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not sufficiently clearly state that the divergent result was only founded on the assumption $\frac{d^2b}{dx^2} = 0$. I knew, however, that VAN DER WAALS in his Continuität II p. 24 has already treated this question. Yet on theoretical considerations I abide by my opinion that in the neighbourhood of the limiting volume, so at very high pressures, $\frac{d^2b}{dx^2}$ must be $= 0$.

And now I think that I for my part, have sufficiently elucidated Mr. KOHNSTAMM's Reply, so that further misunderstanding seems almost precluded.

Baarn, Febr. 21, 1910.

Mathematics. — *"The oscillations about a position of equilibrium where a simple linear relation exists between the frequencies of the principal vibrations."* (1st part). By Mr. H. J. E. BETH. (Communicated by Prof. KORTEWEG).

Introduction.

§ 1. In his paper[1]): "On certain vibrations of higher order of abnormal intensity (vibrations of relation) for mechanisms with more degrees of freedom" (Verhandelingen der Koninklijke Akademie van Wetenschappen, Vol. V. N°. 8, 1897; Archives Néerlandaises Vol. I, series II, pages 229—260) Prof. KORTEWEG has written down the expansions in series for the principal coordinates of an arbitrary mechanism with more degrees of freedom, performing small oscillations about a position of stable equilibrium. From these expansions in series could be deduced that in a certain case it was possible for some vibrations of higher order, having in general a small intensity with respect to the principal vibrations, to obtain an abnormally great intensity; this is the case when between the frequencies n_x, n_y etc. of the principal vibrations a relation exists of the form

$$pn_x + qn_y + \ldots = \varrho ;$$

where p, q etc. are positive or negative integers and ϱ is with respect to n_x, n_y etc. a small quantity, called residue of relation.

Furthermore however it became evident that, when $S \leqq 4$ (S is the sum of the absolute values of p, q etc.) and at the same time $\varrho = 0$,

1) "Over zekere trillingen van hooger orde van abnormale intensiteit (relatie-trillingen) bij mechanismen met meerdere graden van vrijheid".

the above-mentioned expansions in series lost their validity; we must therefore investigate in a different way what becomes of the movement in the case mentioned. In what follows we shall investigate this for a mechanism with two degrees of freedom. As a base for this investigation a very simple mechanism is selected, namely a material point which moves without friction yet under the influence of gravitation on a given surface in the vicinity of its lowest point. Every time one of the cases $S \leqq 4$ is discussed we shall pass to an arbitrary mechanism with two degrees of freedom.

Movement on the bottom of a surface.

§ 2. We shall accordingly first pass on to the treatment of the simple mechanism we have chosen as a base for our investigation. When the surface has positive curvature in the vicinity of its lowest point O, when plane XY is the tangential plane in O, and the XZ- and YZ-planes are the principal sections of the surface in that point, whilst the Z-axis is supposed positive upwards, then the equation of the surface in the vicinity of O takes the form of:

$$z = \frac{1}{g}(c_1 x^2 + c_3 y^2 + d_1 x^3 + d_2 x^2 y + d_3 x y^2 + d_4 y^3 + \ldots); \quad (1)$$

where c_1 and c_3 are positive.

The equations of motion of the material point become:

$$\left.\begin{aligned} \ddot{x} + \frac{\partial z}{\partial x}(g + \ddot{z}) = 0, \\ \ddot{y} + \frac{\partial z}{\partial y}(g + \ddot{z}) = 0. \end{aligned}\right\}$$

Availing ourselves of (1) to eliminate $\ddot{z}$ we find:

$$\left.\begin{aligned} \ddot{x} + \frac{\partial z}{\partial x}\left(g + \frac{\partial^2 z}{\partial x^2}\dot{x}^2 + 2\frac{\partial^2 z}{\partial x \partial y}\dot{x}\dot{y} + \frac{\partial^2 z}{\partial y^2}\dot{y}^2 + \frac{\partial z}{\partial x}\ddot{x} + \frac{\partial z}{\partial y}\ddot{y}\right) = 0, \\ \ddot{y} + \frac{\partial z}{\partial y}\left(g + \frac{\partial^2 z}{\partial x^2}\dot{x}^2 + 2\frac{\partial^2 z}{\partial x \partial y}\dot{x}\dot{y} + \frac{\partial^2 z}{\partial y^2}\dot{y}^2 + \frac{\partial z}{\partial x}\ddot{x} + \frac{\partial z}{\partial y}\ddot{y}\right) = 0. \end{aligned}\right\} \quad (2)$$

Let h be the small quantity (small e.g. with respect to the principal radii of curvature R_1 and R_2 of the surface in O) which determines the order of greatness of x and y, then the equations (2) become, omitting the terms of order h^2 and higher:

$$\left.\begin{aligned} \ddot{x} + 2c_1 x = 0, \\ \ddot{y} + 2c_3 y = 0. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (3)$$

These equations are in general sufficient to arrive at the solution at first approximation. This then becomes:

$$\left.\begin{aligned} x &= Ah \cos (n_1 t + \lambda), \\ y &= Bh \cos (n_2 t + \mu); \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

where $n_1 = \sqrt{2c_1}$, $n_2 = \sqrt{2c_3}$.

Here Ah, Bh, λ and μ are constants of integration; we suppose A and B to be of moderate greatness.

At first approximation therefore the horizontal projection of the moving point describes a Lissajous curve, which is closed when $pn_1 = qn_2$, where p and q are integers. If $pn_1 = qn_2 + \varrho$, the curve described is not closed, but it consists of a succession of parts each of which differs but little from a closed curve. These last closed curves have however various shapes which answer to different values of the difference in phase. They are all described in the rectangle with $2Ah$ and $2Bh$ as sides.

§ 3. If we wish to take into consideration the terms of a higher order appearing in (2) we generally have but to apply small modifications to the first approximation.

