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CONTENTS.

Mathematics. — *"On the orbits of a function obtained by infinitesimal iteration in its complex plane."* By M. J. VAN UVEN. (Communicated by Prof. W. KAPTEYN).

(Communicated in the meeting of November 27, 1909).

When a function $y = \varphi(x)$ is iterated, each iteration $y_n = \varphi_n(x)$ will give rise to a conform representation of the complex planes of x and y_n.

If we suppose $y = \varphi(z)$ to be built up by means of infinitesimal iteration of the function $\lim_{m=\infty} y_{\frac{1}{m}} = \lim_{m=\infty} \varphi_{\frac{1}{m}}(x)$, so that y_n has also a meaning for broken and unmeasurable values of n, then the conform representation of $y = \varphi(x)$ will gradually appear out of the identity belonging to $y_0 = \varphi_0(x) = x$.

We now regard a plane V_0 as complex plane of the quantity x and we place the complex plane V_1 of the quantity $y = \varphi(x)$ parallel to V_0 at a distance h and in such a way, that the real axes and the imaginary axes are each other's orthogonal projection. Then to each point x of V_0 are conjugated by means of the function $y = \varphi(x)$ one or more points y of V_1. By connecting corresponding points x and y by rays a congruence of rays is formed which can serve as the image of the function $y = \varphi(x)$.

For the case $y = \varphi(x) \equiv x$ we should obtain in this way the congruence of rays formed by all the normals on the planes V_0 and V_1 as representative of the identity.

If now we let the function $y = \varphi(x)$ gradually arise from the identity, then to each stage of the generating process a definite congruence of rays will belong. All these congruences form together a complex of rays. It is clear, that the formation of the function $y = \varphi(x)$ will now be represented by this complex of rays.

Let us first examine the complex cones of the points of V_0. Each point $x = u + iv$ of this plane is the vertex of a cone counting in any case the normal in x on V_0 among its generatrices; this edge namely intersects the plane V_1 in $y = u + iv = x$.

The section of this complex cone with V_1 will pass through the point $z = x$ and all points representing the values taken by $y_n = \varphi_n(x)$ when n increases from 0 to 1. So this section also gives us a representation of the generating process of $\varphi(x)$. It goes without saying that we can continue the iteration also past $y = \varphi(x)$ and likewise that we can also regard negative values of n. The whole of the complex cone embraces in fact all functions $y_n = \varphi_n(x)$, where n varies from $-\infty$ to $+\infty$. Also the section regarded as a whole will contain all the values of the function $y_n = \varphi_n(x)$, where x is constant and n varies from $-\infty$ to $+\infty$. Each value of x possesses its own complex cone and therefore also its own section. We shall indicate this section by the orbit $x \rightarrow y_n$.

We might also have indicated the increase of $\varphi(x)$ by allowing the plane V_1 to grow gradually out of V_0 and that by allowing the distance of the planes to increase regularly from 0 to h, so that $\varphi_n(x)$ is represented in the plane V_n at a height nh above V_0. Let us then suppose in each plane V_n the image $y_n = \varphi_n(x)$ belonging to some initial-point $x = u + iv$ to be constructed, then all these points will form in their regular succession a twisted curve. Each of the ∞^2 points x of V_0 gives rise to a suchlike *twisted curve* and the function $y = \varphi(x)$ with its different stages of development is thus represented by a *congruence of twisted curves*.

It is clear that the orthogonal projection of the twisted curve of x on the plane V_1 coincides with the orbit $x \rightarrow y_n$.

We shall for the present occupy ourselves only with the study of such an orbit $x \rightarrow y_n$.

To find the orbit $x \rightarrow y_n$ we have but to solve the functional equation of ABEL. We have namely to find that function $f(x)$ of x increasing with n when for x is substituted $y_n = \varphi_n(x)$; this function increases for the process of iteration with real contributions, i.e. the quantity $\zeta = f(x) = U + iV$ describes in its complex plane the right line $V = c$ parallel to the real axis. If once we know the form of the function $\zeta = f(x)$, then we also know the orbit of the quantity $x = f_{-1}(\zeta)$.

The value of V and the initial value ($n = 0$) of the real part U of ζ represent together two arbitrary constants, of which we do not dispose until we choose the initial value of x.

