Mathematics. — "*Continuous one-one transformations of surfaces in themselves.*" (2nd communication).[1]) By Dr. L. E. J. BROUWER. (Communicated by Prof. D. J. KORTEWEG.)

(Communicated in the meeting of June 26, 1909).

We shall now consider an arbitrary twosided[2]) surface and we shall submit it to an arbitrary continuous one-one transformation in itself with invariant indicatrix.

Under a *limit region* of the transformation we shall understand a region of the surface lying entirely outside its image region, but losing that property by any extension.

However it may happen that a limit region allows of *enlargement*, i.e. can be united after an indefinitely small modification of its boundary with a finite adjacent region into a new limit region, whose surface, measured by a certain system of coordinates, is then of course greater than the old one's. This will be illustrated by the following developments.

Under a *transformation domain* we shall understand a limit region not capable of enlargement, and our intention is to construct such a transformation domain.

To this end we start from two arcs of simple curve[3]), which are each other's image, which have two and not more than two points in common, and which do not cross each other in those points. We shall suppose, that these arcs have no endpoint in common; their situation with respect to each other then still allows of various possibilities, indicated by fig. 1.

With the aid of these two arcs we now construct two regions G and G', bounded by simple closed curves, which regions are each other's image and lie entirely outside each other, whilst their boundaries have two arcs of simple curve in common. In fig. 2 this has been executed for the second possibility indicated by fig. 1.[4])

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1) See these *Proceedings*, Vol. XI, page 788.

2) A onesided surface falls under our result-only when brought into a continuous multivalent correspondence with a twosided one.

3) i.e. "einfache Kurvenbogen" after SCHÖNFLIES. In my preceding communications of this year (these *Proceedings* XI, p. 788, 850) I translated "einfach" by "single". Finding the term "simple closed curve" used by VEBLEN, I shall adopt in future this mode of expression

4) Only in the fifth and eighth case of fig. 1 this might give rise to some difficulty, namely if the two common points of the arcs are each other's image. By a slight modification of the figure this difficulty can then be cancelled.