These modifications are, however, not small in case a relation exists of the form:

$$pn_1 = qn_2 + \varrho;$$

where $S \equiv p + q \leq 4$ and $\frac{\varrho}{n_1}$ is very small (what is meant here by "very small" will be evident later on).

When by applying the method of consecutive approximations, starting from (4) as first approximation, we try to find expansions in series for x and y, we shall find, when substituting the expressions (4) into the terms of higher order of (2) and developing the products and powers of the cosines, in case $\frac{\varrho}{n_1}$ is very small, periodical terms which have about the same period as the principal vibration, to which the equation in which the indicated term appears relates more especially. Such terms in the equations of motion give rise in the expansions in series for x and y to terms with abnormally great amplitude. These amplitudes may reach the order h and even seem to be greater still.

This proves that in the case supposed our first approximation was not correct. It is evident that in the equations of motion there are terms of higher order, which are of influence even on the first approximation. So we shall have to find in the equations (2) which terms give rise to the failure of the application of the method of consecutive approximations. These terms we shall have to include in the abridged equations, serving to determine the first approximation.

We shall consecutively discuss the cases:

$$S=3\ (2n_1=n_2+\varrho), \qquad S=4\ (3n_1=n_2+\varrho), \qquad S=2\ (n_1=n_2+\varrho).$$

$S=3.$[1]) *Strict relation.*

§ 4. We suppose $\varrho=0$; therefore

$$n_2=2n_1.$$

In the equations of motion appear for the first time among the terms of order h^2 terms which, according to what was said in § 3, must be included in the abridged equations. They are: in the first equation $2d_2xy$, in the second d_2x^2. These are the most important among the terms referred to. Omitting the remaining terms of higher order we therefore have to consider:

$$\left.\begin{aligned}\ddot{x}+n_1^2x+2d_2xy&=0,\\ \ddot{y}+4n_1^2y+d_2x^2&=0.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

We may also write this system as follows:

$$\left.\begin{aligned}\ddot{x}+n_1^2x-\frac{\partial R}{\partial x}&=0,\\ \ddot{y}+4n_1^2y-\frac{\partial R}{\partial y}&=0;\end{aligned}\right\}$$

in which:

$$R\equiv -d_2x^2y.$$

To this we apply the method of the variation of the canonical constants. This means, as is known, that the equations, arising when the terms $\frac{\partial R}{\partial x}$ and $\frac{\partial R}{\partial y}$ are omitted, first are solved; in which solution 4 arbitrary constants appear; we then investigate what functions of the time must be the quantities just now regarded as constants, so that the expressions for x and y, taken in this way, represent the solution of the complete equations containing $\frac{\partial R}{\partial x}$ and $\frac{\partial R}{\partial y}$. The equations in which $\frac{\partial R}{\partial x}$ and $\frac{\partial R}{\partial y}$ are lacking, are solved according to

[1]) In a following paper we shall discuss the cases $S=2$ and $S=4$.

the method of HAMILTON-JACOBI in order that the constants we obtain may form a canonical system.

If α_1, α_2, β_1 and β_2 are the canónical constants then by substitution of the expressions found for x and y in R this R will become a function of α_1, α_2, β_1, β_2 and t. The variability of the α's and β's with the time is then given by:

$$\frac{d\alpha_1}{dt}=\frac{\partial R}{\partial \beta_1}\,,\quad \frac{d\alpha_2}{dt}=\frac{\partial R}{\partial \beta_2}\,,\quad \frac{d\beta_1}{dt}=-\frac{\partial R}{\partial \alpha_1}\,,\quad \frac{d\beta_2}{dt}=-\frac{\partial R}{\partial \alpha_2}\quad . . \quad (6)$$

In case R is a function of the α's and the β's alone, and consequently does not contain t explicitly, the system has as an integral:

$$R = constant. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

§ 5. If now we solve the equations

$$\left.\begin{aligned} \ddot{x} + n_1^2\, x = 0, \\ \ddot{y} + 4n_1^2\, y = 0, \end{aligned}\right\}$$

arising from (5) by omission of the terms $\frac{\partial R}{\partial x}$ and $\frac{\partial R}{\partial y}$, according to the method HAMILTON-JACOBI we may arrive at:

$$\left.\begin{aligned} x &= \frac{\sqrt{\alpha_1}}{n_1} \cos\,(n_1 t + 2n_1\beta_1), \\ y &= \frac{\sqrt{\alpha_2}}{2n_1} \cos\,(2n_1 t + 4n_1\beta_2)\,; \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (8)$$

where α_1, α_2, β_1, β_2 form a canonical system of constants. We must suppose α_1 and α_2 to be of order h^2 as the amplitudes of the x- and y-vibrations must be of order h.

Substitution of (8) in $R \equiv -d_2 x^2 y$ furnishes 3 terms:

$$\frac{\alpha_1\sqrt{\alpha_2}}{4n_1^3} \cos\,(2n_1 t + 4n_1\beta_2),\quad \frac{\alpha_1\sqrt{\alpha_2}}{8n_1^3} \cos\,\{4n_1 t + 4n_1\,(\beta_1 + \beta_2)\} \text{ and}$$

$$\frac{\alpha_1\sqrt{\alpha_2}}{8n_1^3} \cos\,4n_1\,(\beta_1 - \beta_2),$$

each term multiplied by $-d_2$.

The first two terms contain t explicitly; setting aside the variability of the α's and β's we can say that those terms are periodical, whilst the period is comparable to that of the principal vibrations. The last term, however, does not contain t explicitly. Only this last term is of importance for the first approximation; the two others we omit (we shall revert to this in § 6).

