that which is brought about the apex and the margin of the leaf and on the leaf-surface as a result of the action of glands.

In many cases the glands are originally mucilage-glands (Colleteren, Keulenzotten, Trichomzotten) which secrete resin or balsam in the bud, as proved to be the case in *Kerria, Sambucus, Corylus, Ulmus, Syringa, Forsythia,* but in other plants they are from the beginning real water-glands: *Philadelphus, Deutzia, Hydrangea, Weigelia,* etc.

Physics. — "*On the theory of the* ZEEMAN-*effect in a direction inclined to the lines of force.*" By Prof. H. A. LORENTZ.

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§ 1. Certain phenomena observed by HALE in sun-spot spectra have induced me to work out the theory of the ZEEMAN-effect on the assumption that the direction of observation is oblique to the lines of force, a problem that has already been treated by VOIGT [1]), but in which some details remained to be examined.

Our subject will be the "inverse" effect, to which the direct one is intimately related, and we shall start from the fundamental equations in the form I have given them in a recent article in the "Mathematische Encyklopädie" [2]), supposing the magnetic field to be homogeneous and parallel to the axis of z.

We shall assume that the particles of the body through which the light is propagated, unless they be magnetically isotropic (i. e. of such a structure that a rotation of a particle in the field has no influence on the frequency of its free vibrations) are turned by the magnetic force in such a manner that a certain "axis" proper to each particle takes the direction of the field. We shall further imagine that each particle contains a certain number of electrons forming by their arrangement some definite and regular configuration, and capable of vibrating about their positions of equilibrium under the joint influence of "quasi-elastic" forces, of resistances and of the action exerted by the external field. Though, on account of the complexity of its structure, the mode of motion of a particle may be far from simple, we can easily treat it mathematically in a general way. This is due to the circumstance that, under certain simplifying restrictions,

1) W. VOIGT, Weiteres zur Theorie der magneto-optischen Wirkungen, Ann. Phys. I (1900), p. 389.

2) H. A. LORENTZ, Theorie der magneto-optischen Phänomene, Encyklopädie d. math. Wiss. V 22, p. 199.

the electromagnetic action exerted by a particle is found to be wholly determined by its electric moment. Therefore, in considering the influence of a particle on the propagation of light, we may replace it by a single electron, to which we may assign an arbitrarily chosen charge $\mathbf{e}$ and whose displacements $\mathbf{x}, \mathbf{y}, \mathbf{z}$ have such values that the products $\mathbf{ex}, \mathbf{ey}, \mathbf{ez}$ are equal to the components of the electric moment of the particle. This imaginary electron may be called the "equivalent electron".

If a particle were outside the magnetic field (but in the position it really has in the field) and if it were free from resistances and from the influence of the other particles, its electrons would be able to vibrate in a number of definite modes. We shall suppose that, in these circumstances, there are certain groups of "fundamental vibrations", of such a kind that all the vibrations belonging to one and the same group have a common frequency n_0, corresponding to a definite spectral line. Whenever it is necessary, we shall distinguish the different groups from each other by the indices $a, b, c, \ldots$, and we shall denote by k the number of modes of vibration in a group.

Let us next suppose the magnetic field to be excited, without, however, as yet introducing the resistances and the mutual actions between the particles. Then, instead of any group consisting of k modes of vibration with equal frequencies n_0, we shall have k modes whose frequencies are unequal, all differing slightly from this original common value. In order to distinguish these k modes, we shall assign to each of them an index ($\varkappa$), which we shall write on the right-hand side and at the top of the symbols relating to the mode in question.

Now, in each of these fundamental vibrations that can go on in the magnetic field under the circumstances just stated, the equivalent electron will have a motion which, according to the theory of vibrating systems, must be, generally speaking, a harmonic elliptic vibration. It can be further specified, if we take into account the states of polarization observed in the ZEEMAN-effect. From these one can infer that the path of the equivalent electron must be, either a straight line in the direction of the field, or a circle whose plane is at right angles to it. The index $\varkappa_1$ will be applied to those fundamental modes for which the first case occurs, and similarly the index $\varkappa_2$ to the second case; if we want to distinguish whether the circular motion of the equivalent electron is in the direction corresponding to that of the lines of force, or in the opposite one, we shall use the index $\varkappa_{2+}$ or $\varkappa_{2-}$. However, in order not to encumber our formulae with too many indices, we shall omit them whenever this can be done without fear of confusion.

The states of motion $\varkappa_2$ always occur in even number; in fact, corresponding to each state $\varkappa_{2+}$, there will be a state $\varkappa_{2-}$ whose frequency $n^{(\varkappa_2+)}$ lies at the same distance from n_0 as $n^{(\varkappa_2-)}$, but on the other side of it. The modes of motion $\varkappa_1$ are conjugate two by two in the same way, with the exception, however, when k is odd, of one of them, which has the original frequency n_0.

The introduction of complex expressions, after the manner generally followed in problems of this kind, will be found very convenient. By this method one finds, for each mode of vibration, definite ratios between the quantities representing the components of the displacements of the several electrons from their positions of equilibrium. These ratios determine what may be called the "forms" of the vibrations, and it is especially to be noticed that, whereas in a particle subjected to the magnetic field only, the vibrations of the k modes mentioned above have unequal frequencies, a periodic external electric force can produce forced vibrations in these different modes, all taking place with the period of the force itself.

§ 2. This case of an impressed electric force occurs when a ray of homogeneous light is propagated through the system, so that, if n is its frequency, the complex expressions for the dependent variables all contain the factor e^{int}. While a particle is vibrating, in all its different modes at the same time, under the influence of the alternating electric force $\mathfrak{E}$ existing in the beam of light, it has an electric moment whose components are

$$\mathfrak{p}_x = \mathbf{ex},\ \mathfrak{p}_y = \mathbf{ey},\ \mathfrak{p}_z = \mathbf{ez},$$

and the body is therefore the seat of an electric polarization (electric moment per unit of volume) for which we may write

$$\mathfrak{P} = N\mathfrak{p},$$

where N is the number of particles in unit of volume.