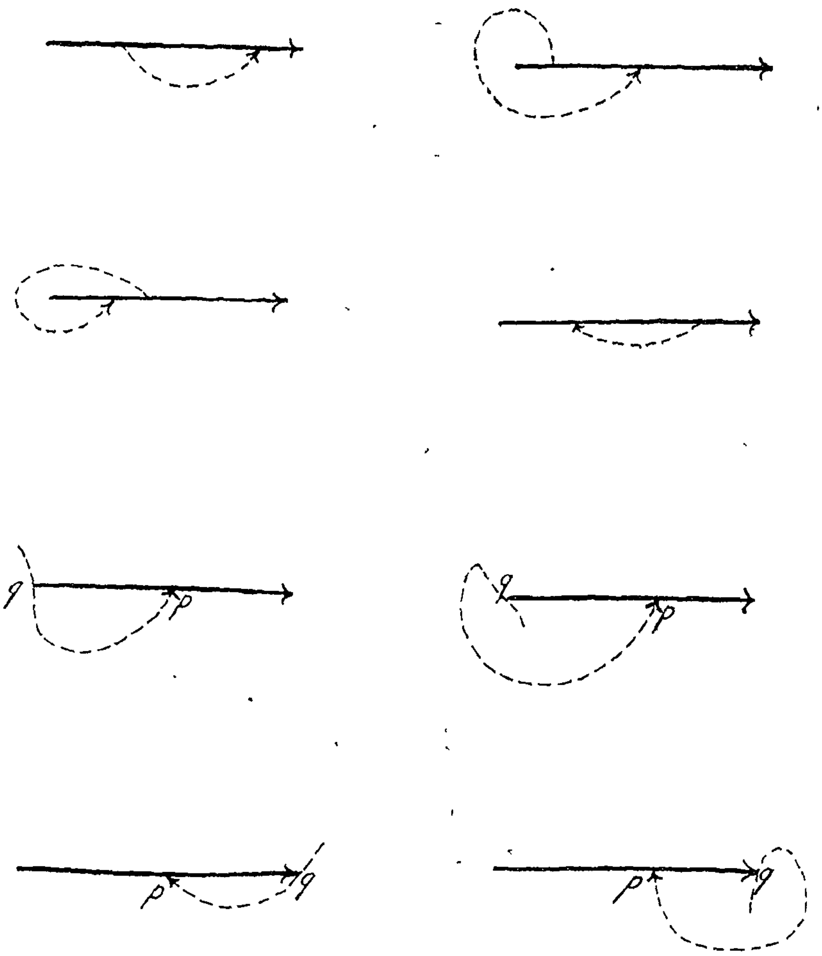

Fig. 1.

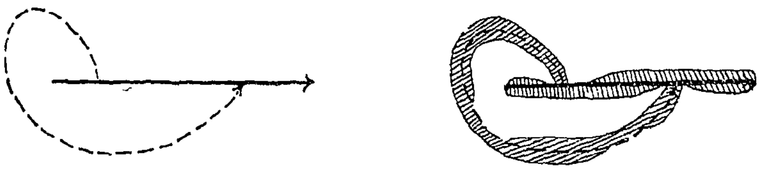

Fig. 2.

On the boundary of each of the two regions G and G' then directly lie some [1]) arcs of simple curve, which will remain *accessible* [2])

[1]) In the first five cases of fig. 1 this number is 4; in the sixth it is 4 or 3; in the seventh 3 or 2; in the eighth it is 3.

[2]) i. e. "erreichbar" after SCHÖNFLIES, Bericht über die Mengenlehre II, p. 176.

for any limit region, to which the region G may be extended, and likewise for its image region.

We now extend G in such a way, i. e. we replace it in such a way by a region in which it is contained, that, when G' has obtained the corresponding extension, the extended regions G and G' still lie outside each other. We repeat this extending process so often, until we get a limit region, and this will be the case after a denumerable number of extensions.

If possible, we then, after an indefinitely small modification of the boundary, execute an enlargement of this limit region; by this its property of being a limit region will in general be lost, but can be regained by a denumerable number of new extensions. This new limit region we again try to enlarge, and in this way go on, until by a denumerable number of operations a *transformation domain* O is obtained.

If in the surface no region exists, which at once with all points of its boundary is invariant for the transformation, the domain O can at most determine two rest regions, namely a rest region R_1, in which O' lies, and a rest region R_2, identical to R_1 or not, in whose image region R'_2 lies O.

If namely a third rest region G_3 existed, G_3 as well as G'_3 would be free of O, as well as of O'. Let P be an arbitrary point on the boundary of G_3 not coinciding with its image P'. Let us construct about P and P' simple closed curves, chosen as small as one likes, which are each other's image and bound regions π and π', then $\mathfrak{M}\ (O, G_3, \pi)$ and $\mathfrak{M}\ (O', G'_3, \pi')$ each contain one of a pair of regions, which are each other's image, and which one can make to contain of O and G_3 resp. of O' and G'_3 as closely as one likes *approximating* [1]) partial regions, to which O and O' might be *enlarged*, but this would clash with the property of a transformation domain. The same reasoning holds with a slight modification for the supposition that G_3 and G'_3 coincide.

We shall say, that a region of a surface *does not break the connection of the surface*, if it determines only one rest region, possessing for analysis situs the character of a rest region of a trema. Then of course the region itself is singly connected.

We shall now assume, that O does not break the connection of the surface. Then it is bounded by a *closed curve* [2]) K, which on account of the commencement of the construction according to fig. 2 is a *non-singular closed curve*, and therefore does allow of

[1]) Schönflies, Bericht über die Mengenlehre II, p. 104 sqq.

[2]) Id., ibid., p. 118 sqq.

division into two proper arcs, and not into two improper ones [1]).

