Mathematics. — "*On continuous vector distributions on surfaces*". (2nd communication)[1]). By Dr. L. E. J. BROUWER. (Communicated by Prof. D. J. KORTEWEG).

(Communicated in the meeting of February 26, 1910).

§ 1.

The tangent curves to a finite, uniformly continuous vector distribution with a finite[2]) *number of singular points in a singly connected inner domain of a closed curve.*

Let γ be the domain under consideration, then we can represent it on a sphere, so we can immediately formulate on account of the property deduced in the first communication (see there page 855):

THEOREM 1. *A tangent curve, which does not indefinitely approach a point zero, is either a simple closed curve, or its pursuing as well as its recurring branch shows one of the following characters:* 1st. *stopping at a point of the boundary of* γ; 2nd. *spirally converging to a simple closed tangent curve;* 3rd. *entering into a simple closed tangent curve.*

We now shall farther investigate the form (in the sense of analysis situs) of a tangent curve r, of which we assume, that at least *one* of the two branches (e.g. the pursuing branch) approaches indefinitely one or more points zero, i. e singular points of the vector distribution.

We start the tangent curve in a point A_0 (not a point zero) and we pursue that curve in the following way: By β_ε we understand a distance with the property that in two points lying inside the same geodetic circle described with a radius β_ε, and possessing both a distance $> \varepsilon$ from the points zero, the vectors certainly make an angle $< \frac{1}{8}\pi$ with each other. We farther choose a fundamental series of decreasing quantities $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ converging to 0, and of corresponding decreasing distances $\beta_{\varepsilon_1}, \beta_{\varepsilon_2}, \ldots$, which all we suppose, if α is the distance of A_0 from the points zero, to be smaller than $\alpha - \varepsilon_1$.

We then prove in the manner indicated in the first communication p. 852, that, when pursuing r from A_0, a point B_0 is reached, possessing a distance β_{ε_1} from A_0; we call the arc $A_0 B_0$ a β_{ε_1}-arc. According to our supposition there now exists a finite number n_1 in

1) For the first communication see these Proceedings Vol. XI 2, p. 850.

2) This restriction we shall drop in a following communication.

such a way, that after having completed n_1 β_{ε_1}-arcs, but not yet n_1+1 β_{ε_1}-arcs, we reach a point A_1, where for the first time we have approached the points zero as far as a distance ε_1. Then again there is a finite number n_2 in such a way that, having completed from A_1 n_2, but not yet n_2+1 β_{ε_2}-arcs, we reach a point A_2, where for the first time we have approached the points zero as far as a distance ε_2. From there we pursue r with β_{ε_3}-arcs and continue this process indefinitely.

If we understand by $m(\varepsilon_n)$ the maximum distance from the points zero, which r reaches when being pursued *after* having for the first time approached the points zero as far as a distance ε_n, then a first possibility is, that $m(\varepsilon_n)$ converges with ε_n to zero.

In that case the pursuing branch converges to one single point zero and it is *an arc of simple curve, stopping at that point zero.*

We now suppose the second possibility, that $m(\varepsilon_n)$ surpasses for each ε_n a certain finite quantity e. Then we can effect (by eventually omitting a finite number of terms of the series of ε_n's), that each $\varepsilon < \frac{1}{2} e$ and each $\beta_{\varepsilon_n} < \frac{1}{2} e$.

On the pursuing branch then certainly two points P_1 and Q_1 can be indicated both at a distance e from the points zero, and separated on r by at least one point at a distance ε_1 from the points zero, whilst the distance between P_1 and Q_1 is $\leq \frac{1}{4} \beta_{\varepsilon_1}$. Let $P_1 S$ and $Q_1 U$ be pursuing β_{ε_1}-arcs, and $P_1 R$ and $Q_1 T$ recurring β_{ε_1}-arcs.

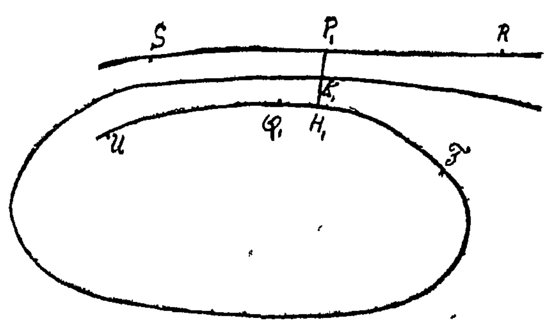

Fig. 1.

