectrum where $R = 0$, or $n = 1$.

The effect of absorption and scattering may be considered from two points of view. First we may inquire into the influence of these two damping causes on the motion of the electrons, and, consequently, on the optical properties of the medium (as characterized by n and κ). The second point of view is that in which, the incident light being given, we desire to study the intensity and the composition of the light that leaves the absorbing medium.

So far we have only considered the first question. We concluded that for waves, belonging to the nearest vicinity of those corresponding to the free vibrations of a gas, the damping parameter h must be the sum of two terms:

$$h = \frac{2m}{\tau} + \frac{2}{3} \frac{v^2 e^2}{c^3}; \quad \dots \dots \dots (11)$$

that in the middle part of that narrow region the first term is great compared with the second one; that very probably, however, the value of the first term sinks rapidly to zero at a short distance from each of the proper frequencies, so that in the rest of the spectrum it is only the effect of the second term which remains.

The parameter h , therefore, is not a constant, even if we confine our attention to a part of the spectrum so small, that the variation of the factor v^2 may be neglected. Nevertheless, our fig. 1 (p. 884) gives the principal features of the index of refraction as a function of the frequency, with a fair degree of exactness, because the character of the dispersion curve is much the same for different values of h . The relation, for instance, that the maximum and the minimum of n are found where $\mu = \pm \frac{1}{2} v'$, will continue to hold, if at these points of the spectrum the second (constant) term of h already prevails. But if at the points where μ passes the values $\pm \frac{1}{2} v'$, the function v' (or h) increases rapidly with decreasing absolute value of μ , then the distance between the maximum and the minimum of n must be greater than v' .

§ 5. *Radiation through an extensive atmosphere.* — We shall now

proceed to a discussion of the second question, that which regards the composition and properties of the light that has traversed a very thick layer of gas, the atmosphere of a celestial body, if we suppose the emission curve of the original source of light to be continuous.

Evidently the solution cannot be found by simply putting the value

$$v' = \frac{h}{m} = \frac{2}{\tau} + \frac{2}{3} \frac{v^2 \rho^2}{c^3 m}$$

into the formulae (3) or (6a), and then, for each wave-length separately, substituting the resulting value of nx into an equation of the form

$$I = I_0 e^{-\frac{4\pi}{\lambda} n' z} \dots \dots \dots (12)$$

(in which z represents the distance travelled by the beam through the layer of gas).

For this would lead to an entirely erroneous result, even if the layer of gas were perfectly homogeneous. It is true that the part of the attenuation, which is due to *absorption*, conforms to the law expressed by (12), proceeding in a geometrical progression when the path through the gas increases in an arithmetical progression; but the same does not apply to the part that is caused by *scattering*. If the source of light and the layer of gas are very extensive, we must take into consideration that each electron emits a certain quantity of scattered light owing to irradiation from all directions, and partly joining the directly transmitted beam. The attenuation of the beam must therefore proceed less quickly than it would do according to the law expressed by (12), which holds for the loss of intensity by absorption¹⁾.

SCHESTER²⁾ was the first to discuss in an ample way the combined influence of scattering, absorption, and emission of light in extensive masses of gas. Basing his conclusions on KIRCHHOFF's law, and making various suppositions as to the ratio between the coefficients of absorption and scattering, he examined into the circumstances that would make an atmosphere of a certain depth produce either dark or bright spectral lines.

¹⁾ RAYLEIGH, in deducing the formula (9), has not taken this into consideration; his result only applies to the attenuation which the original beam suffers by scattering, and does not include the scattered light itself.

²⁾ SCHESTER, Radiation through a foggy atmosphere. *Astroph. Journ.* 21, p. 1, (1905).

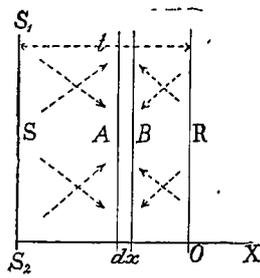


Fig. 3

The way of attacking the problem was as follows. Let an atmosphere of thickness t be irradiated by a surface S_1S_2 (fig. 3), which per unit emits a quantity of energy S within the limit of wave-lengths λ and $\lambda + d\lambda$, and uniformly distributed over all directions. Now SCHUSTER begins with calculating the change which the total flow of radiant energy suffers in a thin layer dx of

that atmosphere. The layer receives from the left a quantity A per unit surface (which in general must be smaller than S , although it includes, besides the radiation directly coming from S_1S_2 , also the radiation emitted by the part of the atmosphere lying between S_1S_2 and the layer dx). Of this quantity A the layer absorbs κAdx ¹⁾, and scatters $sAdx$, the latter part not being lost as radiant energy of the given wave-length, but proceeding half to the right, half to the left. From the right side the layer receives a quantity of energy B per unit surface (composed of scattered light and proper radiation due to the outer part of the atmosphere); it absorbs κBdx and scatters $sBdx$, of which $\frac{1}{2} sBdx$ goes to the right and $\frac{1}{2} sBdx$ to the left. The layer also radiates energy in both directions, amounting to κEdx , if E represents the emission power of the black body within the chosen limit of wave lengths and at the temperature of the layer.