The equations of motion of a particle lead to the values [1])

$$\begin{aligned}\mathfrak{P}_x &= Q_{2+}(\mathfrak{E}_x + i\mathfrak{E}_y) + Q_{2-}(\mathfrak{E}_x - i\mathfrak{E}_y),\\ \mathfrak{P}_y &= Q_{2+}(\mathfrak{E}_y - i\mathfrak{E}_x) + Q_{2-}(\mathfrak{E}_y + i\mathfrak{E}_x),\\ \mathfrak{P}_z &= Q_1\,\mathfrak{E}_z,\end{aligned}$$

the coefficients Q_1, Q_{2+}, Q_{2-} indicating to what amount the vibrations in the modes $\varkappa_1, \varkappa_{2+}, \varkappa_{2-}$ contribute to the polarization $\mathfrak{P}$. The first of these coefficients is given by

[1]) Cf. Math. Encykl. V 22, § 47.

$$Q_1 = -\underset{ab..}{S}\ \underset{\varkappa_1}{\Sigma} \frac{B^{(\varkappa)}}{\left(n + \frac{n_0^{\,2}}{n^{(\varkappa)}}\right)(n - n^{(\varkappa)}) - ing}, \quad . \ . \ . \ . \ . \quad (1)$$

and for Q_{2+} and Q_{2-} we have similar expressions, containing sums which relate to the modes of motion $\varkappa_{2+}$ and $\varkappa_{2-}$. The symbols $\underset{\varkappa_1}{\Sigma}$, $\underset{\varkappa_{2+}}{\Sigma}$, $\underset{\varkappa_{2-}}{\Sigma}$ serve to indicate sums that must be extended either over the modes $\varkappa_1$, or over the modes $\varkappa_{2+}$ or $\varkappa_{2-}$ of one of the groups $a, b, \ldots$, and all these groups have their share in the sums $\underset{ab..}{S}$. The coefficient g, which we shall suppose to have the same value for all terms belonging to one and the same group, represents the influence of the resistances, and may be regarded as a measure of the breadth of the absorption lines. As to the quantities $B^{(\varkappa)}$, these are all real and positive; their value depends on the structure of the particles and is the same for two conjugate vibrations.

Let us denote by $\mathfrak{D}$ the dielectric displacement, so that

$$\mathfrak{D} = \mathfrak{E} + \mathfrak{P}, \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (2)$$

and let us abbreviate by putting

$$1 + Q_1 = S_1,$$
$$1 + Q_{2+} + Q_{2-} = S_2,$$
$$Q_{2+} - Q_{2-} = R.$$

Then we have the relations

$$\left.\begin{aligned} \mathfrak{D}_x &= S_2\, \mathfrak{E}_x + iR\, \mathfrak{E}_y, \\ \mathfrak{D}_y &= S_2\, \mathfrak{E}_y - iR\, \mathfrak{E}_x, \\ \mathfrak{D}_z &= S_1\, \mathfrak{E}_z, \end{aligned}\right\} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (3)$$

which are to be combined with the general equations of the electromagnetic field

$$rot\ \mathfrak{H} = \frac{1}{c}\dot{\mathfrak{D}}, \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (4)$$

$$rot\ \mathfrak{E} = -\frac{1}{c}\dot{\mathfrak{H}}. \quad . \ . \ . \ . \ . \ . \ . \ . \quad (5)$$

In these latter formulae $\mathfrak{H}$ denotes the magnetic force that belongs to the beam of light and alternates with its frequency.

§ 3. We shall examine the propagation of plane waves in a direction lying in the plane xz, and making a positive sharp angle ϑ with the axis of z. Let the variable quantities which determine the state of the system contain the coordinates and the time only in the factor

$$e^{in\left[t-\frac{(\mu)}{c}(z\cos\vartheta+x\sin\vartheta)\right]} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The quantity (μ), which we shall have to determine further on, may properly be called the complex index of refraction, and if

$$(\mu)=\mu-\frac{ich}{n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

μ will be the real index of refraction, and h the index of absorption, the amplitude diminishing in the ratio of 1 to e^{-hl} when a distance l is travelled over.

From (5) we infer

$$\mathfrak{H}_x=-(\mu)\,\mathfrak{E}_y\cos\vartheta,$$
$$\mathfrak{H}_y=(\mu)\,(\mathfrak{E}_x\cos\vartheta-\mathfrak{E}_z\sin\vartheta),$$
$$\mathfrak{H}_z=(\mu)\,\mathfrak{E}_y\sin\vartheta,$$

and then from (4)

$$\left.\begin{aligned}\mathfrak{D}_x&=(\mu)^2\,(\mathfrak{E}_x\cos\vartheta-\mathfrak{E}_z\sin\vartheta)\cos\vartheta,\\ \mathfrak{D}_y&=(\mu)^2\,\mathfrak{E}_y,\\ \mathfrak{D}_z&=-(\mu)^2\,(\mathfrak{E}_x\cos\vartheta-\mathfrak{E}_z\sin\vartheta)\sin\vartheta.\end{aligned}\right\} \quad . \quad . \quad . \quad (8)$$

If here we substitute the values (3), and if we put

$$\frac{(\mu)^2}{S_2}=1+\xi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\frac{S_1}{S_2}=1+\eta,\ \frac{R}{S_2}=\zeta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

we get the relations

$$\mathfrak{E}_x+i\zeta\,\mathfrak{E}_y=(1+\xi)\,(\mathfrak{E}_x\cos\vartheta-\mathfrak{E}_z\sin\vartheta)\cos\vartheta,$$
$$\mathfrak{E}_y-i\zeta\,\mathfrak{E}_x=(1+\xi)\,\mathfrak{E}_y,$$
$$(1+\eta)\,\mathfrak{E}_z=-(1+\xi)\,(\mathfrak{E}_x\cos\vartheta-\mathfrak{E}_z\sin\vartheta)\sin\vartheta.$$

Before proceeding further it will be well to turn the axes of x and z in their plane over an angle ϑ, so that the second of them takes the direction of the rays. Calling the new coordinates x' and z', we have

$$\mathfrak{E}_x=\mathfrak{E}_{x'}\cos\vartheta+\mathfrak{E}_{z'}\sin\vartheta,\quad \mathfrak{E}_z=-\mathfrak{E}_{x'}\sin\vartheta+\mathfrak{E}_{z'}\cos\vartheta,$$

by which our last three equations become

$$\mathfrak{E}_{z'}\sin\vartheta+i\zeta\,\mathfrak{E}_y=\xi\,\mathfrak{E}_{x'}\cos\vartheta,$$
$$-i\zeta\,(\mathfrak{E}_{x'}\cos\vartheta+\mathfrak{E}_{z'}\sin\vartheta)=\xi\,\mathfrak{E}_y,$$
$$(1+\eta)\,\mathfrak{E}_{z'}\cos\vartheta=(\eta-\xi)\,\mathfrak{E}_{x'}\sin\vartheta.$$