We shall indicate the current point (y_n) of the orbit $x \rightarrow y_n$ by z, whilst we shall point out x by z_0; we then have

$$f(z) = f(z_0) + n$$

or

$$U + iV = U_0 + iV_0 + n,$$

so that

$$U = U_0 + n \quad , \quad V = V_0.$$

The choice of the initial point z_0 now determines the values U_0 and V_0.

When working out some examples we shall not always follow the systematic way sketched above, as it is unnecessarily lengthy in simple cases.

In reference to the broken linear function $y = \dfrac{\alpha x + \beta}{x\gamma + \delta}$ we notice that this has been thoroughly investigated already by POINCARÉ [1]) and KLEIN [2]), the latter having also included complex values of α, β, γ, and δ in the study. KLEIN too allows the function $y = \dfrac{\alpha x + \beta}{\gamma x + \delta}$ to arise gradually out of x and regards the orbit described thereby. For the non-parabolic cases he builds up the function by infinitesimal iteration in the sense indicated by us. For the parabolic case, on the other hand, he takes as parameter of the function in its orbit not the iteration-index n, but a complex multiple of it. In consequence of

1) POINCARÉ. Acta Mathematica I (1882), p. 1.

2) KLEIN—FRICKE. Vorl. ü. d. Theorie der ell. Modulfunktionen (TEUBNER, 1890), p. 165.

this the orbit of z found by KLEIN differs a little from ours. Although after stating and annulling this difference we might suffice with a reference to the results of KLEIN, we will dwell a little longer on the function $y = \frac{\alpha x + \beta}{\gamma x + \delta}$, the more so as, differing from KLEIN, who treats first simple cases and then applies the principle of transformation of the circle correspondence, we shall immediately investigate the most general case.

Examples:

I. $y = x + \beta, \quad y_n = x + n\beta$ or $z = z_0 + n\beta$.

The point z describes the *right line* connecting the points $z = z_0$ and $z = z_0 + \beta$, in such a way that the distance from z to z_0 is proportional to n.

II. $y = \alpha x, \quad y_n = \alpha^n x$ or $z = \alpha^n z_0$.

Let us put $z = \varrho e^{i\theta}$, $z_0 = \varrho_0 e^{i\theta_0}$, $\alpha = \sigma e^{i\tau}$, then

$$\varrho e^{i\theta} = \sigma^n e^{in\tau} \varrho_0 e^{i\theta_0},$$

from which ensues

$$\varrho = \sigma^n \varrho_0, \quad \theta = \theta_0 + n\tau \quad . \quad . \quad . \quad . \quad , \quad . \quad (1)$$

or

$$\varrho = \varrho_0 \, \sigma^{\frac{\theta - \theta_0}{\tau}} = c e^{i\theta}.$$

Point z describes a *logarithmic spiral* round the origin. The polar angle θ increases uniformly with n, i.o.w. the polar angle θ increases *arithmetically uniformly*, it is clear that the radius vector ϱ increases *geometrically uniformly*.

If α is real, then $\tau = 0$. The second equation (1) tells us that the polar angle remains constant, so that point z moves along the *line connecting* O and z_0 and that with a geometrically uniform increase of ϱ.

If $mod\, \alpha = 1$, then $\sigma = 1$. The first equation (1) then indicates, that the radius vector remains constant, so that point z describes the circle round O as centre, passing through point z_0. The polar angle θ increases arithmetically uniformly.

If τ is commensurable with π, i. e. if α is a root out of unity, then $y = \alpha x$ leads back to x after a whole number of iterations.

$$III. \quad y = \alpha x + \beta, \quad f(x) = \frac{\log\left(x + \frac{\beta}{\alpha - 1}\right)}{\log \alpha}.$$

$$f(x) = \frac{-\infty}{\log \alpha} \text{ for } x = -\frac{\beta}{\alpha - 1} = g, \text{ therefore } f(z) = \frac{\log(z - g)}{\log \alpha}.$$

If we displace the origin to g and if accordingly we call $z - g = \varrho' e^{i\theta'}$, we find for the orbit of z the *logarithmic spiral* $\varrho' = ce^{c\theta'}$ round the point g. If, however, α is real, then z describes the line from z_0 to $\alpha z_0 + \beta$, containing also point $g = -\dfrac{\beta}{\alpha - 1}$. Is on the contrary $mod\, \alpha = 1$, then the orbit of z is a circle round g as centre.