We shall now consider on each of the curves K and K' the cyclically ordered sets u and u' of their outwardly accessible points. These determine on each other certain segments σ_n and σ'_n, forming in pairs certain cyclically ordered sets $\varkappa_n$ of points accessible from a rest region of O and O', as is schematically illustrated by fig. 3.

If we make a circuit along u and u' together in such senses, that segments σ_n and σ'_n are reached simultaneously, this circuit is made *in opposite senses* for u and u'.

So if we make a circuit along u and u' together in such a way that points corresponding for the transformation are reached simultaneously, then with respect to the former order they must meet twice in that course, and at such a meeting either corresponding

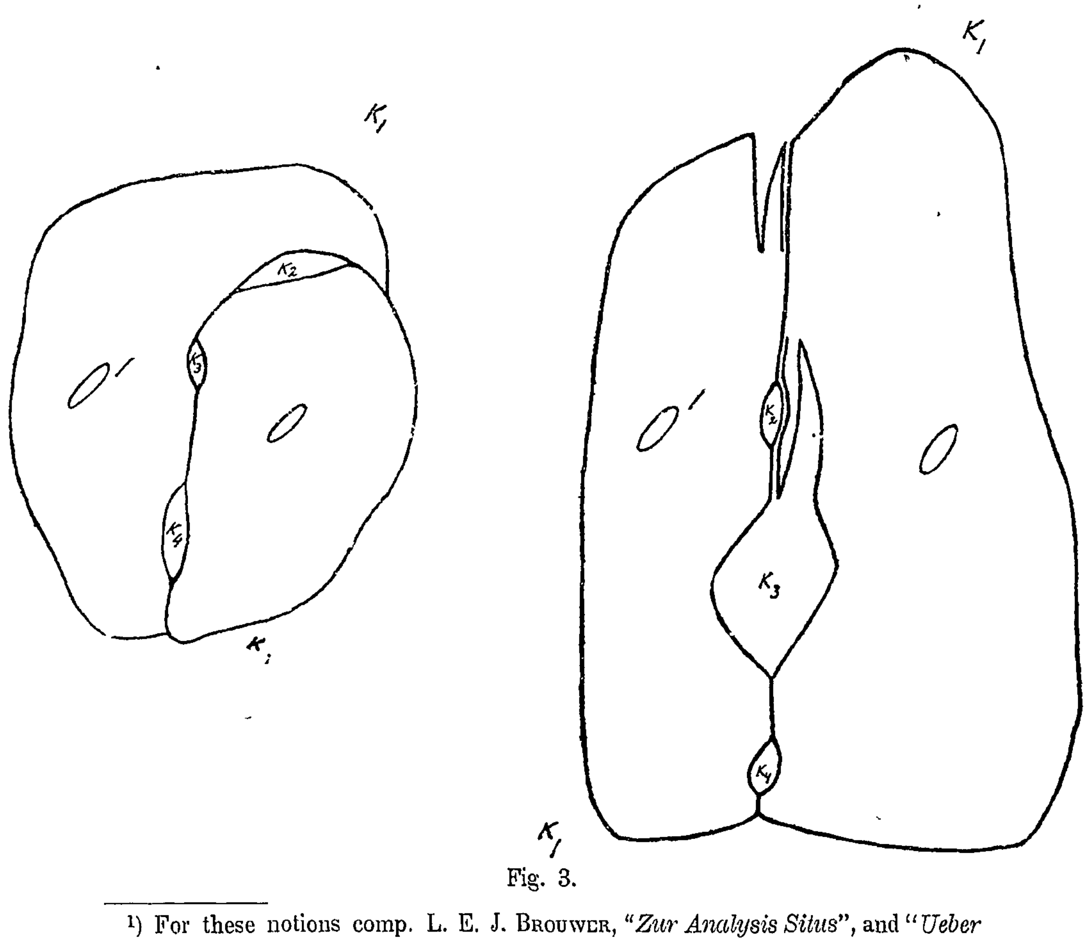

Fig. 3.