We therefore take:

$$R = -\frac{d_2}{8n_1^3} a_1 \sqrt{a_2} \cos 4n_1 (\beta_1 - \beta_2).$$

Consequently system (6) takes this form:

$$\left.\begin{aligned} \frac{da_1}{dt} &= 2\, Nm_1\, a_1\, a_2^{\frac{1}{2}} \sin\varphi, \\ \frac{da_2}{dt} &= -\, 2\, Nm_1\, a_1\, a_2^{\frac{1}{2}} \sin\varphi, \\ \frac{d\beta_1}{dt} &= m_1\, a_2^{\frac{1}{2}} \cos\varphi, \\ \frac{d\beta_2}{dt} &= \tfrac{1}{2}\, m_1\, a_1\, a_2^{-\frac{1}{2}} \cos\varphi; \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (9)$$

where N is written for $n_2 = 2n_1$; further:

$$m_1 = \frac{d_2}{N^3},$$

$$\varphi = 2\, N (\beta_1 - \beta_2).$$

As t does not appear explicitly in R we get according to what has been said at the close of § 4 as an integral:

$$a_1 \sqrt{a_2} \cos\varphi = constant. \quad \ldots \ldots \quad (10)$$

Furthermore it appears at once from (9) that:

$$\frac{da_1}{dt} + \frac{da_2}{dt} = 0.$$

Therefore:

$$a_1 + a_2 = constant \quad \ldots \ldots \ldots \quad (11)$$

is another integral.

The latter gives us reason to introduce a new variable ζ, in such a way that:

$$a_1 = \frac{1}{4} R_0^2\, N^2\, h^2\, \zeta \quad , \quad a_2 = \frac{1}{4} R_0^2\, N^2\, h^2 (1-\zeta);$$

ζ is then always situated between 0 and 1, R_0 is of moderate greatness.

By this (10) obtains the form:

$$\zeta \sqrt{1-\zeta} \cos\varphi = K; \quad \ldots \ldots \ldots \quad (12)$$

in which K represents a constant.

The first equation of (9) becomes:

$$\frac{d\zeta}{dt} = \frac{d_2 R_0}{N} \zeta \sqrt{1-\zeta} \sin\varphi \, . \, h. \quad \ldots \ldots \ldots \quad (13)$$

By elimination of φ between (12) and (13) we arrive at:

$$\frac{d\zeta}{\sqrt{\zeta^2 (1-\zeta) - K^2}} = \pm \frac{d_2 R_0}{N} h \, . \, dt.$$

Now put:

$$f(\zeta) \equiv \zeta^2 (1-\zeta) - K^2,$$

then for the initial value of ζ we find $f(\zeta) > 0$. For $\zeta = 0$ and $\zeta = 1$ we find $f(\zeta) < 0$. Thus the equation $f(\zeta) = 0$ has two roots between 0 and 1.

So K^2 cannot have all values; the possible values of K^2 lie between two limits; in § 9 we shall revert to this and to the special cases, corresponding to the limiting values of K^2.

The roots between 0 and 1 which the equation

$$\zeta^2 (1-\zeta) - K^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

has in the general case will be called ζ_1 and ζ_2, where we suppose $\zeta_2 > \zeta_1$. The third root is negative, we call it $-\lambda$.

The differential relation between ζ and t may now be written:

$$\frac{d\zeta}{\sqrt{(\zeta_2-\zeta)(\zeta-\zeta_1)(\zeta+\lambda)}} = \pm \frac{d_2 R_0}{N} h \, . \, dt \ldots \quad . \quad . \quad . \quad (15)$$

So with the aid of elliptic functions ζ may be expressed in t. It changes periodically between the limits ζ_1 and ζ_2.

Now with the aid of (12) we can also calculate φ as function of t. And β_1 and β_2 likewise, it being possible to write the last two equations of (9):

$$\frac{d\beta_1}{dt} = \frac{d_2 R_0 K}{2 N^2} \cdot \frac{h}{\zeta},$$

$$\frac{d\beta_2}{dt} = \frac{d_2 R_0 K}{4 N^2} \cdot \frac{h}{1-\zeta}.$$

So now x and y are also known as functions of t [1]).

In fig. 1 the relation (12) between ζ and φ is represented in polar coordinates, φ is taken as polar angle, $\sqrt{1-\zeta}$ as radius vector. The circle drawn has unity as radius. The curves change with the value of K. For $K > 0$ the curves lie to the right of the straight line $\varphi = \frac{\pi}{2}$, for $K < 0$ to the left of it; $K = 0$ furnishes degeneration into the straight line $\varphi = \frac{\pi}{2}$ and the circle $\zeta = 0$. By the maximal positive and negative value of K ($K = \pm \frac{2}{9}\sqrt{3}$) the curve has contracted into an isolated point. The special cases of the motion belonging to $K = 0$ and to $K = \pm \frac{2}{9}\sqrt{3}$ will be discussed in § 9.

1) These calculations will be found in my dissertation, which will appear before long.

§ 6. When astronomers try to obtain in the Theory of the disturbances of the movements of the planets by the application of the method of LAGRANGE expansions in series for the coordinates of the planets or the elements of their orbits, then terms may appear with abnormally large coefficients in consequence of small divisors, originating from the integration. This takes place when between the inverse values of the periods of revolution of some planets a linear relation with integer coefficients is almost fulfilled. Besides some other properties the terms are also distinguished according to their class, by which is meant:

$$\alpha - \frac{m}{2} - \frac{m'}{2};$$

where α represents the exponent of μ (a small quantity indicating the order of greatness of the disturbing function), m the exponent of t, m' the exponent of the small divisor, as they appear in the coefficient of the term indicated. Now it is the terms of the lowest class which we have to take into consideration if we wish to make the expansions in series to hold for a long space of time. By DELAUNAY a method is indicated to determine the terms of the lowest class. It consists principally in omitting all terms of short period (period comparable to the periods of the revolution of the planets) in the disturbing function and retaining the most important of the others. (Comp. e. g. H. POINCARÉ, Leçons de mécanique céleste, vol. I, page 341).

The problem under discussion has much resemblance with the one mentioned from the theory of disturbances. In the preceding § in omitting some terms in R we have imitated what is done in the theory of disturbances.