IV. $y = \dfrac{\alpha x + \beta}{\gamma x + \delta}$, where $(\alpha - \delta)^2 + 4\beta\gamma \gtrless 0$.

$$f(x) = \frac{1}{\lambda} log \frac{px+1}{qx+1}, \text{ where}$$

$$\lambda = log \frac{(\alpha+\delta)+\sqrt{(\alpha-\delta)^2+4\beta\gamma}}{(\alpha+\delta)-\sqrt{(\alpha-\delta)^2+4\beta\gamma}} = log \frac{\{(\alpha+\delta)+\sqrt{(\alpha-\delta)^2+4\beta\gamma}\}^2}{4(\alpha\delta-\beta\gamma)},$$

$$p = \frac{(\alpha-\delta)+\sqrt{(\alpha-\delta)^2+4\beta\gamma}}{2\beta} \quad , \quad q = \frac{(\alpha-\delta)-\sqrt{(\alpha-\delta)^2+4\beta\gamma}}{2\beta}.$$

We shall take as general case, that β, γ, and δ are all complex; then λ, p, and q will also be complex.

$$f(z) = \frac{1}{\lambda} log \frac{pz+1}{qz+1} = \frac{1}{\lambda} log \frac{p}{q} + \frac{1}{\lambda} log \frac{z+p^{-1}}{z+q^{-1}} = \frac{1}{\lambda} log \frac{p}{q} + \frac{1}{\lambda} log \frac{z_0+p^{-1}}{z_0+q^{-1}} + n.$$

From

$$log \frac{z+p^{-1}}{z+q^{-1}} = log \frac{z_0+p^{-1}}{z_0+q^{-1}} + \lambda n \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

ensues that for an infinite value of n the point z takes either the value $-p^{-1}$ or the value $-q^{-1}$. We shall call the points $z = -p^{-1}$ and $z = -q^{-1}$ the *limiting points* and we shall put $-p^{-1} = g'$, $-q^{-1} = g''$.

Thus our equation (2) becomes

$$log \frac{z-g'}{z-g''} = log \frac{z_0-g'}{z_0-g''} + (\mu + i\nu)n, \quad . \quad . \quad . \quad . \quad (3)$$

where we have replaced λ by $\mu + i\nu$.

Let us choose g' and g'' as auxiliary origins and let us call

$$z - g' = z' = \varrho' e^{i\theta'} \quad , \quad z - g'' = z'' = \varrho'' e^{i\theta''},$$

we then find out of (3)

$$log \frac{\varrho'}{\varrho''} + i(\theta' - \theta'') - log \frac{\varrho'_0}{\varrho''_0} - i(\theta'_0 - \theta''_0) = \mu n + i\nu n,$$

where by separating the real part from the imaginary we find

$$log \frac{\varrho'}{\varrho''} - log \frac{\varrho'_0}{\varrho''_0} = \mu n \quad , \quad (\theta' - \theta'') - (\theta'_0 - \theta''_0) = \nu n, \ldots \quad (4)$$

or

$$\frac{\varrho'}{\varrho''}=\frac{\varrho'_0}{\varrho''_0}e^{\mu n}\quad,\quad \theta'-\theta''=\theta'_0-\theta''_0+\nu n. \quad . \quad . \quad . \quad . \quad (5)$$

Elimination of n leads, when μ and ν are neither of them equal to zero, to

$$\varrho' e^{-\frac{\mu}{\nu}\theta'} : \varrho'' e^{-\frac{\mu}{\nu}\theta''} = \varrho'_0 e^{-\frac{\mu}{\nu}\theta'_0} : \varrho''_0 e^{-\frac{\mu}{\nu}\theta''_0} = C. \quad . \quad . \quad (6)$$

By putting

$$\varrho' = c'e^{\frac{\mu}{\nu}\theta'} \;,\; \varrho'' = c''e^{\frac{\mu}{\nu}\theta''} \;, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

we find

$$c' = Cc''. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The equations (7) and (8) determine together a so-called *logarithmic double spiral* [1]), with the points g' and g'' as poles.