[1]) For these notions comp. L. E. J. Brouwer, "*Zur Analysis Situs*", and "*Ueber ein-eindeutige stetige Transformationen von Flächen in sich*", Mathem. Annalen, Bd. 68.

junctures [1]) Z (lying in u) and Z' (lying in u') form together a coherent set determining *one* rest region, or a point P of u, and its image P' in u' will find themselves on the same $\varkappa_n$ in such a way, that P lies outside K', and P' outside K. Then however in those points P and P' an enlargement of O and O' would be possible, which would clash with the property of a transformation domain. So we are sure, that both meetings take place in the first-mentioned way.

Let us discern in a juncture Z, performing such a meeting, its *right end* Z_r and its *left end* Z_l. Let us represent by K_r resp. K_l the branch of u approaching to Z_r resp. Z_l; by ζ_r resp. ζ_l the part of the *circumference* [2]) of Z_r resp. Z_l, approximated by K_r resp. K_l; and by ζ the complete circumference of Z. Let us further indicate m (Z, Z') by T, the circumference of T by τ, (Z_r, Z'_l) by T_r, m (Z_l, Z'_r) by T_l, and the part of the circumference of T_r resp. T_l, approximated by K_r and K'_l resp. K_l and K'_r, by τ_r resp. τ_l.

We at once see, that of the sets T_r and T_l at least one is coherent. For in the opposite case we could choose on K_r in the vicinity of Z_r a point of u outside K', whose image on u' would lie outside K, and an enlargement of O and O' would be possible there (see the schematic fig. 4).

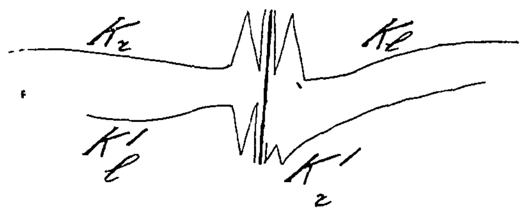

Fig. 4.

In the same way it is evident, that *either* ζ_r and ζ'_l or ζ'_r and ζ_l must be identical to each other, *or* Z must be a part of Z' or Z' a part of Z.

Otherwise namely *either* ζ_r would have to lie partly outside ζ'_l and at the same time ζ'_r partly outside ζ_l *or* ζ_l partly outside ζ'_r, and at the same time ζ'_l partly outside ζ_r, which free segments of circumference would partly correspond to each other, and would admit an enlargement of O and O' in their vicinity.

Of the two possibilities obtained we shall first discuss:

[1]) Two arcs of curve cohere to a new arc of curve by means of a "*juncture*", which contains an *end* of each of them. In the following reasoning we suppose the considered juncture to be composed of two non-singular ends. For singular ends it needs a slight modification.

[2]) i. e. the cyclically ordered set of its accessible points.

1. ζ_r *and* ζ'_l *are identical to* τ_r. We then further notice, that ζ_l and ζ'_r cannot leave a part s of the circumference of T_r free between K_l and K'_r (so that T_l also is coherent, though of τ_l this is not yet certain).

Otherwise namely s would be a part of the circumference of T_r, outside τ_r, ζ_l and ζ'_r; if we then regard T_r as Z_r, there will correspond to s a part s' of the circumference of Z'_r outside ζ'_r, ζ_l and τ_r. So about s and s' the domains O and O' could be enlarged.

We now distinguish the following two cases:

Ia. τ_r *and* ζ_l *do not cohere on* τ [1]; then τ_r and ζ'_r do not cohere either. We can then represent Z schematically according to fig. 5, where each of the segments represents an arc of curve, to which further ramifications may be attached; however no ramifications cohere with the schema between *ap* and *ah*, and neither between *bq* and *bk*. Furthermore we notice that in a as well as in b ab unites itself with each of the two other arcs of curve ending there to a new arc of curve. In an analogous way the sets Z' and T are represented in this figure [2]).