It is easy to see that the terms omitted have really no influence on the first approximation, when we consider the terms which appear e. g. in α_1 by introduction of such a term.

Osculating curves.

§ 7. In § 5 we have found that the movement of the horizontal projection of the material point might be represented by:

$$x = \frac{\sqrt{a_1}}{n_1} \cos(n_1 t + 2n_1\beta_1),$$

$$y = \frac{\sqrt{a_2}}{2n_1} \cos(2n_1 t + 4n_1\beta_2);$$

where α_1, α_2, β_1 and β_2 are slowly variable; for $\frac{d\alpha_1}{dt}$ and $\frac{d\alpha_2}{dt}$ are of order h^3, $\frac{d\beta_1}{dt}$ and $\frac{d\beta_2}{dt}$ of order h. (Comp. (9)).

For every arbitrary moment the α's and the β's have a definite value. These values determine a certain Lissajous curve. This curve we shall call the osculating curve for the moment indicated, which name is in use in the theory of disturbances. (See among others H. Poincaré, Leçons de mécanique céleste, vol. I, page 90). Thus in our problem the osculating curves are the wellknown Lissajous figures for 2 octaves.

By the change of the origin of time we may write the equations of an osculating curve:

$$x = R_0 h \sqrt{\zeta} \cos n_1 t,$$
$$y = \tfrac{1}{2} R_0 h \sqrt{1-\zeta} \cos (2n_1 t - \varphi);$$

where as in § 5 we have introduced ζ instead of α_1 and α_2; here too φ means $4n_1(\beta_1 - \beta_2)$.

We now see that φ is the value of the difference in phase, to which the osculating curve corresponds when the phase is calculated from the moment of the greatest deviation to the right.

The amplitudes of the x- and y-vibrations being respectively $R_0 h \sqrt{\zeta}$ and $\frac{1}{2} R_0 h \sqrt{1-\zeta}$, the vertices of the rectangles, in which the osculating curves are described lie on the circumference of an ellipse with its great axis along the x-axis and having a length of $2\ R_0 h$, and its small axis along the y-axis and having a length of $R_0 h$.

Now ζ changes its value between ζ_1 and ζ_2, so the rectangles in which the osculating curves are described also lie between two extremes.

Moreover as according to (12) to each value of ζ a value of $\cos \varphi$ belongs all osculating curves may now be constructed.

It follows from (13) that for the extreme values of ζ we find $\sin \varphi = 0$; so in the extreme rectangles parabolae are described.

The distance from OX of the node of an arbitrary osculating curve is $-\frac{R_0 h K}{2\zeta}$, from which it is evident that the nodes and also the vertices of the parabolae lie all on the same side of O lying below O for positive values of K (see fig. 2).

Envelope of the osculating curves.

§ 8. If we perform the elimination of t and φ from:

$$x = R_0 h \sqrt{\zeta} \cos n_1 t, \quad y = \frac{1}{2} R_0 h \sqrt{1-\zeta} \cos (2n_1 t - \varphi) \text{ and}$$

$$\zeta \sqrt{1-\zeta} \cos \varphi = K,$$

we find for the equation of the osculating curves with ζ as parameter:

$$\zeta^2 (X^2 + Y^2) + \zeta (KY - X^2 - X^4) + \left(\frac{1}{4} K^2 - 2 K X^2 Y + X^4\right) = 0 ;$$

where for the sake of a simplified notation is put:

$$X \text{ for } \frac{x}{R_0 h}, \qquad Y \text{ for } \frac{y}{R_0 h}.$$

Thus the envelope has as equation

$$4 (X^2 + Y^2) \left(\frac{1}{4} K^2 - 2 K X^2 Y + X^4\right) - (K Y - X^2 - X^4)^2 = 0.$$

After reduction and division by X^2 (the Y-axis is the locus of the nodes) it may be written:

$$(K - 4 Y^3 - 3 X^2 Y + Y)^2 = (X^2 + 4 Y^2 - 1)^2 (X^2 + Y^2),$$

or if we solve K:

$$K = -(Y \pm \sqrt{X^2 + Y^2}) + (Y \pm \sqrt{X^2 + Y^2})^3.$$

Putting

$$Y \pm \sqrt{X^2 + Y^2} = \frac{K}{U},$$

it passes into

$$K = -\frac{K}{U} + \frac{K^3}{U^3},$$

$$U^2 (1 - U) - K^2 = 0.$$

Now this cubic equation has the same coefficients as (14), so it also has the same roots. So the envelope is degenerated into the 3 parabolae having as equations:

$$U = \zeta_1 \quad , \quad U = \zeta_2 \quad , \quad U = -\lambda ;$$

which after reduction and reintroduction of x and y take the form of:

$$2 \frac{y}{R_0 h} - \frac{\zeta_1}{K} \cdot \frac{x^2}{R_0^2 h^2} + \frac{K}{\zeta_1} = 0 \qquad \zeta_1 \textit{ parabola},$$

$$2 \frac{y}{R_0 h} - \frac{\zeta_2}{K} \cdot \frac{x^2}{R_0^2 h^2} + \frac{K}{\zeta_2} = 0 \qquad \zeta_2 \textit{ parabola},$$

$$2 \frac{y}{R_0 h} + \frac{\lambda}{K} \cdot \frac{x^2}{R_0^2 h^2} - \frac{K}{\lambda} = 0 \qquad \lambda \textit{ parabola}.$$

The parabolae are confocal and have O as focus. When K is positive the ζ_1 and the ζ_2 parabolae have their openings turned upwards, the λ parabola has its opening turned downwards (this case is represented in fig. 2, where besides some osculating curves the enveloping parabolae are also given).

Special cases.

§ 9. At the close of § 5 we saw that two special cases may occur, viz. when $K=0$ and when $K=\pm\frac{2}{9}\sqrt{3}$.

A. $K=0$. We deduce from the relation

$$\zeta\sqrt{1-\zeta}\cos\varphi=K$$

three possibilities:

1. $\zeta=0$. The movement remains confined to the YZ-plane.