Finally, if the value of $\mathfrak{E}_{z'}$ drawn from the third equation is substituted in the first and the second, we get the following relations between the transverse components of the electric force

$$\left.\begin{array}{l}\{\xi(1+\eta\cos^2\vartheta)-\eta\sin^2\vartheta\}\,\mathfrak{E}_{x'}=i\,\zeta\,(1+\eta)\,\mathfrak{E}_y\cos\vartheta,\\ -i\zeta\{(\cos^2\vartheta+\eta)-\xi\sin^2\vartheta\}\,\mathfrak{E}_{x'}=\xi\,(1+\eta)\,\mathfrak{E}_y\cos\vartheta,\end{array}\right\}\quad .\quad (11)$$

from which we deduce by eliminating these components

$$\xi^2(1+\eta\cos^2\vartheta)-\xi(\eta-\zeta^2)\sin^2\vartheta-\zeta^2(\cos^2\vartheta+\eta)=0,\quad .\ (12)$$

or

$$(\xi+\zeta^2)(\xi-\eta)\sin^2\vartheta+(\xi^2-\zeta^2)(1+\eta)\cos^2\vartheta=0.\quad .\ (13)$$

If the frequencies n_0, the frequencies $n^{(\nu)}$ (such as they are under the influence of the external magnetic field), the resistances g and the coefficients $B^{(\nu)}$ are known, the quantities S_1, S_2, R and, by (10), η and ζ will be wholly determined for any chosen value of the frequency n. After having calculated the value of ξ from equation (12), we can deduce from it, first, by means of (9), the complex index of refraction (μ) and then, by means of (7), the real index of refraction μ and the index of absorption h.

Moreover, when ξ has been found, the equations (11) give the ratio between the components $\mathfrak{E}_{x'}$ and $\mathfrak{E}_y$, and also that between $\mathfrak{D}_{x'}$ and $\mathfrak{D}_y$, which has the same value, because, on account of (8),

$$\mathfrak{D}_{x'}=(\mu)^2\,\mathfrak{E}_{x'}.$$

The result is

$$\frac{\mathfrak{D}_y}{\mathfrak{D}_{x'}}=-\,i\,\frac{\zeta(\cos^2\vartheta-\xi\sin^2\vartheta+\eta)}{\xi(1+\eta)\cos\vartheta}\ ;\quad .\ .\ .\ .\ (14)$$

it determines the state of polarization for any beam that can be propagated in the manner specified by (6), for any "principal beam", as we shall say.

Whereas the component $\mathfrak{E}_{z'}$ may very well be different from zero, the equations (8) show that $\mathfrak{D}_{z'}=0$, as might have been expected beforehand. Hence, at every point of the system, the extremity of the vector $\mathfrak{D}$ describes an ellipse in a plane perpendicular to the direction of propagation. This line, which shows us the state of polarization of the principal beam, may be called its "characteristic ellipse"; equation (14) determines, not only its shape and position, but also the direction in which it is described.

It must further be noticed that, on account of the relation (2), the ratio $\frac{\mathfrak{P}_y}{\mathfrak{P}_{x'}}$ is equal to the ratios $\frac{\mathfrak{D}_y}{\mathfrak{D}_{x'}}$ and $\frac{\mathfrak{E}_y}{\mathfrak{E}_{x'}}$, the equality of which has already been mentioned. Hence, remembering that the components of $\mathfrak{P}$ are proportional to those of the displacement of the equivalent electron in a particle, one easily sees that, while a particle is made to vibrate in its different modes of motion (in the way determined by the sums in Q_1, Q_{2+} and Q_{2-}) the projection of the equivalent electron on the wave-front moves in an ellipse of the same form and position as the characteristic ellipse, and described in the same direction.

As (12) gives two values of ξ, there are *two* principal beams, differing from each other by their states of polarization, their velocities of propagation and their indices of absorption. All these details depend on the angle ϑ and in general on the value chosen for the frequency n.

§ 4. The magnetic components belonging to one of the members of a definite group a, b, or c etc. lie within a narrow strip of the spectrum, which we shall likewise denote by the letter a, b, or c etc. We shall confine ourselves to the propagation of light belonging to one of these parts, say to a, and we shall assume that the distances of this part from the parts b, c etc. are very great in comparison both with the breadth of a and with that of b, c etc. On this assumption the part

$$-\sum_{'1} \frac{B^{(\prime)}}{\left(n+\frac{n_0^2}{n^{(\prime)}}\right)(n-n^{(\prime)})-ing}$$

of (1), which relates to the group b for instance, may be simplified by writing n_{0a} instead of n, n_{0b} for each $n^{(\prime)}$, and

$$n^2_{0a} - n^2_{0b}$$

for each denominator. The result is

$$\frac{1}{n^2_{0b}-n^2_{0a}} \sum_{'1} B^{(\prime)}.$$

The quantities Q_{2+}, Q_{2-} may be treated in the same way and we can repeat for the groups c, d, .. what we have done with b. If we assume that for each group [1])

$$\sum_{'1} B^{(\prime)} = \sum_{'2+} B^{(\prime)} + \sum_{'2-} B^{(\prime)} = 2 \sum_{'2+} B^{(\prime)}, \quad . \qquad . \quad (15)$$

the parts contributed by the groups b, c, .. taken together, to the quantities Q_1, Q_{2+}, Q_{2-} may be represented by s, $\frac{1}{2}s$, $\frac{1}{2}s$, where s is a real quantity, constant through the region a. As for the parts due to the group a, in these we may replace every denominator by

$$2\,n_0\,(n - n^{(\prime)}) - i\,n_0\,g,$$

understanding by n_0 the value n_{0a}. We shall simplify still further by assuming that the group a is a magnetic triplet, so that it comprises but one mode of vibration $\varkappa_1$, one mode $\varkappa_{2+}$ and one $\varkappa_{2-}$. The frequencies of the free vibrations in these modes are

$$n_1 = n_0, \quad n_{2+} = n_0 + \nu, \quad n_{2-} = n_0 - \nu, \quad . \quad . \quad . \quad (16)$$

[1]) See § 51 of the Article in the Math. Encykl. cited above.