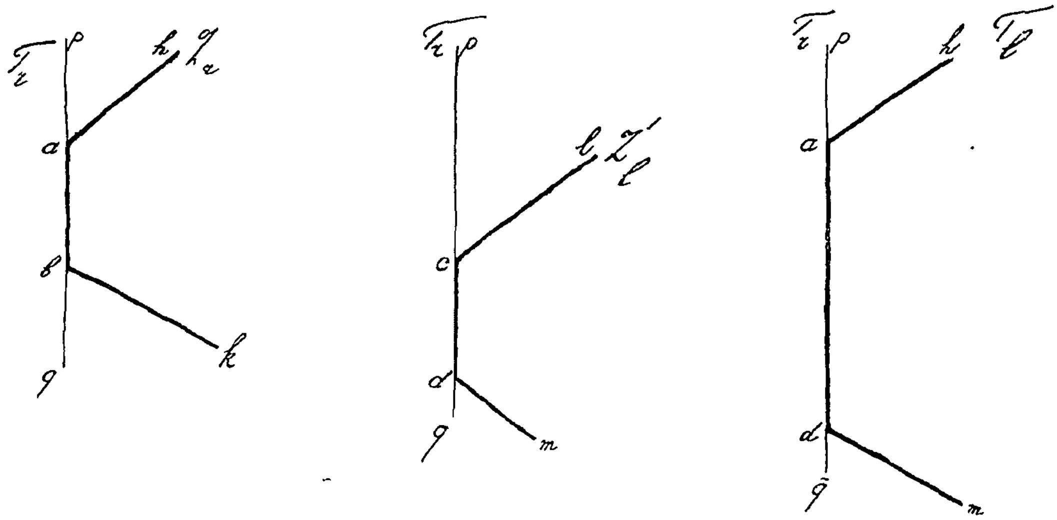

Fig. 5.

Now the situation of the three sets T_r, Z'_r, and Z_l with respect to each other can be in different ways in this case; the two essential possibilities are represented in fig. 6.

1) *Ia* may also be treated in a less direct way, analogously to *Ib*.

2) Some of the segments of these figures may shrink to zero; this however does not injure the correctness of our conclusions.

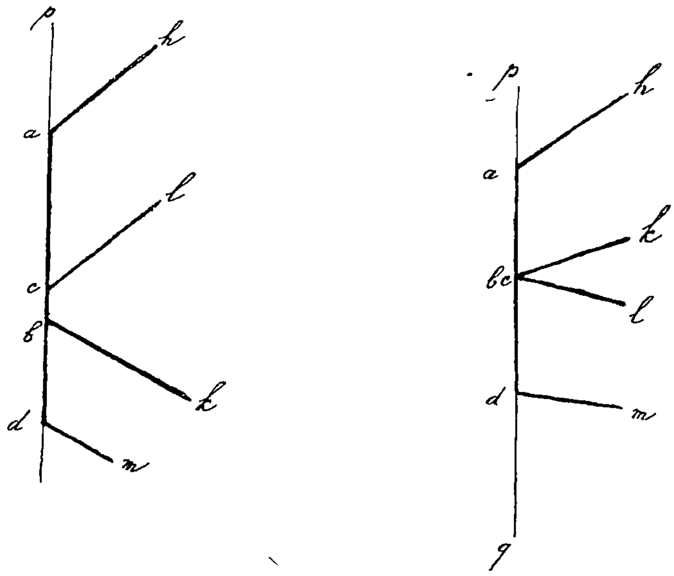

Fig. 6.

I$a\alpha$. When the first possibility of this figure appears, it follows from the correspondence of the arcs of curve ab and dc for the transformation, that on the arc of curve bc must lie somewhere a *juncture invariant for the transformation.*

I$a\beta$. When the second possibility appears, we notice that in bc the arcs of curve dc and bk certainly cohere. For otherwise ab and cl would cohere there neither, and from both would follow, that in bc between ζ'_r and ζ_l a segment s of the circumference of T_r would be free of both ζ'_r and ζ_l, the impossibility of which we have proved.