2. $\zeta=1$. The movement remains confined to the XZ-plane. This form of motion however proves to be impossible when $x\neq 0$ and $y=0$ is substituted in (5).

3. $\cos\varphi=0$, therefore $\varphi=\frac{\pi}{2}$ or $\varphi=\frac{3\pi}{2}$ invariably. The osculating curves have their nodes at O. The form of movement approaches asymptotically to a motion in the YZ-plane. What becomes of the enveloping parabolae has been represented in fig. 3, in which some osculating curves have been drawn too.

B. $K=\pm\frac{2}{9}\sqrt{3}$. Then $\zeta_1=\zeta_2=\frac{2}{3}$, $\lambda=\frac{1}{3}$. Now $\cos\varphi=\pm 1$ invariably, thus $\varphi=0$ or $\varphi=\pi$. The same parabola is continuously described, in which also the ζ_1 and ζ_2 parabolae have coincided. (Fig. 4). When K undergoes a slight change, ζ_1 and ζ_2 fall close together. So this form of movement is stable.

$$S=3\ ,\ \frac{\varrho}{n_1}\ \textit{is of order}\ \frac{h}{R_1}.$$

§ 10. The expansions in series written down by Prof. KORTEWEG lose for $S=3$ their convergency as soon as $\frac{h}{R_1}$ passes into order $\frac{\varrho}{n_1}$ (page 18 of his paper) or i. o. w. as soon as $\frac{\varrho}{n_1}$ sinks into order $\frac{h}{R_1}$.

We shall now discuss this case.

We again take as first approximation:

$$x=\frac{\sqrt{a_1}}{n_1}\cos(n_1 t+2n_1\beta_1),$$

$$y=\frac{\sqrt{a_2}}{2n_1}\cos(2n_1 t+4n_1\beta_2);$$

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and we must investigate what form the function R now assumes.

As we have supposed that

$$2n_1 = n_2 + \varrho\,,$$

the terms of the order h in the equations of motion would become

$$\ddot{x} + n_1^2\, x$$

and

$$\ddot{y} + (2n_1 - \varrho)^2\, y.$$

Because $\frac{\varrho}{n_1}$ is of order $\frac{h}{R_1}$ and we take no terms of higher order than h^2 in the equations, we may write for the latter:

$$\ddot{y} + 4n_1^2\, y - 4n_1\, \varrho y.$$

If we thus take the above expression for x and y as first approximation, then we must admit in the function R besides the term $-d_2\, x^2 y$ also a term $2n_1\, \varrho y^2$.

In the expression

$$-d_2\, x^2 y + 2n_1\, \varrho y^2$$

we substitute the above expressions for x and y and omit the terms containing t explicitly. In this way we arrive at:

$$R = -\frac{d_2}{N^3}\, \alpha_1 \sqrt{\alpha_2}\, \cos\varphi + \frac{\varrho}{2N}\, \alpha_2\,;$$

where again N is put for $n_2 + \varrho = 2n_1$.

The equations which serve to determine the α's and β's become:

$$\frac{d\alpha_1}{dt} = 2N\, m_1\, \alpha_1\, \alpha_2^{\frac{1}{2}} \sin\varphi,$$

$$\frac{d\alpha_2}{dt} = -\, 2N m_1\, \alpha_1\, \alpha_2^{\frac{1}{2}} \sin\varphi,$$

$$\frac{d\beta_1}{dt} = m_1\, \alpha_2^{\frac{1}{2}} \cos\varphi,$$

$$\frac{d\beta_2}{dt} = -\, \varrho' h + \tfrac{1}{2}\, m_1\, \alpha_1\, \alpha_2^{-\frac{1}{2}} \cos\varphi\,;$$

where

$$m_1 = \frac{d_2}{N^3} \quad , \quad \varrho' = \frac{\varrho}{2Nh}.$$

We again see that

$$\frac{d\alpha_1}{dt} + \frac{d\alpha_2}{dt} = 0,$$

so

$$\alpha_1 + \alpha_2 = constant;$$

for which reason we put:

$$\alpha_1 = \frac{1}{4} R_0^2 N^2 h^2 \zeta \quad , \quad \alpha_2 = \frac{1}{4} R_0^2 N^2 h^2 (1 - \zeta).$$

Further we have according to § 4 as an integral of the system :

$$-\frac{d_2}{N^3} \alpha_1 \sqrt{\alpha_2} \cos \varphi + \frac{\varrho}{2N} \alpha_2 = constant.$$

Introducing ζ, it becomes

$$\zeta \sqrt{1-\zeta} \cos \varphi - \varrho'' (1-\zeta) = K;$$

where K is a constant and

$$\varrho'' = \frac{\varrho N}{d_2 R_0 h}.$$

In the same way as this was done for the case $\varrho = 0$ we may write down the differential relation between ζ and t and find x and y in the way indicated there as functions of the time; they get quite the same form as for $\varrho = 0$ [1]).

In general ζ keeps changing periodically between two limits ζ_1 and ζ_2; ζ_1 and ζ_2 being the positive roots of

$$\zeta^2 (1-\zeta) - \{K + \varrho'' (1-\zeta)\}^2 = 0.$$

Yet there is a considerable difference between the cases $\varrho = 0$ and ϱ of order h.

§ 11. We notice this difference most distinctly when we represent the relation established between ζ and φ in polar coordinates.

If we put

$$\varrho''' = -\varrho'',$$

then we find :

$$\cos \varphi = \frac{K - \varrho''' + \varrho''' \zeta}{\zeta \sqrt{1-\zeta}}.$$

We take φ as polar angle, $\sqrt{1-\zeta}$ as radius vector and we investigate the site and shape of the curves for positive values of ϱ''' and for all possible values of K.