From the second equation (5) ensues that the angle $\theta' - \theta'' = \varphi$ between the two auxiliary radii vectores $g'z$ and $g''z$ increases arithmetically uniformly, whilst the first equation (5) shows us that the quotient of the auxiliary radii vectores increases geometrically uniformly.

For the case α, β, γ, and δ real, some simplifications appear.

We shall distinguish three cases.

A. $(\alpha - \delta)^2 + 4\beta\gamma > 0$, $\alpha\delta - \beta\gamma < 0$.

The quantities p and q are real, so the points g' and g'' lie on the real axis. Farthermore we have $e' < 0$, so that $\nu = \pi$.

Hence the orbit of z is a *logarithmic double spiral*, whose two poles lie on the real axis.

A special case is furnished by the condition $\alpha + \delta = 0$, or $\mu = 0$.

From the first equation (5) now ensues that the quotient of the auxiliary radii vectores is constant, so that the point z describes a circle of APOLLONIUS of the triangle $g'g''z_0$, whilst the angle $g'zg''$ increases uniformly with n. An example of the latter case is furnished by $y = \frac{1}{x}$; here $g' = +1$, $g'' = -1$.

B. $(\alpha - \delta)^2 + 4\beta\gamma > 0$, $\alpha\delta - \beta\gamma > 0$.

The points g' and g'' lie on the real axis, whilst $e' > 0$, thus $\nu = 0$.

Now the second equation (5) shows us, that $\theta' - \theta'' = \varphi$ is constant, so that the point z describes the circle passing through g', g'' and z_0.

[1]) For the logarithmic double spirals the reader may consult: HOLZMÜLLER, Ueber die logarithm. Abbildung etc Zeitschr. f. Math. u. Physik., Vol. 16. (1871), p. 281.

All ∞^2 initial points z_0 furnish thus together all circles of the pencil of which g' and g'' are the base points.

Let us suppose point z determined on its orbit as point of intersection of this orbit with an element of the conjugated pencil of circles, intersecting the real axis a. o. in a point s, then evidently $g'z : g''z = g's : g''s$ holds, so that the equation (5) expresses that the quotient $g's : g''s$ increases geometrically uniformly. (This property enables us to construct easily the points z belonging to given values of n). Farthermore holds $g' = z_{-\infty}$ and $g'' = z_{+\infty}$.

$C \quad (\alpha - \delta)^2 + 4\beta\gamma < 0.$

The points g' and g'' lie symmetrically with respect to the real axis, p and q being conjugate complex. As mod. $e' = 1$ we have $\mu = 0$. The ratio $\varrho' : \varrho''$ is now constant, so that the point describes a circle of APOLLONIUS of $\Delta g' g'' z_0$, i.e. a circle of the pencil with g' and g'' as point circles. We can again regard the point z as if originated by intersection of the orbit with a circle of the conjugated pencil of circles. As the angle $g'zg''$ increases uniformly with n we can easily construct with the aid of the conjugated pencil of circles the points z belonging to definite values of n. It is clear that the orbit of z when n increases indefinitely is described innumerable times, so that the function $\varphi_n(x)$ has as a function of n a real period. If ν is commensurable with π, then this period is a mensurable number.

If particularly $\alpha + \delta = 0$ holds, then $\nu = \pi$. This case is a. o. realized in the function $y = \frac{-1}{x}$; here $g' = i$, $g'' = -i$.

$V. \quad y = \frac{\alpha x + \beta}{\gamma x + \delta}$, where $(\alpha - \delta)^2 + 4\beta\gamma = 0$.

Here we are in the *parabolic* case.

$$f(x) = \frac{\alpha x + \beta}{\frac{\alpha - \delta}{2} x + \beta},$$

or, if we put $-\frac{\beta}{\alpha} = a, \; \frac{-2\beta}{\alpha - \delta} = \frac{\alpha - \delta}{2\gamma} = g,$

$$f(x) = \frac{2\alpha}{\alpha - \delta} \cdot \frac{x - a}{x - g},$$

so

$$f(z) = \frac{2\alpha}{\alpha - \delta} \cdot \frac{z - a}{z - g} = \frac{2\alpha}{\alpha - \delta} \cdot \frac{z_0 - a}{z_0 - g} + n,$$

so that

$$\frac{z-a}{z-g}=\frac{z_0-a}{z_0-g}+\frac{a-\delta}{2a}\cdot n=\frac{z_0-a}{z_0-g}+(\mu+i\nu)n. \quad . \quad . \quad . \quad (9)$$

The difference between our method and that of KLEIN arises from the fact that KLEIN allows the quantity $\frac{a-\delta}{2a}\cdot n$ to increase *really*.