Let us furthermore in bc call L_{dc}, L_{lc}, and L_{ab} the ends of the arcs of curve dc, lc, and ab, then, $habq$ having for the transformation as its image $qdcl$, and L_{lc} and L_{ab} cohering with each other, the situation of L_{dc}, L_{lc} and L_{ab} with respect to each other is quite the same, as that of τ_r, ζ'_r and ζ_l; the former thus allow of quite the same investigation as the latter, in which they are contained. If by this investigation we arrive again at the case I$a\beta$, we can investigate still further partial sets which are in the same case, and can go on in this way. After a denumerable number of repetitions of this process we must then either have reduced this case to an other, or have found *a point invariant for the transformation.*

Ib. *τ_r and ζ_l partly cover each other, or at least cohere on τ*; then the same holds for τ_r and ζ'_r.

Then, as τ_r is the image of ζ_l, and ζ'_r the image of τ_r, we can, starting at ζ_l, divide the part of τ, approximated by K and K', into successive "transformation domains", separated by Schnitte in this circumference, so that in a circuit each following domain is the image of the preceding one; of the last domain of the series then in general only a part appears, but three at least must be complete; fartheron the derived set of this last partial domain must cohere with that of the first domain; and we may suppose the number of the domains to be finite, as otherwise we should at once fall back on case II.

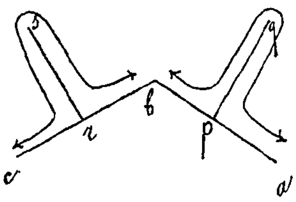

Fig. 7.

These domains of the circumference we can reduce by destroying in each of them the subsets of T, belonging to no other part of τ. If e.g. in fig. 7 the first domain extends from a to b, the second from b to c, we can deprive the first of the arc pq, the second of the arc rs.

On the circumference of the rest set R_1 we have then still a division into domains with the same properties, as the original division of the circumference of T. By a displacement of the separating Schnitte between the domains (after which in general the first as well as the last domain will be a partial one) it may occur, that the same process of reduction is once more applicable to R_1, and in this way it is repeated, until after a denumerable number of reductions a set R_0 is left, which by no displacement of the separating Schnitte can be made fit for further reduction.

A domain d then spreads over a part of the circumference h of an arc of curve k in such a manner, that the domain itself as well as the rest of the circumference possesses the whole k as its derived set..

Let the next domain d' follow of that rest of the circumference of k a part e', before it leaves k, and let f be the part of h belonging neither to d, nor to e'. Let e' be the image of e, and let us call g the segment of h approximated by k'. These notations are illustrated by two examples in fig. 8.

We now distinguish the following two cases:

Ibα. f does not lie everywhere dense on k. Then e must lie everywhere dense on k.

For in the opposite case we might take the part of h enclosed between the end Schnitt of e and the end Schnitt of e' as a domain, which would be capable of reduction, and this is impossible.

But if e lies everywhere dense on k, the image of k is a part of k; so that we fall back on case II.

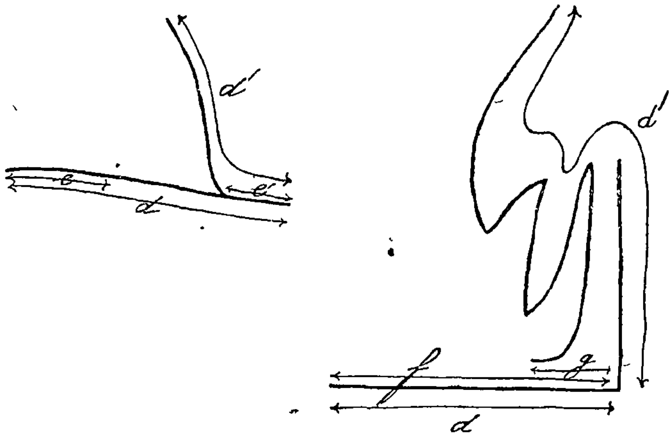

Fig. 8.