For $K = \varrho'''$ there is degeneration into the circle $\zeta = 0$ and a straight line normal to the origin of the angles at a distance ϱ''' from pole O_1. We have two cases now : $\varrho''' < 1$ and $\varrho''' > 1$.

$\varrho''' < 1$. Let us now investigate the shape of the curves for different values of K. For $K > \varrho'''$ they lie to the left of the straight line just mentioned, for increasing value of K they contract more and more until for the maximal value of K, belonging to a certain value of

[1]) Vide Chapter V of my dissertation.

ϱ''' we get an isolated point. If $0 < K < \varrho'''$ the curves surround point O_1; if $K = 0$ we have a curve through O_1, for $K < 0$ they lie to the left of O_1; for the minimal value of K we again get an isolated point (fig. 5).

For increasing values of ϱ''' the straight line separating the domains $K > \varrho'''$ and $K < \varrho'''$ moves to the right. The domain $K > \varrho'''$ becomes smaller and vanishes for $\varrho''' = 1$. For $\varrho''' \geq 1$ we therefore have curves surrounding O_1 and curves to the left of O_1 only. When ϱ''' increases still more the remaining isolated point approaches to O_1 and the curves farther from O_1 approach to circles.

For $\varrho = 0$ we had (with the exception of the special case $K = 0$) only curves to the right of O_1, and curves to the left of O_1. For ϱ of order h we have moreover curves around O_1, which are even more frequent for great values of $\frac{\varrho}{h}$.

The curves around O_1 point to a form of motion, where φ takes all values, the nodes of the osculating curves lie then above as well as below the point O of fig. 2; the osculating parabolae have their openings turned to opposite sides.

That for increasing values of ϱ''' the curves in general begin to resemble circles more and more, indicates that ζ is about constant; it changes between narrow limits.

This also appears in this way. From (16) we deduce:

$$K - \varrho'''(1 - \zeta_1) = \pm \zeta_1 \sqrt{1 - \zeta_1},$$

$$K - \varrho'''(1 - \zeta_2) = \pm \zeta_2 \sqrt{1 - \zeta_2}.$$

By subtraction we find:

$$\zeta_2 - \zeta_1 = \frac{\pm \zeta_2 \sqrt{1 - \zeta_2} \mp \zeta_1 \sqrt{1 - \zeta_1}}{\varrho'''}.$$

For greater values of ϱ''' we find $\zeta_2 - \zeta_1$ becoming very small. In this way we approach the general case where there is no question about relation.

§ 12. How the transition to this general case takes place is also clearly evident from the limitation of the domain of motion, which limitation we find by determining the envelope of the osculating curves. In the same way as this was done for the case $\varrho = 0$, we find that the envelope degenerates into three parabolae, of which the equations are:

$$2\frac{y}{R_0 h}+\frac{K-\varrho'''}{\zeta_1}+\varrho'''=\frac{\zeta_1}{K-\varrho'''}\cdot\frac{x^2}{R_0{}^2h^2}\qquad \zeta_1 \text{ parabola},$$

$$2\frac{y}{R_0 h}+\frac{K-\varrho'''}{\zeta_2}+\varrho'''=\frac{\zeta_2}{K-\varrho'''}\cdot\frac{x^2}{R_0{}^2h^2}\qquad \zeta_2 \text{ parabola},$$

$$2\frac{y}{R_0 h}-\frac{K-\varrho'''}{\lambda}+\varrho'''=-\frac{\lambda}{K-\varrho'''}\cdot\frac{x^2}{R_0{}^2h^2}\qquad \lambda \text{ parabola}$$

The points of intersection of the λ parabola with the ζ_1 and ζ_2 parabolae lie again on the ellipse having $R_0 h$ and $2R_0 h$ as axes. The parabolae are confocal; the focus lies on the y-axis at the height of $-\frac{1}{2}R_0 h\,.\,\varrho'''$. In fig. 6^a, 6^b, 6^c we find those parabolae (and also the osculating parabolae) corresponding to the cases $\varrho'''<1$ and $K \gtreqless \varrho'''$.

In fig. 7 we see how the limitation approaches more and more to a rectangle for increasing ϱ'''.

The ζ_1 and ζ_2 parabolae coincide for maximal and minimal K.

Arbitrary mechanism with 2 degrees of freedom for which $S=3$.

§ 13. Let q_1 and q_2 be the principal coordinates of the mechanism; they remain during the movement of order h and are zero in the position of equilibrium.

The kinetic energy T and the potential energy U may be written:

$$T=\frac{1}{2}\dot q_1{}^2+\frac{1}{2}\dot q_2{}^2+T_3\,;\qquad U=\frac{1}{2}(n_1{}^2q_1{}^2+n_2{}^2q_2{}^2)+U_3,$$

where T_3 and U_3 are expressions in whose terms h appears at least to the 3rd degree.

Let us write down the terms of order h^3 in T_3:

$$T_3\equiv\frac{1}{2}(aq_1\dot q_1{}^2+bq_2\dot q_1{}^2+2\,cq_1\dot q_1\dot q_2+2\,dq_2\dot q_1\dot q_2+eq_1\dot q_2{}^2+fq_2\dot q_2{}^2)+\ldots$$

As far as and inclusive of the terms of order h^2 the equations of LAGRANGE now become:

$$\left.\begin{aligned}\ddot q_1+n_1{}^2q_1 &= -\frac{1}{2}a\dot q_1{}^2-aq_1\ddot q_1-bq_2\ddot q_1-b\dot q_1\dot q_2-cq_1\ddot q_2-dq_2\ddot q_2+\\ &\qquad +\left(\frac{1}{2}e-d\right)\dot q_2{}^2-\frac{\partial U_3}{\partial q_1},\\ \ddot q_2+n_2{}^2q_2 &= \left(\frac{1}{2}b-c\right)\dot q_1{}^2-cq_1\ddot q_1-dq_2\ddot q_1-e\dot q_1\dot q_2-eq_1\ddot q_2-fq_2\ddot q_2+\\ &\qquad -\frac{1}{2}f\dot q_2{}^2-\frac{\partial U_3}{\partial q_2}.\end{aligned}\right\}$$

In case the relation $n_2 = 2n_1$ is strictly satisfied or nearly so, the disturbing terms are:

in the first equation those with $q_2 q_1$, $\dot{q}_1 \dot{q}_2$, $q_1 \ddot{q}_2$, $\ddot{q}_1 q_2$,

„ „ second „ „ „ $\dot{q}_1^2$, $q_1 \ddot{q}_1$, q_1^2.