where ν has a value proportional to the strength of the field. Moreover, if the relation (15) is supposed also to hold for the group a, we may put

$$B^{(\varkappa_2+)} = B^{(\varkappa_2-)} = a n_0 \quad , \quad B^{(\varkappa_1)} = 2\, a n_0 \ ,$$

with a positive constant a, so that we find

$$Q_1 = s - \frac{2a}{2(n-n_1)-ig} \ ,$$

$$Q_{2+} = \tfrac{1}{2}\, s - \frac{a}{2(n-n_{2+})-ig} \ ,$$

$$Q_{2-} = \tfrac{1}{2}\, s - \frac{a}{2(n-n_{2-})-ig} \ .$$

By this the values of S_1, S_2 R, η and ζ have likewise become known. It is easily seen that $1 + s = \mu_0{}^2$, when μ_0 is the real index of refraction that would be found for $n = n_0$ if the particles were not put in motion in the modes of the group a, but only in those of the groups $b, c, \ldots$ If, finally, we put

$$a = \mu_0{}^2\, \beta \ ,$$

$$u_1 = \frac{\beta}{2(n-n_1)-ig} = \beta\, \frac{2(n-n_1)+ig}{4(n-n_1)^2+g^2} \ , \quad . \quad . \quad . \quad . \quad (17)$$

$$u_{2+} = \beta\, \frac{2(n-n_{2+})+ig}{4(n-n_{2+})^2+g^2} \ , \ u_{2-} = \beta\, \frac{2(n-n_{2-})+ig}{4(n-n_{2-})^2+g^2} \ , \quad . \quad . \quad (18)$$

we get

$$\eta = \frac{u_{2+}+u_{2-}-2u_1}{1-(u_{2+}+u_{2-})} \ , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$\zeta = \frac{u_{2-}-u_{2+}}{1-(u_{2+}+u_{2-})} \ , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and, after having calculated ξ by means of (12),

$$(\mu)^2 = \mu_0{}^2 \{1 - (u_{2+} + u_{2-})\} (1 + \xi) \quad . \quad . \quad . \quad . \quad (21)$$

In the large majority of cases the absorption, even at the place in the spectrum where it is strongest, is very feeble along a distance of a wave-length. Consequently, the quantities u are very much smaller than 1. Equations (19) and (20) show that η, ζ are very small, and by (12) ξ is so likewise, so that (19), (20) and (21) may be written [1])

$$\eta = u_{2+} + u_{2-} - 2\, u_1 ,$$

$$\zeta = u_{2-} - u_{2+} ,$$

$$(\mu) = \mu_0 \left\{1 - \frac{1}{2}(u_{2+} + u_{2-}) + \frac{1}{2}\, \xi\right\} \quad . \quad . \quad . \quad . \quad (22)$$

[1]) Many of VOIGT's equations are free from these approximations. See also § 11 below.

§ 5. By putting $\vartheta = 0$ or $\frac{1}{2}\pi$, we are led back to the well known theory of the ZEEMAN-effect for directions parallel or perpendicular to the lines of force. Indeed, from (13) we deduce for the first case

$$\xi = \pm \zeta \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and for the second

$$\xi = -\zeta^2 \text{ or } \xi = \eta \, .$$

Further, when $\vartheta = 0$, we have by (14)

$$\frac{\mathfrak{D}_y}{\mathfrak{D}_{x'}} = \mp i,$$

so that in this case, whatever be the value of n, one of the principal beams, corresponding to the upper sign, is characterized by a left-handed, and the other by a right-handed circular polarization. This will require no further explanation. We may, however, say some words about the rotation of the plane of polarization that is observed along the lines of force, and especially about its amount for $n = n_0$. In this case

$$u_1 = i\frac{\beta}{g}, \; u_{2+} = \beta\,\frac{-2\,\nu + i\,g}{4\,\nu^2 + g^2}, \; u_{2-} = \beta\frac{2\,\nu + i\,g}{4\,\nu^2 + g^2}, \quad . \quad . \quad (24)$$

$$\zeta = \frac{4\,\beta\,\nu}{4\,\nu^2 + g^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Hence, according to the formulae (23) and (22), the complex index of refraction is

$$(\mu_+) = \mu_0 \left\{ 1 - i\,\frac{\beta\,g}{4\,\nu^2 + g^2} + \frac{2\,\beta\,\nu}{4\,\nu^2 + g^2} \right\}$$

for the left-handed beam, and

$$(\mu_-) = \mu_0 \left\{ 1 - i\,\frac{\beta\,g}{4\,\nu^2 + g^2} - \frac{2\,\beta\,\nu}{4\,\nu^2 + g^2} \right\}$$

for the right-handed one.

Comparing these expressions with (7), we see that the two rays are equally absorbed, the index of absorption being for both of them

$$h = \frac{n_0\,\mu_0\,\beta\,g}{c\,(4\,\nu^2 + g^2)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

but that their real indices of refraction are unequal. Their difference is given by

$$\mu_+ - \mu_- = \mu_0\,\frac{4\,\beta\,\nu}{4\,\nu^2 + g^2},$$

and, corresponding to it, there is a rotation of the plane of polarization amounting to

$$\psi = \frac{n_0}{2c}(\mu_- - \mu_+) = -\frac{n_0\,\mu_0}{c}\,\frac{2\,\beta\,\nu}{4\,\nu^2 + g^2} \quad . \quad . \quad . \quad . \quad (27)$$

per unit of length.

As for the case $\vartheta = \frac{1}{2}\pi$, it will suffice to mention here that the two principal beams are rectilinearly polarized. For the one, whose vibrations are parallel to the lines of force, the maximum of absorption, which occurs when $n = n_0$, has an intensity determined by

$$h_I = \frac{n_0 \mu_0 \beta}{cg} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

For the other beam, whose vibrations are at right angles to the lines of force, the absorption for $n = n_0$ may be calculated by the formula

$$h_{II} = \frac{n_0 \mu_0 \beta g}{c(4\nu^2 + g^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

§ 6. Let us now pass on to consider the propagation in a direction making an angle ϑ with the lines of force. In doing so we shall, however, exclude cases in which this angle is very near 0 or $\frac{1}{2}\pi$, because for these directions some terms which may in general be omitted, might become of influence [1]).