If we take a and g as auxiliary origins and if we put

$$z-a=z'=\varrho' e^{i\theta'}, \quad z-g=z''=\varrho'' e^{i\theta''},$$

then the equation (9) takes the form of

$$\frac{\varrho'}{\varrho''}e^{i(\theta'-\theta'')}=\frac{\varrho'_0}{\varrho''_0}e^{i(\theta'_0-\theta''_0)}+(\mu+i\nu)\,n$$

or

$$\frac{\varrho'}{\varrho''}\{\cos(\theta'-\theta'')+i\sin(\theta'-\theta'')\}=\frac{\varrho'_0}{\varrho''_0}\{\cos(\theta'_0-\theta''_0)+i\sin(\theta'_0-\theta''_0)\}+(\mu+i\nu)n,$$

from which ensues

$$\left.\begin{aligned}\frac{\varrho'}{\varrho''}\cos(\theta'-\theta'')&=\frac{\varrho'_0}{\varrho''_0}\cos(\theta'_0-\theta''_0)+\mu n,\\ \frac{\varrho'}{\varrho''}\sin(\theta'-\theta'')&=\frac{\varrho'_0}{\varrho''_0}\sin(\theta'_0-\theta''_0)+\nu n.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (10)$$

If we put $\mu=\sigma\cos\tau$, $\nu=\sigma\sin\tau$ $\left(\text{i. o. w. } \frac{a-\delta}{2a}=\sigma e^{i\tau}\right)$ we find out of (10) when eliminating n:

$$\frac{\varrho'}{\varrho''}\sin(\theta'-\theta''-\tau)=\frac{\varrho'_0}{\varrho''_0}\sin(\theta'_0-\theta''_0-\tau)=c. \quad . \quad . \quad . \quad (11)$$

It is clear, that the orbit as found by KLEIN follows from ours by putting $\tau=0$. The orbit of KLEIN can thus serve as iteration-orbit for real values of the quantity $\frac{a-\delta}{2a}$, thus of $\frac{\delta}{a}$.

To investigate the curve determined by the equation (11) we imagine the circle passing through g and a and of which the arc ga amounts to 2τ, so that from each point of the supplementary arc the line ga is seen under the angle τ. (See fig. p. 511).

If we connect g with z_0 and z, the connecting lines will meet the circle in m_0 and m.

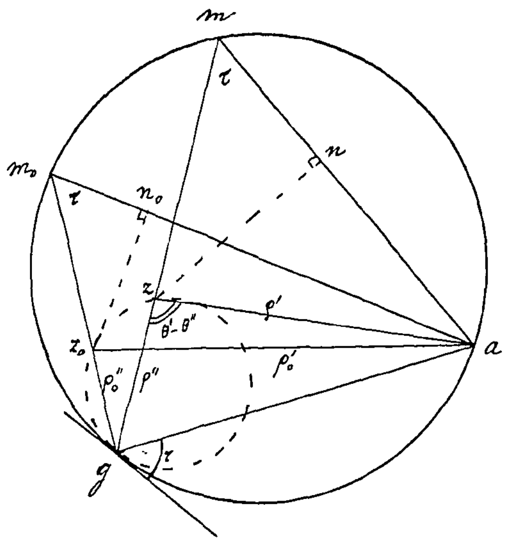
Now $\angle gma=\angle gm_0a=\tau$

Farthermore $\angle zam=\theta'-\theta''-\tau$, $\angle z_0am_0=\theta'_0-\theta''_0-\tau$.

If we let fall the normals z_0n_0 and zn on am_0 and am, then $z_0n_0=\varrho'_0\sin(\theta'_0-\theta''_0-\tau)$ and $zn=\varrho'\sin(\theta'-\theta''-\tau)$.