If at first approximation we try to satisfy the equations by:

$$q_1 = Ah \cos (n_1 t + \lambda) \quad , \quad q_2 = Bh \cos (n_2 t + \mu)$$

where A, B, λ and μ are functions of t, however in such a manner that $\dot{A}$, $\dot{B}$, $\dot{\lambda}$, $\dot{\mu}$ are of order h or smaller, we may replace in the second member of the equations:

$\dot{q}^2_1$ by $n_1^2 (A^2 h^2 - q^2_1)$, $\dot{q}_2^2$ by $n_2^2 (B^2 h^2 - q_2^2)$,

$\ddot{q}_1$ by $-n_1^2 q_1$, $\ddot{q}_2$ by $-n_2^2 q_2$.

If we take this into account for the disturbing terms and if we omit the non-disturbing terms, the equations become:

$$\left. \begin{aligned} \ddot{q}_1 + n_1^2 q_1 &= (bn_1^2 + cn_2^2 + 2p)\, q_1 q_2 - b\dot{q}_1 \dot{q}_2 \\ \ddot{q}_2 + n_2^2 q_2 &= \left(2\, cn_1^2 - \frac{1}{2}\, bn_1^2 + p\right) q_1^2. \end{aligned} \right\}$$

The terms $2pq_1 q_2$ in the first equation and pq^2_1 in the second originate from a term $-pq^2_1 q_2$, appearing in U_3.

To get rid of the term with $\dot{q}_1 \dot{q}_2$ we use the new variable q' so that:

$$q'_1 = q_1 + \frac{1}{2}\, bq_1\, q_2.$$

Then:

$$\ddot{q}'_1 = \ddot{q}_1 + \frac{1}{2}\, b\ddot{q}_1\, q_2 + \frac{1}{2}\, bq_1\, \ddot{q}_2 + b\dot{q}_1\, \dot{q}_2 =$$

$$= \ddot{q}_1 + b\dot{q}_1\, \dot{q}_2 - \frac{1}{2} b\, (n_1^2 + n_2^2)\, q_1\, q_2.$$

Therefore:

$$\ddot{q}_1 + b\dot{q}_1\, \dot{q}_2 = \ddot{q}'_1 + \frac{1}{2}\, b\, (n_1^2 + n_2^2)\, q_1\, q_2.$$

The equations now pass into:

$$\left\{ \begin{aligned} \ddot{q}'_1 + n_1^2\, q'_1 &= (b\, n_1^2 + c\, n_2^2 - \frac{1}{2}\, b\, n_2^2 + 2\, p)\, q'_1\, q_2, \\ \ddot{q}_2 + n_2^2\, q_2 &= (2\, c\, n_1^2 - \frac{1}{2}\, b\, n_1^2 + p)\, q'^2_1. \end{aligned} \right.$$

For we may replace in the second members q_1 by q_1', as their difference is of order h^2.

H. J. E. BETH. "The oscillations about a position of equilibrium where a simple linear relation exists between the frequencies of the principal vibrations." (1st part).

Fig. 1.

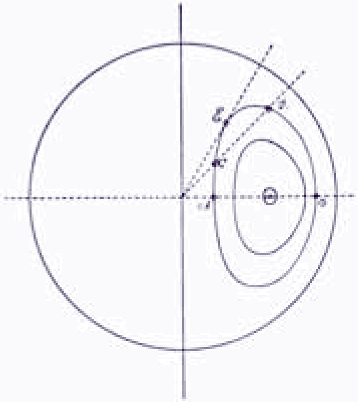


Fig. 2.

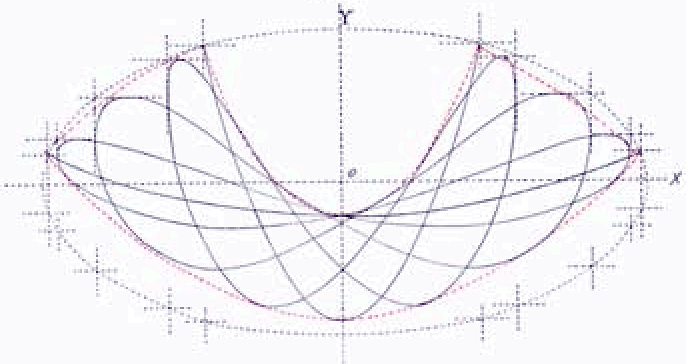


Fig. 3.

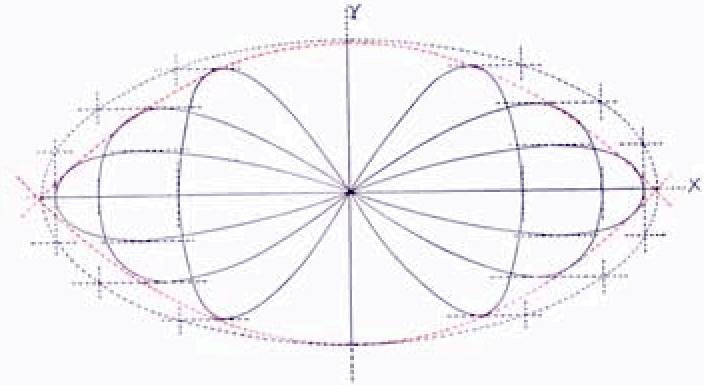


Fig. 4.