When both $\sin\vartheta$ and $\cos\vartheta$ are large in comparison with the small quantities occurring in our calculations, formula (12) may be replaced by

$$\xi^2 - \xi\,\eta \sin^2\vartheta - \zeta^2 \cos^2\vartheta = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

so that

$$\xi = \frac{1}{2}\,\eta \sin^2\vartheta \pm \sqrt{\frac{1}{4}\,\eta^2 \sin^4\vartheta + \zeta^2 \cos^2\vartheta} \quad . \quad . \quad . \quad (31)$$

At the same time (14) becomes

$$\frac{\mathfrak{D}_y}{\mathfrak{D}_{x'}} = -\,i\,\frac{\zeta \cos\vartheta}{\xi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

We have, therefore, when the quantities relating to the two principal beams are distinguished by the indices I and II,

$$\left(\frac{\mathfrak{D}_y}{\mathfrak{D}_{x'}}\right)_I \left(\frac{\mathfrak{D}_y}{\mathfrak{D}_{x'}}\right)_{II} = -\frac{\zeta^2 \cos^2\vartheta}{\xi_I\,\xi_{II}},$$

or, on account of (30),

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[1]) Notwithstanding this, we shall find that, if we put $\vartheta = 0$ or $\vartheta = \frac{1}{2}\pi$, we can deduce from some of our formulae results that are true for a propagation along the lines of force or at right angles to these lines.

$$\left(\frac{\mathfrak{D}_y}{\mathfrak{D}_{x'}}\right)_I \left(\frac{\mathfrak{D}_y}{\mathfrak{D}_{x'}}\right)_{II} = 1.$$

This means that one of the characteristic ellipses can be considered as the reflected image of the other with respect to a line bisecting the angle $X'OY$, a rule which also applies to the direction of motion in the two cases.

The imaginary parts of u_1, u_{2+}, u_{2-}, on which the absorption ultimately depends, have their maximum values for $n = n_1, n_{2+}, n_{2-}$, and have diminished to a small part only of the maximum value, when $|n-n_1|$, $|n-n_{2+}|$ or $|n-n_{2-}|$ is equal to a moderate multiple of the coefficient g.

From this we can infer that, when $\mathfrak{v}$ is sufficiently great in comparison with g, there will be three maxima of absorption at the points of the spectrum determined by (16), and that, if $\mathfrak{v}$ greatly surpasses g, we have three absorption bands that are completely separated, the body being practically transparent to rays of the interlying wave-lengths. At a point where the imaginary part of one of the quantities u_1, u_{2+}, u_{2-} has its maximum value, both the real and the imaginary parts of the two other quantities may, under these circumstances, be neglected in comparison with that maximum value. For $n = n_{2+}$, for instance, we may put

$$u_1 = 0 \quad , \quad u_{2+} = i\frac{\beta}{g} \quad , \quad u_{2-} = 0,$$

$$\eta = i\frac{\beta}{g} \quad , \quad \zeta = -i\frac{\beta}{g},$$

by which the roots of equation (30) become

$$\xi = -\eta \cos^2 \vartheta \text{ and } \xi = \eta.$$

Choosing the first root, we find

$$\frac{\mathfrak{D}_y}{\mathfrak{D}_{x'}} = -\frac{i}{\cos\vartheta},$$

$$(\mu) = \mu_0 \left\{1 - \frac{1}{2} i \frac{\beta}{g} (1 + \cos^2 \vartheta)\right\},$$

and if we take the second

$$\frac{\mathfrak{D}_y}{\mathfrak{D}_{x'}} = i \cos\vartheta,$$

$$(\mu) = \mu_0.$$

It appears from these results that only the first of the two principal rays is absorbed, and that the axes of its characteristic ellipse are parallel to OY and OX', being to each other in the ratio of 1 to $\cos\vartheta$; this ellipse can be considered as the projection on the wave-front of a circle whose plane is at right angles to the lines of force. If we imagine a point moving along this circle in the direction in which the equivalent electron moves in the mode of motion $\varkappa_2+$, the projection of this motion on the wave-front indicates the direction in which the characteristic ellipse is described.

All this agrees with the elementary theory of the ZEEMAN-effect and similar remarks apply to the other outer component of the triplet.

§ 7. We shall now enter upon some more details concerning the propagation of rays whose frequency n_0 corresponds to the middle point of the triplet. In order not to exclude cases in which the components of the triplet are not neatly separated or hardly so, we shall *not* assume that ν is much greater than g.

For $n = n_0$ we may use the values (24) and (25), whereas

$$\eta = -\frac{2i\nu}{g}\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Hence, if we abbreviate by putting

$$\frac{\nu}{g}\frac{\sin^2\vartheta}{\cos\vartheta} = q, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

we find from (31) and (32)

$$\frac{\mathfrak{D}_{x'}}{\mathfrak{D}_y} = q \pm i\sqrt{1-q^2}\,, \quad . \quad . \quad . \quad . \quad (35)$$

and

$$\xi = -i\frac{\mathfrak{D}_{x'}}{\mathfrak{D}_y} \cdot \frac{4\beta\nu\cos\vartheta}{4\nu^2+g^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

In discussing these results we shall suppose the quantities ν and q, which are related to each other in the manner shown by (34), to be positive [1]).

The nature of the phenomena that will be observed greatly depends on whether q is greater or less than 1. Both cases may occur. Indeed, if we determine an angle ϑ_1 by the equation

$$\frac{\nu}{g}\frac{\sin^2\vartheta_1}{\cos\vartheta_1} = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

as we can always do, whatever be the value of $\frac{\nu}{g}$, we shall have $q >$ or < 1, according as $\vartheta >$ or $< \vartheta_1$.

[1]) The quantity ν is positive when the magnetic field has the direction of the positive axis of z (so that the direction of the rays makes a sharp angle with the lines of force) and when, besides, the right- and left-handed circularly polarized components of the spectral line in the longitudinal ZEEMAN-effect have the ordinary relative positions. The sign of ν is changed both by an inversion of this relative position and by an inversion of the field.