Ib β. f lies everywhere dense on k. Then *g* must also lie everywhere dense on *k*.

For in the opposite case the arcs of curve k, k', k'', etc. form an arc of curve *B*, in whose circumference the domains d, d', d'', etc. leave free a part everywhere dense in *B*, whilst each arc $k^{(n)}$ coheres with $k^{(n-1)}$ and $k^{(n+1)}$, but with none of the other. This however is impossible, as we have shown above, that the last and the first domain must cohere with each other.

But if *g* lies everywhere dense on *k*, *k* is a part of k', and again we are in case II.

II. *Z' is a part of Z.* We then, and also in the preceding cases reduced to this, have an *open system of curves*[1]) *S*, having as its image S' a part of itself. The image S'' of S' is again a part of S', etc. If $S^{(\omega)}$ is the set common to all sets $S^{(n)}$, it is *an open system of curves i, invariant for the transformation.*

From the commencement of the domain construction according to fig. 2 is evident, that the two sets *T*, in which *u* and u' meet each other, and therefore also the two invariant sets *i*, lie isolated from each other.

At the commencement of the domain construction we have presupposed, that the two arcs of curve of fig. 1 have no endpoint in common. If that *is* the case, the method needs but slight modifications, giving no difficulties. We then start not with common arcs of simple curve accessible for both *O* and O', but only with common points accessible for both *O* and O', which is sufficient too.

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1) This means a perfect, coherent, nowhere dense set, determining in a region of the connection of the inner region of a circle or parabola only *one* rest region.

Furthermore we have supposed in our reasoning, that K_r etc. have nothing more in common with T, than Z_r etc. However it might be possible, that the last part of e. g. the branch of K_r approximating Z_r is contained in Z'. One can easily be convinced, that this pecularity does not harm the proof.

We can sum up our results as follows:

THEOREM 1. *An arbitrary continuous one-one transformation of a twosided surface in itself with invariant indicatrix possesses a transformation domain, which either breaks the connection of the surface, or joins two isolated open systems of curves, invariant for the transformation.*

As furthermore such an invariant open system of curves possesses at least *one* invariant point, as I shall prove in another communication, the following holds likewise:

THEOREM 2. *An arbitrary continuous one-one transformation of a twosided surface in itself with invariant indicatrix possesses a transformation domain, which either breaks the connection of the surface, or joins two points invariant for the transformation.*

The fundamental importance of these theorems for the theory of transformations and transformation groups I shall show lateron. Here I only wish to indicate, how the theorem of the invariant point of the sphere, proved in my first communication upon this subject, is contained in them, and how this theorem can be extended to the Cartesian plane.

For, if the transformation domain of a sphere or Cartesian plane has more than two rest regions, then surely one of them must be invariant together with all the points of its boundary.

If we thus exclude this case, a transformation domain O on the sphere, which breaks its connection, is either annular, or singly connected; fig. 9 shows either of these possibilities.

Fig. 9.

In both cases O determines on the sphere two regions, V and W, in one of which lies its image O'; then in the other lies O'_i, i.e. its image for the inverse transformation. If we now repeat the transformation itself as well as its inverse an indefinite number of times, we obtain on one hand in V a series of domains $O', O'', O''',\ldots$, and on the other hand in W a series of domains $O'_i, O''_i, O'''_i,\ldots$

In neither of these series exist intermediary regions between the successive terms, and each series converges to an invariant limit set, which causes the existence of at least one invariant point. These invariant limit sets may cohere with each other (in the second case of fig. 9); then we are sure of only *one* invariant point, otherwise always of *two*.

If in the Cartesian plane we have a transformation domain breaking its connection, either one of the cases of fig. 9 appears again, or the domain is bounded by an arc of curve running from infinite to infinite, or by two such arcs of curve.