Collecting these effects, one obtains the equations

$$\begin{aligned} \frac{dA}{dx} &= \kappa (E - A) + \frac{1}{2} s (B - A) \\ \frac{dB}{dx} &= \kappa (B - E) + \frac{1}{2} s (B - A) \dots \dots \dots (13) \end{aligned}$$

If now the temperature and the composition of the atmosphere are supposed to be everywhere the same, so that E , κ , and s may be considered as constants, A and B can be solved as functions of x .

Let x be reckoned positive toward the right, and the origin of coördinates taken in the outer surface of the medium, then we shall find the emergent radiation R equal to the value which A takes for $x = 0$, while at the same time we have $B = 0$. Another condition is, that for $x = -t$, we have $A = S$.

Performing the calculations, SCHUSTER obtains

¹⁾ I am using here SCHUSTER's notation, and therefore must draw the reader's attention to the fact, that the above coefficient κ is not quite the same as that occurring in § 2, § 3 and § 4, but corresponds to the expression $\frac{4\pi}{2} \cdot n\kappa$ of formula (12).

$$R = 2\alpha \frac{[(1 + \alpha)e^{\alpha(\nu+s)t} + (1 - \alpha)e^{-\alpha(\nu+s)t}] E + 2(S - E)}{(1 + \alpha)^2 e^{\nu(\nu+s)t} - (1 - \alpha)^2 e^{-\nu(\nu+s)t}} \quad (14)$$

where α means $\sqrt{\frac{z}{z+s}}$.

In order to study the nature of this rather intricate relation, SCHUSTER assigned a number of different values to the ratio $\frac{z}{s} = \beta$ and to the product $s \cdot t$, and constructed several diagrams in which the corresponding values of $\frac{R}{S}$ and $\frac{E}{S}$ were taken as ordinates and abscissae respectively.

As to these results, and a great many other interesting conclusions, we refer to the original paper.

SCHUSTER made no special assumptions connecting z and s with frequencies.

It lies in our line to bring the selective character of these coefficients to the front. The simple relation to which (14) may be reduced for waves suffering no absorption at all, will prove very important and useful in this connection. Denoting by R_0 the value which R assumes for $z = 0$, we obtain¹⁾

$$R_0 = \frac{2}{2 + s \cdot t} S \quad \dots \quad (15)$$

Let us call to mind, before applying this formula, that in deducing (14) SCHUSTER supposed the temperature and the composition of the mass of gas to be uniform, and the intensity of the radiation not to depend on the angle between any direction considered and the normal to the radiating surface. These conditions evidently not being satisfied in the atmospheres of celestial bodies, (14) and (15) only give a first approximation; the influence of the said circumstances will afterwards have to be separately discussed.

In §4 we introduced the hypothesis that the first term of the damping parameter vanishes at a short distance from the proper frequencies, which means that the region of real absorption is confined to the middle-part of each dark line in all cases, where the conditions are such as to make scattering effects appreciable. On that score we assume the equation (15) to hold good for the rest of the spectrum, including — in the case of the solar spectrum — the outer parts of the Fraunhofer lines.

Equation (15) shows that, with increasing thickness t of the

¹⁾ SCHUSTER, l. c. p. 6.

scattering layer, the intensity of the emergent radiation diminishes, but at a slower rate than it would do if scattering acted in the same way as absorption. Putting for instance, $s.t = 98$, we obtain $R_0 = 0.02 S$; and then doubling the layer, we find $R_0 = 0.01 S$; while, if in the original layer an equal loss of 98 percent had been caused by *absorption*, the layer of double thickness would only have transmitted 0.0004 S.

In a vast mass of gas, like the solar atmosphere, even an exceedingly small absorption-coefficient would suffice to produce a very sensible attenuation of the light. We therefore think it much easier to understand the narrowness of most of the Fraunhofer lines, and their appearance in general, if we assume the absorption coefficient to vanish at a very short distance from the middle of each line, so that in the rest of the spectrum the distribution of the light only depends on scattering¹⁾ and other influences (refraction, diffraction, etc.).