In the first case, i.e. when the angle between the ray and the lines of force is not too small, the ratio (35) is real, namely

$$\frac{\mathfrak{D}_{x'}}{\mathfrak{D}_y} = q \mp \sqrt{q^2 - 1} \quad \ldots \ldots \ldots \quad (38)$$

The vibrations of the two principal beams will therefore be rectilinear, the angles χ_I and χ_{II} which they make with the axis OX', and which we shall reckon positive in the direction from OX' towards OY, being given by

$$\sin 2\chi_I = \sin 2\chi_{II} = \frac{1}{q}, \quad \ldots \ldots \ldots \quad (39)$$

Both angles lie between 0 and $\frac{1}{2}\pi$. and the smaller of the two, which we shall call χ_I, corresponds to the under sign in (38), so that we may write

$$\left(\frac{\mathfrak{D}_{x'}}{\mathfrak{D}_y}\right)_I = q + \sqrt{q^2 - 1},$$

$$\left(\frac{\mathfrak{D}_{x'}}{\mathfrak{D}_y}\right)_{II} = q - \sqrt{q^2 - 1}.$$

Equation (36) shows that ξ has now an imaginary value for both principal beams, and, since the same is true of $u_{2+} + u_{2-}$, we see from (22) and (7) that the two beams have the same real index of refraction μ_0 (and therefore the same velocity of propagation), but different indices of absorption, namely

$$h_I = \frac{n_0 \mu_0 \beta}{c(4v^2 + g^2)} \{g + 2(q + \sqrt{q^2 - 1})\, v \cos \vartheta\}, \quad \ldots \quad (40)$$

$$h_{II} = \frac{n_0 \mu_0 \beta}{c(4v^2 + g^2)} \{g + 2(q - \sqrt{q^2 - 1})\, v \cos \vartheta\} \quad \ldots \quad (41)$$

It appears from this that the absorption is strongest for the beam whose vibrations make the smaller angle with the lines of force. This might have been expected on the ground of the elementary theory of the ZEEMAN-effect.

The difference between the expressions $q + \sqrt{q^2-1}$ and $q - \sqrt{q^2-1}$ which occur in h_I and h_{II} increases as q becomes greater. Now, if for a fixed value of $\frac{v}{g}$, the angle ϑ is made to approach the limit $\frac{1}{2}\pi$, (34) shows that q increases indefinitely. When it has become very great, we may replace $q + \sqrt{q^2-1}$ by $2q$, and since $q \cos \vartheta$ tends towards the value $\frac{v}{g}$, as may be seen from (34), we have at the limit

$$h_I = \frac{n_0 \mu_0 \beta}{cg},$$

agreeing with the value (28) which we have given for a ray perpendicular to the lines of force and having its vibrations along these lines. At the same time $q - \sqrt{q^2-1}$ may be replaced by $\frac{1}{2q}$, so that h_{II} approaches the limit (29).

On the other hand, when ϑ is made smaller and ultimately becomes equal to ϑ_1 ($q = 1$), both h_I and h_{II} have the limiting value

$$h = \frac{n_0 \mu_0 \beta}{c} \frac{2v \cos \vartheta_1 + g}{4v^2 + g^2}, \quad . \quad . \quad . \quad . \quad . \quad (42)$$

or, if (37) is taken into account,

$$h = \frac{n_0 \mu_0 \beta}{cg} \frac{2v^2 \sin^2 \vartheta_1 + g^2}{4v^2 + g^2}.$$

This lies between the values (28) and (29).

As for the directions of the vibrations in the principal beams, these are determined by $\chi_I = 0$ and $\chi_I = \frac{1}{4}\pi$ in the extreme cases $\vartheta = \frac{1}{2}\pi$ and $\vartheta = \vartheta_1$. The former of these results was to be expected, and the latter shows that for $\vartheta = \vartheta_1$ both directions coincide with the line bisecting the angle $X'OY$. We shall denote this line by OL.

§ 8. It appears from what precedes that for $\vartheta > \vartheta_1$ the state of things is wholly different from the one existing when $\vartheta = 0$, which is characterized by a circular polarization of the principal beams. The transition between these phenomena is formed by those which are observed when $\vartheta < \vartheta_1$.

In this case $q < 1$, so that we may put

$$q = \frac{v}{g} \frac{\sin^2 \vartheta}{\cos \vartheta} = \cos \omega ,$$

by which some of our formulae are simplified. The mode of vibration of the principal beams is determined by the relation

$$\frac{\mathfrak{D}_{x'}}{\mathfrak{D}_y} = e^{\pm i \omega} , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

following from (35), and we may therefore say that if we have at some point of the system

$$\mathfrak{D}_y = a e^{i(nt+p)},$$

with real a and p, the other component of the dielectric displacement will be given by

$$\mathfrak{D}_{x'} = a e^{i(nt+p\pm\omega)}.$$

Taking the real parts of these expressions, namely

$$\mathfrak{D}_{x'} = a \cos (n\,t + p \pm \omega) \quad , \quad \mathfrak{D}_y = a \cos (n\,t + p),$$

we may conclude that both principal beams are elliptically polarized, with the same characteristic ellipse, one of whose axes has the direction of the line OL mentioned above. The difference between the two beams lies merely in this, that the characteristic ellipses are described in opposite directions. In order to see this, we have only to observe that, if $\varkappa$ is the angle between $\mathfrak{D}$ and OX',

$$tg\,\varkappa = \frac{\cos (nt + p)}{\cos (nt + p \pm \omega)},$$

$$\frac{1}{\cos^2 \varkappa} \frac{d\varkappa}{dt} = \pm \frac{n \sin \omega}{\cos^2 (nt + p \pm \omega)}.$$

In the beam to which the upper signs refer, the direction of the motion corresponds to that of propagation. For this reason we shall distinguish all quantities relating to it by the index $+$ and those which relate to the other beam by the index $-$.

We need hardly add that the characteristic ellipse coincides with the straight line OL when $\vartheta = \vartheta_1$, and that it becomes a circle when $\vartheta = 0$.

We can further deduce from (43), (36) and (22)

$$\left.\begin{aligned} \mu_+ &= \mu_0 \left(1 + \frac{2\beta v \cos \vartheta}{4v^2 + g^2} \sin \omega \right), \\ \mu_- &= \mu_0 \left(1 - \frac{2\beta v \cos \vartheta}{4v^2 + g^2} \sin \omega \right), \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (44)$$

$$h_+ = h_- = \frac{n_0 \mu_0 \beta}{c(4v^2 + g^2)} (g + 2v \cos \vartheta \cos \omega),$$

showing that, for $n = n_0$ and for any direction between $\vartheta = \vartheta_1$ and $\vartheta = 0$, the two principal beams are equally absorbed, just like the two circularly polarized beams in the extreme case $\vartheta = 0$. The common index of absorption, for which we shall henceforth write h, diminishes as ϑ increases; for $\vartheta = \vartheta_1$ $(\omega = 0)$ it takes the value (42), and for $\vartheta = 0$ $(\omega = \frac{1}{2}\pi)$ the value (26). How far these extreme values are different, depends on the relative magnitude of v and g.