If it is bounded by one arc, the existence of an invariant open system of curves can be deduced according to the proof of theorem 1.

If it is bounded by two such arcs B_1 and B_2, we can, if no

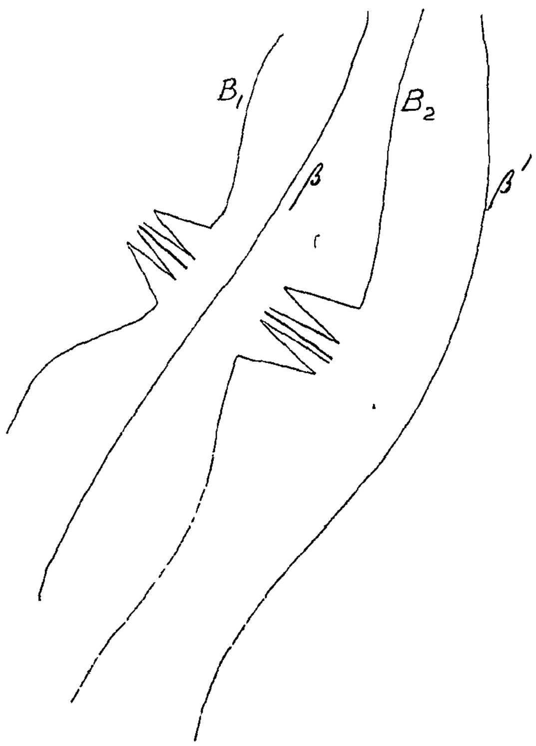

Fig. 10

invariant point exists, always arrange that these arcs do not cohere, and within O (see fig. 10) an arc of simple curve β can be constructed, running from infinite to infinite, lying entirely outside its image β', and enclosing with β' a new transformation domain ω. According to the SCHÖNFLIES process of representation [1]) the domains $\omega', \omega'', \omega''', \ldots,$ determined by a series of successive repetitions of the transformation, can then be represented on regions enclosed by straight lines $x = na$ and $x = (n+1)\,a$ in such a way that a series of corresponding points in $\omega, \omega', \omega'', \omega''',\ldots$ answers to a series of points (p,q), $(p+a,q)$, $(p+2a,q)$, $(p+3a,q),\ldots$; furthermore, if no invariant point exists, we can arrange, that the just-mentioned series of images of ω, continued indefinitely on both sides, covers the whole Cartesian plane, i. o. w. we have proved:

THEOREM 3. *An arbitrary continuous one-one transformation of the Cartesian plane in itself with invariant indicatrix either leaves at least one point invariant, or is a continuous one-one image of a translation.*

ERRATUM.

In my paper: *"The force field of the non-Euclidean spaces with positive curvature"*, these *Proceedings* IX 1, p. 261, l. 6 from top:
for: The unilateral elliptic Sp_n is enclosed by a plane Sp_{n-1},
read: The elliptic Sp_n is enclosed by a unilateral plane Sp_{n-1}.

Chemistry. — *"On the nitration of diethylaniline"*. By Prof. P. VAN ROMBURGH.

(Communicated in the meeting of September 25, 1909).

A considerable time ago I communicated to the Academy the results of the action of nitric acid on dimethylaniline dissolved in a large excess of sulphuric acid [2]). The difference in behaviour which dimethyl- and diethylaniline exhibit when nitrated under other conditions induced me to study also the action of nitric acid on diethylaniline in presence of an excess of sulphuric acid, and indeed this amine and some of its nitro-derivatives appeared to behave in many respects differently from dimethylaniline.

If we dissolve diethylaniline in double its volume of concentrated sulphuric acid and pour this solution into an excess of nitric acid

[1]) Mathem. Ann. 62, p. 319—324

[2]) Verslagen Akad. v. W. Febr. 23, 1895 and These Proc. Dec. 30, 1900.