Our confidence in the validity of the hypothesis is, however, chiefly based on the fact, that it enables one to explain concisely and in mutual coherence a great many astrophysical phenomena, e.g. the systematic displacements of the Fraunhofer lines, and, if also refraction effects are considered, several irregularities in the behaviour of the lines, together with many particulars revealed by the spectro-heliograph.

Let us now substitute the value of the scattering coefficient as given by (10) into the equation (15); it thus becomes

$$R_0 = \frac{3 \lambda^4}{3 \lambda^4 + 16 \pi^3 \Delta f t R^2} S.$$

We wish to investigate how R_0 varies with λ . If however, we only consider a small part of the spectrum at once, comprising no more than a few Ångström units, we are free to treat λ^4 and S as constants, and may write

$$R_0 = \frac{a}{a + b R^2} S \quad (16)$$

¹⁾ The question may arise whether there are perhaps indications from which one might obtain some idea about the magnitude of scattering effects, reasonably to be expected in a gaseous medium of the dimensions of the solar atmosphere. Now, according to RAYLEIGH'S theory, the average sunlight loses about 5% of its intensity by molecular scattering in passing through our terrestrial atmosphere. Substituting $R_0 = 0.95 S$ in our formula (15), we find $s.t = 0.1$. If we make the very rough estimate, that the solar atmosphere is 50 times as thick as the atmosphere of the earth, and has the same average density, we must write for the sun: $s.t = 5$, and, consequently, $R_0 = \frac{2}{7} S$. This is not an unreasonable result. It proves that even with a much smaller density the solar atmosphere would be able to produce sensible scattering effects, especially near absorption lines.

The constant b is proportional to the density of the medium, to the thickness of the layer, and to the average mass per scattering particle. The greater each of these quantities is, the smaller will be the intensity of the emergent light (for any value of λ). The only quantity strongly variable with λ in the small spectral region considered, is the factor R^2 , in case there is an absorption line.

The upper part of fig. 4 represents $R = \frac{n-1}{\Delta}$ as a function of λ ¹⁾.

The origin of co-ordinates corresponds to the wave-length λ_0 of a free vibration; the line $P_1 P_2$, having the approximately constant ordinate $\frac{n_0-1}{\Delta}$, would be the dispersion curve if there were no absorption line at λ_0 .

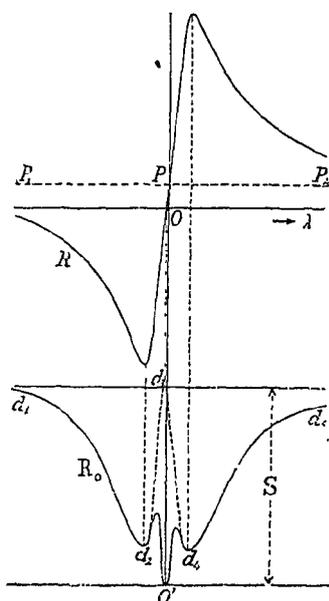


Fig. 4.

Supposing t to be sufficiently great, and R not too small, we may, in the denominator of (16), neglect a in comparison with bR^2 ; so R_0 is about inversely proportional to R^2 . The light is therefore more weakened by scattering on the red side than on the violet side of the absorption lines, if (as supposed in the figure) we have $n_0 > 1$. With *very* strong lines this difference will, however, partly be neutralized, because, according to § 3, the minimum of the refractive index sinks further below n_0 than the maximum rises above n_0 .

Taking this into consideration, and preliminarily fixing our attention on the effect of scattering only, leaving that of absorption aside, we may represent the intensity of the emergent light by a curve of the shape $d_1 d_2 d_3 d_4 d_5$ (fig. 4, lower part). The top d_3 (corresponding to $R=0$) does not coincide with λ_0 , but is a little displaced toward the violet. If, therefore, one would imagine the region between d_3 and d_4 , where the loss of light due to scattering passes through a minimum, to be an "emission line", one would have to assign to it a smaller wave-length than to the absorption

¹⁾ In this figure λ increases from the left toward the right; the succession of the kinds of light is therefore opposite to that in the figures 1 and 2, where the frequencies ν were chosen as abscissae.

line, and, applying DOPPLER'S principle, would conclude that the radiating vapour moves toward the observer.