§ 9. The difference between the velocities with which a left-handed and a right-handed circularly polarized beam travel along the lines of force, leads to the well known rotation of the plane of polarization. On account of the unequality of the velocities of propagation determined by (44), there is a similar rotation in the interval from $\vartheta = 0$ to $\vartheta = \vartheta_1$, with some difference in the details, however, owing to

23*

the fact that the principal beams are not circularly, but elliptically polarized.

Let one of the beams be represented by

$$\mathfrak{D}_{x'} = a\, e^{-h z' + i\left[n_0\left(t - \frac{\mu_+ z'}{c}\right) + p + \omega\right]},$$

$$\mathfrak{D}_y = a\, e^{-h z' + i\left[n_0\left(t - \frac{\mu_+ z'}{c}\right) + p\right]},$$

and the other by similar expressions containing μ_- instead of μ_+ and $-\omega$ instead of $+\omega$. Or, in other terms, let us write for one beam

$$\mathfrak{D}_{x'} = a\, e^{-hz'} \cos\left[n_0\left(t - \frac{\mu_+ z'}{c}\right) + p + \omega\right],$$

$$\mathfrak{D}_y = a\, e^{-hz'} \cos\left[n_0\left(t - \frac{\mu_+ z'}{c}\right) + p\right],$$

and for the other

$$\mathfrak{D}_{x'} = a\, e^{-hz'} \cos\left[n_0\left(t - \frac{\mu_- z'}{c}\right) + p - \omega\right],$$

$$\mathfrak{D}_y = a\, e^{-hz'} \cos\left[n_0\left(t - \frac{\mu_- z'}{c}\right) + p\right].$$

Then, compounding the two, and putting

$$\psi = \frac{n_0}{2c}(\mu_- - \mu_+),$$

we get

$$\mathfrak{D}_{x'} = 2a\, e^{-hz'} \cos(\psi z' + \omega) \cos\left[n_0 t - \frac{n_0}{2c}(\mu_+ + \mu_-) z' + p\right],$$

$$\mathfrak{D}_y = 2a\, e^{-hz'} \cos \psi z' \cos\left[n_0 t - \frac{n_0}{2c}(\mu_+ + \mu_-) z' + p\right].$$

Hence, at any point z', the resultant vibration is rectilinear, and its amplitude, considered as a vector, may be represented by

$$2\, a\, e^{-hz'}\, \mathfrak{A},$$

$\mathfrak{A}$ being a vector in the wave front with the components

$$\mathfrak{A}_{x'} = \cos(\psi z' + \omega) \quad , \quad \mathfrak{A}_y = \cos \psi z'.$$

If this vector is drawn from a fixed point, its extremity describes an ellipse when z' is made to increase continually. The corresponding rotation of the plane of polarization is similar to the one observed in more familiar cases inasmuch as it goes on in a constant direction, but when z' is made to increase at a constant rate, the velocity of the rotation is variable. Its changes are determined by the rule that the vector $\mathfrak{A}$ describes equal areas in equal times; consequently, the velocity of rotation is greatest when the vibration has the direction of the minor axis of the ellipse.

Let $\varkappa$ be the angle between the vector $\mathfrak{A}$ and the axis of x'. Then the above formulae give the following value for the rotation of the plane of polarization per unit of length

$$\frac{d\varkappa}{dz'} = \frac{\psi \sin \omega}{1 + \cos \omega \cos (2\psi z' + \omega)}, \quad . \quad . \quad . \quad . \quad . \quad (45)$$

an expression that is constant only for $\vartheta = 0$ $(\omega = \frac{1}{2}\pi)$. For any other value of the angle ϑ we may also consider the *mean* value of the rotation. As the vector $\mathfrak{A}$ makes a complete revolution while z' increases by $\frac{2\pi}{\psi}$, we find for this mean rotation

$$\psi = \frac{n_0}{2c}(\mu_- - \mu_+) = -\frac{n_0 \mu_0}{c} \cdot \frac{2\beta\, v \cos \vartheta \sin \omega}{4v^2 + g^2}.$$

It takes the value (27) for $\vartheta = 0$ $(\omega = \frac{1}{2}\pi)$ and vanishes for $\vartheta = \vartheta_1$ $(\omega = 0)$.

It must be noticed that, even in the neighbourhood of this latter direction of propagation, whereas the mean rotation per unit of length becomes very small, the rotation $\frac{d\varkappa}{dz'}$, may very well have an appreciable magnitude, if the direction of vibration be properly chosen. In fact, the maximum value of (45) is

$$\frac{\psi \sin \omega}{1 - \cos \omega} = \psi \cot \tfrac{1}{2}\omega = -\frac{n_0 \mu_0}{c} \cdot \frac{4\beta v \cos \vartheta \cos^2 \frac{1}{2}\omega}{4v^2 + g^2},$$

and this can be of the same order of magnitude as (27), even for a value of ϑ very near ϑ_1 $(\omega = 0)$.

The ellipse described by the extremity of the vector $\mathfrak{A}$ is similar in form and position to the characteristic ellipse of which we have spoken in § 8.

§ 10. Summing up the above results (and always confining ourselves to the particular frequency n_0) we may say that in the interval between $\vartheta = \vartheta_1$ and $\vartheta = \frac{1}{2}\pi$ the phenomena are in the main of the same kind as the true transverse ZEEMAN-effect that is observed at right angles to the lines of force; the principal beams present a rectilinear polarization and differ from each other by their indices of absorption, whereas the velocity of propagation is the same for both of them. For values of ϑ smaller than ϑ_1, on the contrary, the effect is similar to the true longitudinal one. In this interval it is only by their velocities of propagation, but not by the intensity of the absorption, that the two beams are different.

We can diminish ϑ_1 by making v greater. Hence, the region of the transverse effect expands when the magnetic field is strengthened, and finally, when $\frac{v}{g}$ has become very large, i. e. when the distance between the magnetic components far surpasses their breadth, the longitudinal effect is confined to a very narrow interval.