The equation (11) now demands

$$\frac{zn}{zg}=\frac{z_0n_0}{z_0g} \quad \text{or} \quad \frac{zg}{z_0g}=\frac{zn}{z_0n_0}=\frac{zm}{z_0m_0}.$$



It is therefore evident that we arrive from points m to points z by diminishing or enlarging the chords gm in a definite ratio.

So the orbit of z is a circle touching the auxiliary circle (m) in g, whose tangent in g forms in that way the angle τ with the line ga.

If the quantities α, β, γ, and δ are real, then a and g are real, whilst $\nu = 0$, therefore also $\tau = 0$. The points a and g therefore lie on the real axis and the orbit of z touches the real axis in the point g. If on the other hand $\mu = 0$, then the centre of the orbit lies on the line ga.

The way in which z changes with n we can read from the equations (10).

If we suppose the point z to be furnished by the circle, which passes through g and z and whose centre lies on ga, then the first equation (10) tells us that the *reciprocal* value of the radius of that circle increases arithmetically uniformly that i. o. w. the radii of the circles through g whose centres lie on ga and which pass through z_1, z_2 etc., form an *harmonic series*. If on the other hand we suppose that the point z is constructed as point of intersection of its orbit with the circle through z touching the line ga in g, it then follows easily out of the second equation that also the *reciprocal* value of the radius of this circle increases arithmetically uniformly, that i.o.w. the radii of the circles touching ga in g and passing through the points z_1, z_2 etc. form an *harmonic series.*

It is clear that for the case α, β, γ, and δ real, thus $\nu = 0$ and $\tau = 0$, only the first determination of the course of z can serve, whilst in the case $\mu = 0$ only the second determination retains its validity.

VI. $y = x^{\alpha}$, $y_n = x^{\alpha^n}$, $f(x) = \dfrac{\log \log x}{\log \alpha}$.

$$f(z) = \frac{\log \log z}{\log \alpha} = \frac{\log \log z_0}{\log \alpha} + n.$$

Let us put $\log \alpha = \mu + i\nu$, we then have

$$\log \log z = \log \log z_0 + (\mu + i\nu)\, n,$$

so

$$\log z = \log z_0 \,.\, e^{\mu n} (\cos \nu n + i \sin \nu n),$$

or

$$\log \varrho + i\theta = (\log \varrho_0 + i\theta_0)\, e^{\mu n} (\cos \nu n + i \sin \nu n)\,, \quad . \quad . \quad (12)$$

from which ensues

$$\left.\begin{aligned} (\log \varrho)^2 + \theta^2 &= \{(\log \varrho_0)^2 + \theta_0^{\,2}\}\, e^{2\mu n}\,, \\ \frac{\log \varrho}{\theta} &= \frac{\log \varrho_0 \cos \nu n - \theta_0 \sin \nu n}{\log \varrho_0 \sin \nu n + \theta_0 \cos \nu n}\,. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Out of these equations follows by elimination of n the orbit of z.

For the case α positive, so $\nu = 0$, the second equation passes into

$$\frac{\log \varrho}{\theta} = \frac{\log \varrho_0}{\theta_0} = \varkappa,$$

or

$$\varrho = e^{\varkappa \theta}.$$

The orbit of z is in this case a logarithmic spiral around the origin, which is *independent* of α.

If *mod* $\alpha = 1$, then $\mu = 0$, so that the first equation (13) tells us that

$$(\log \varrho)^2 + \theta^2 = (\log \varrho_0)^2 + \theta_0^{\,2} = \tau^2$$

or

$$\varrho = e^{\pm \sqrt{\tau^2 - \theta^2}}.$$

This curve is likewise independent of the argument of α.

The function $y = x^{-1}$, which we have regarded on one hand under IV A, $\mu = 0$, and which then furnished for the orbit of z a circle, we can also range under the case treated last. If namely we take $y = x^{-1}$ as a special case of $y = x^{\alpha}$ (*mod.* $\alpha = 1$, *arg.* $\alpha = \pi$), we then find for the orbit of z quite a different curve.

To this remarkable property of $y = x^{-1}$ we hope to refer more explicitly later on.