Properly speaking, we are not allowed to apply the formulae (15) and (16) to all waves between d_2 and d_4 , for where x is not equal to zero, equation (14) should be used. Let us suppose that only in the middle part of that region x has appreciable values; then we probably shall obtain a fairly true intensity curve, when subtracting the ordinates of an absorption curve (supposed to be symmetrical with respect to OO') from the ordinates of the displaced, unsymmetrical curve $d_2 d_3 d_4$. The result is a sharp drop in the intensity curve, representing a narrow dark line in the spectrum; whose "centre of gravity" is somewhat displaced toward greater wave-lengths (with respect to λ_0), and which is partly caused by absorption, partly by scattering. If one should mistake this line for a mere absorption line, its displacement toward the red would make one think, that the absorbing vapour recedes from the observer. ¹⁾

The above particulars which, according to our theory, the distribution of the light in a wide dispersion band must show, bear a striking resemblance to the phenomena really observed by CHARLES E. ST. JOHN ²⁾ in the calcium lines H and K of the solar spectrum. And if, besides the consequences of anomalous scattering, we also consider those of anomalous refraction (not noticed in this paper), the agreement between the results of theory and of observation proves to extend to almost every detail of the phenomena described by ST. JOHN. So it is possible to explain the rather intricate peculiarities exhibited by the components $H_1, H_2, H_3, K_1, K_2, K_3$ of the well-known broad calcium lines in the spectrum of the various parts of the solar disk, *without* having to admit with ST. JOHN, that there is a general radial circulation of the calcium vapour going on in the solar atmosphere, with velocities that would amount to 1,97 kilometer per second in the mean for the ascending, and to 1,14 kilometer per second in the mean for the descending motion. I must refer the fuller discussion of these interesting observations to a subsequent paper.

With most lines of the solar spectrum the total region of the dispersion anomaly, from d_1 to d_5 , is so narrow, that the particulars concerning the part included between d_2 and d_4 escape our observation. What then remains visible, is only the *asymmetry* of the

¹⁾ In a former communication (Proc. Roy. Acad. Amst. XIII, p. 10; Astroph. Journ. 31, p. 428, 1910) I wrote that the central part of the K -line, the true absorption line, cannot be displaced by anomalous dispersion. I did not yet realize, at that time, that even the central line might be an impure absorption line.

²⁾ CHARLES E. ST. JOHN, The general circulation of the mean and high-level calcium vapor in the solar atmosphere. Astrophysical Journ. 32, p. 38—82 (1910).

dispersion bands enveloping the absorption lines. How the systematic displacements of the Fraunhofer lines toward the red, the obliquity of the lines in the spectra of sun-spots, and some other phenomena, may be explained from this point of view, has been shown in former publications¹⁾.

Anatomy. — “Notes on the trochlear and oculomotor nuclei and the trochlear root in the lower vertebrates”. By Dr. W. G. HUWT.
(Communicated by Prof. L. BOIK).

(Communicated in the meeting of January 28, 1911).

In the course of the past year I made several observations regarding the oculomotor and trochlear nuclei and their roots, in *Petromyzon*, *Lophius*, *Gadus*, *Hippoglossus*, *Rhombus*, *Pleuronectes*, *Selache maxima* and *Scyllium Canicula*. The results of my researches can be best demonstrated by comparing the relations of the said nuclei in *Petromyzon*, *Selache*, and *Lophius*.

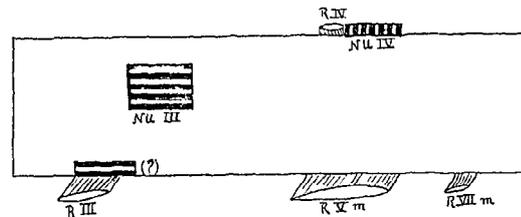


Fig. 1. *Petromyzon*.

Fig. 1 shows the topographic relation of the oculomotor nucleus and root, the trochlear nucleus and root and the motor V and VII root in *Petromyzon*.

As will be seen from this figure, the oculomotor nucleus in this animal lies partly on the level of its root-entrance, partly behind. The principal nucleus (Fig. 7), the only III nucleus according to some investigators, lies with its dorsal edge not far from the aqueduct. Whether the so-called “ventral III nucleus” be a III nucleus or not, I will not state positively. Its topography speaks strongly for this view, as can be seen in Figs. 1 and 6. The cell-type is, however, somewhat smaller than that of the dorsal nucleus. I have not been able to obtain sufficient certainty about the course of its axis-cylinders to enable me to decide this question.²⁾

¹⁾ Proc. Roy. Acad. Amsterdam, XII, p. 266 and 466 (1909); XIII, p. 2 (1910); Les raies de Fraunhofer et la dispersion anormale de la lumière. *Le Radium*, t. VII Oct. 1910.

²⁾ This cellgroup is regarded by AILBORN, JOHNSTON, and SCHILLING as being a part of the III nucleus, but by ТРАТТАКОФФ, on the other hand, as a cell group independent of the oculomotor.