We may add that any definite direction of propagation may be made to fall within the limits of the transverse effect by properly strengthening the field. At the same time, the phenomena become more and more like those that are observed at right angles to the lines of force. Indeed, when v is made greater, q increases continually, as may be seen from (34). The angle χ_I whose value is given by (39), tends towards the limit 0, so that finally the two principal directions of vibration will be perpendicular to each other, the second of them being also normal to the field. Formula (41) shows that, at the limit, the index of absorption of the second beam becomes 0, and equation (40) may be replaced by

$$h_I = \frac{n_0 \mu_0 \beta}{c(4v^2+g^2)}(g + 4gv \cos \vartheta),$$

or, if we take into account the formula (34) and fact that v becomes very much greater than g, by

$$h_I = \frac{n_0 \mu_0 \beta}{cg} \sin^2 \vartheta$$

(cf. formula (28)). These conclusions may be compared with well known results obtained in the elementary theory of the ZEEMAN-effect in the radiated light, namely that the direction of vibration lies in the plane passing through the ray and a line of force, and that the amplitude is proportional to $\sin \vartheta$.

§ 11. From a theoretical point of view it is interesting to examine somewhat more closely the special case in which the rays have the direction determined by the angle ϑ_1. Our formulae would lead us to infer that in this case the two principal beams are polarized in the same way, so that after all there would be only one kind of vibration that can be propagated. Of course this cannot be true. The difficulty can be overcome by pushing our approximations a step farther than has been done in the preceding calculations, in which we have omitted all quantities that are of an order higher than the first with respect to u_1, u_{2+}, u_{2-}.

If we use for η and ζ the exact formulae (19) and (20) — again confining ourselves to the case $n = n_0$ — we still find an imaginary value for the ratio $\frac{\zeta}{\eta}$. We can therefore define a real angle in the first quadrant by the equation

$$\frac{\sin^2 \vartheta_1}{\cos \vartheta_1} = -2i \frac{\zeta}{\eta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

and it is for the direction of propagation determined by *this* angle, that we shall perform the following calculations.

It follows from (33) that the angle ϑ_1 which we introduce now becomes equal to the angle originally denoted by the same symbol when we take for η and ζ their former values. These are a little different from those which we must now ascribe to these quantities, and therefore the direction of propagation assumed in our present calculation does not exactly coincide with the direction which we considered in the preceding article as the boundary between the regions of the longitudinal and the transverse effect. The deviation of one direction from the other is, however, insignificant; it will even be found to be small in comparison with the new terms that will now become of importance.

We shall again begin with the determination of ξ. For this purpose we have to use equation (12), for which, on account of (46), we may write

$$\xi^2 + 2i\,\xi\zeta \cos \vartheta_1 - \zeta^2 \cos^2 \vartheta_1 = -\xi^2\,\eta \cos^2 \vartheta_1 - \xi\zeta^2 \sin^2 \vartheta_1 + \eta\zeta^2.$$

Here, the terms on the left-hand side, the only ones with which we were concerned in our first approximation, form a complete square, and this is the reason why we found two principal beams identical with each other.

In the approximation now required we must retain the terms on the right-hand side, but it will suffice to substitute in them the values of ξ, η, ζ obtained in our former calculation. Distinguishing these by the index 0, we get the following equation for ξ

$$\xi + i\zeta \cos \vartheta_1 = \pm \sqrt{-\xi_0^2\eta_0 \cos^2 \vartheta_1 - \xi_0\zeta_0^2 \sin^2 \vartheta_1 + \eta_0\zeta_0^2}\,. \quad (47)$$

As to the three quantities ξ_0, η_0, ζ_0, the last of them is given by (25), and we have, in virtue of (46),

$$\eta_0 = -2i\zeta_0 \frac{\cos \vartheta_1}{\sin^2 \vartheta_1},$$

and, on account of (36), since the value of $\frac{\mathfrak{D}_{z'}}{\mathfrak{D}_y}$ was 1,

$$\xi_0 = -i\frac{4\beta\nu\cos\vartheta_1}{4\nu^2+g^2} = -i\,\zeta_0\cos\vartheta_1.$$

Substituting these values in (47), we get

$$\xi = -i\,\zeta\cos\vartheta_1 \pm (1-i)\,\zeta_0\,\sigma, \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

where we have put

$$\sigma = \frac{1+\cos^2\vartheta_1}{\sin\vartheta_1}\sqrt{\tfrac{1}{2}\zeta_0\cos\vartheta_1}.$$

The new term to which our present approximation leads us, namely $(1-i)\,\zeta_0\sigma$, is thus seen to be of the order of magnitude $\zeta_0{}^{3/2}$, so that it will be allowable to neglect quantities of the order $\zeta_0{}^2$. Such are: first, the difference between the values of ϑ_1 given by (37) and (46), and secondly, the change that would be brought about in our results if we took into account the difference between η,ζ and η_0,ζ_0. Moreover, we may neglect the products $\xi\zeta$, $\eta\zeta$, and $\xi\eta$ in formula (14), and we may again use for the complex index of refraction the equation (22), substituting in it, on account of (24), (25) and (37),

$$\frac{1}{2}(u_{2+} + u_{2-}) = \frac{i\beta g}{4\nu^2+g^2} = \frac{ig}{4\nu}\zeta_0 = \frac{1}{4}\,i\,\zeta_0\frac{\sin^2\vartheta_1}{\cos\vartheta_1}.$$

Finally we deduce from (48)

$$\xi = -i\,\zeta_0\cos\vartheta_1 \pm (1-i)\,\zeta_0\,\sigma,$$

from (14)

$$\frac{\mathfrak{D}_y}{\mathfrak{D}_{x'}} = 1 \mp (1+i)\frac{\sigma}{\cos\vartheta_1},$$

and from (22)

$$(\mu) = \mu_0\left\{1 - \frac{1}{4}\,i\,\zeta_0\frac{1+\cos^2\vartheta_1}{\cos\vartheta_1} \pm \frac{1}{2}(1-i)\,\zeta_0\,\sigma\right\}.$$

The result is that in this special case, like in the general one, there are two distinct principal beams, with different characteristic ellipses, both deviating somewhat from the straight line OL mentioned at the end of § 7. Between the two there is a slight difference, both in velocity of propagation and in index of absorption.

The regions of the longitudinal and the transverse ZEEMAN-effect are thus found not to be sharply separated from each other, as we concluded in § 10, but to overlap to a certain extent. This shows that, strictly speaking, the consideration of additional terms of the order $\zeta_0{}^{3/2}$ is necessary, not only in the case $\vartheta = \vartheta_1$, but also for other directions of propagation lying within a small angle on both sides of the direction determined by the angle ϑ_1.