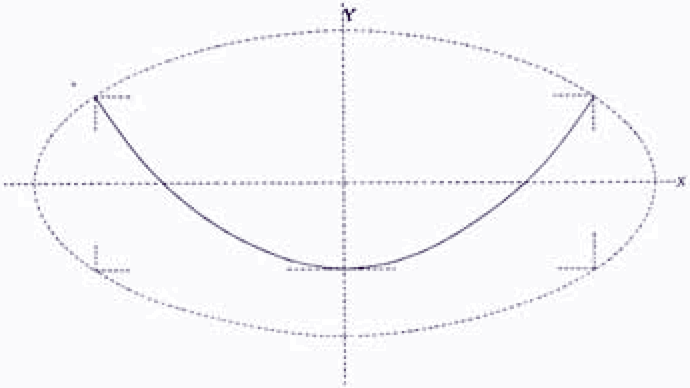


Fig. 5.

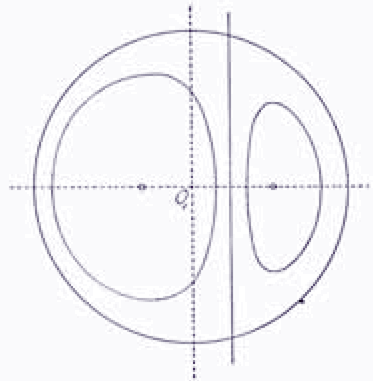


Fig. 6a.

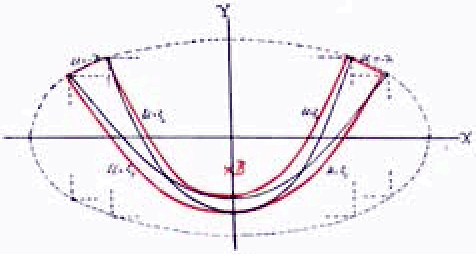


Fig. 6b.

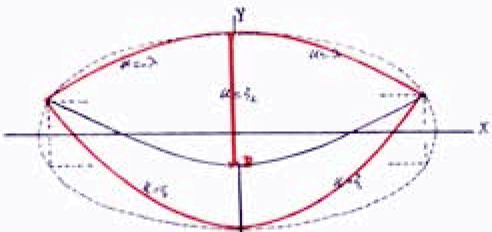


Fig. 6c.

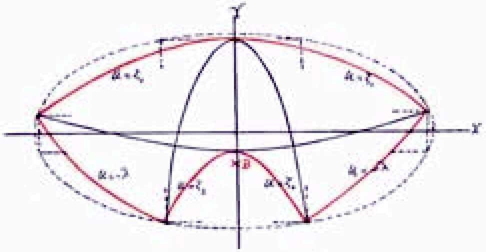
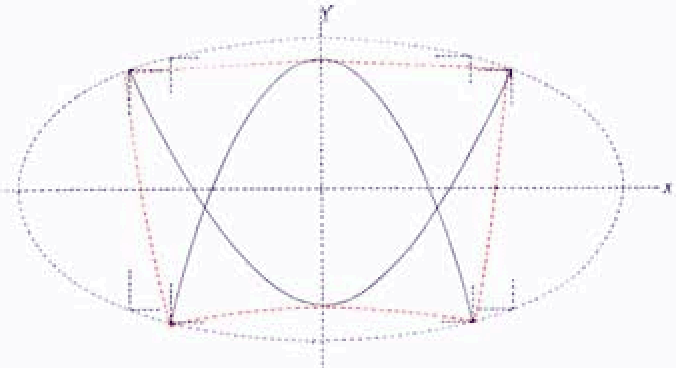


Fig. 7.



Let us now suppose n_2 to be $= 2n_1$; then we get:

$$\ddot{q}'_1 + n_1^2 q'_1 = (4 c n_1^2 - b n_1^2 + 2 p) q'_1 q_2,$$

$$\ddot{q}_2 + n_2^2 q_2 = (2 c n_1^2 - \frac{1}{2} b n_1^2 + p) q'^2_1.$$

So we find:

$$\begin{cases} \ddot{q}_1' + n_1^2 q'_1 + 2 d_2 q'_1 q_2 = 0, \\ \ddot{q}_2 + n_2^2 q_2 + d_2 q'^2_1 = 0; \end{cases}$$

where

$$d_2 = - 2 c n_1^2 + \frac{1}{2} b n_1^2 - p$$

The equations determining the first approximation have exactly the same form as those found in § 4. What was formerly deduced for the simple mechanism holds consequently, if $n_2 = 2n_1$, for an arbitrary mechanism with two degrees of freedom in such a sense that the horizontal projection of the point moving over the surface may be regarded as the representative point for the arbitrary mechanism.

We finally observe that any mechanism for which

$$- 2cn_1^2 + \tfrac{1}{2}bn_1^2 - p = 0$$

is not sensitive for the relation $n_2 = 2n_1$. So this is the condition requisite to make the mechanism for $n_2 = 2n_1$ a mechanism of exception in the sense indicated by Prof. KORTEWEG (§ 26 of his paper).

Mechanisms of exception therefore are among others the symmetrical mechanisms (§ 31 of that paper); for here c, b, and p are all equal to zero.

Microbiology. — "*Viscosaccharase, an enzyme which produces slime from cane-sugar*". By Prof. Dr. M. W. BEIJERINCK.

The emulsion reaction.

Many spore-producing and a few non spore-producing bacilli, cause, when growing in presence of cane-sugar or raffinose on neutral or feebly alkaline agarplates, a very peculiar "colloidreaction", which is also valuable for the diagnosis of these bacteria. This reaction consists in the formation, in and also on the surface of the agar around the colonies or streaks, of a liquid-"precipitate", i. e. an emulsion, which can best be recognised in transmitted light, and at the same time in a swelling of the agar caused by the increase of volume produced by the emulsion.

The emulsion consists of drops (see plate) of different size, mostly very small, but sometimes growing to 0,2 mm. so that they may