Let H_1 be a point of TU, having from P_1 the smallest possible distance, then H_1 cannot coincide with T or U, so that the geodetic arc $P_1 H_1$ is in H_1 normal to the vector direction, and the vector directions in all points of that geodetic arc, forming with each other an angle $< \frac{1}{8}\pi$, are directed to the same side of the geodetic arc P_1H_1.

Let K_1 be the last point of intersection of the arc P_1H_1 of r with the geodetic arc P_1H_1. Then the arc K_1H_1 of r and the geodetic arc K_1H_1 form a simple closed curve, and we prove in the manner indicated on page 853 of our first communication, that either the pursuing branch of r from H_1 lies in the inner domain, and the recurring branch from K_1 in the outer domain, or the pursuing branch from H_1 in the outer domain, and the recurring branch from K_1 in the inner domain.

Let us first assume that the pursuing branch lies in the *inner domain*, then certainly two points P_2 and Q_2 can be chosen on it, both at a distance e from the points zero and separated on r by at least one point at a distance ε_2 from the points zero, whilst the distance between P_2 and Q_2 is $\leqq \frac{1}{4}\beta_{\varepsilon_2}$. With the aid of those two points we construct in the same way as above now a simple closed curve, consisting of an arc K_2H_2, of r and a geodetic arc K_2H_2, in whose inner domain lies the pursuing branch of r from H_2.

Going on in this way we construct a fundamental series of closed curves $u_1, u_2, u_3, \ldots$ lying inside each other. If there is a domain or set of domains G, common to all the inner domains of these curves (which, as we shall presently show, is really the case) then the boundary of G can only be formed by points belonging to none of the curves $u_1, u_2, u_3, \ldots$ but being limit points of fundamental series of points lying on those curves

We assume $q > p$, and B to be a point of u_q having a distance $> 3\,\varepsilon_p$ and $> 3\,\beta_{\varepsilon_p}$ from the points zero. Let C be the first point when recurring from B, and D the first point when pursuing from B, which reaches a distance $\frac{1}{2}\beta_{\varepsilon_p}$ from B, then we shall assume for a moment that there exists on u_q, but not on the arc CD, a point S lying at a distance $\leqq \frac{1}{4}\beta_{\varepsilon_p}$ from B, and we shall show that this assumption leads to an absurdity.

Let SV be a recurring $\frac{1}{2}\beta_{\varepsilon_p}$-arc and SW a pursuing $\frac{1}{2}\beta_{\varepsilon_p}$-arc on u_q, then the arcs CD and VW can have no point in common, and the geodetic arc $K_q H_q$, belonging to u_q, has either no point in common with VW, or none with CD.

In the first case we determine on VW a point M, having from B a distance as small as possible. The geodetic arc BM is then in M normal to VW, and has a last point of intersection N with CD, so that the geodetic arc NM forms with one of the arcs NM of u_q, not containing e.g. the point C, a closed curve; u_q, taken with a certain sense of circuit, would at M enter one of the two domains determined by this closed curve, to leave it no more; further C would lie outside that domain; thus u_q would never be able to reach C, with which the absurdity of our assumption has been proved.

In the second case we determine on CD a point M having from S a distance as small as possible, and on the geodetic arc SM the last point of intersection N with VW. The further reasoning remains analogous to the one just followed: the parts of the arcs VW and CD are only interchanged.

Let now B_ω be the only limit point of a certain fundamental series of points $B_1, B_2, B_3, \ldots$, lying respectively on $u_1, u_2, u_3, \ldots$ We assume that B_ω is not a point zero; it has then for a suitably selected p a distance $> 4\,\varepsilon_p$ and $> 4\,\beta_{\varepsilon_p}$ from the points zero.

Let further each m_k be $> p$ and let $B_{m_1}, B_{m_2}, B_{m_3}, \ldots$ be a fundamental series contained in the series just mentioned, whose points have all from B_ω a distance $< \frac{1}{8}\,\varepsilon_p$ and $< \frac{1}{8}\,\beta_{\varepsilon_p}$.

If then further on the different u_{m_k} $B_{m_k} D_{m_k}$ are pursuing, $B_{m_k} C_{m_k}$ recurring $\frac{1}{2}\,\beta_{\varepsilon_p}$-arcs, we prove by the reasoning followed in the first communication p. 854, that there exists a series $C_{n_1} D_{n_1}$, $C_{n_2} D_{n_2}$, $C_{n_3} D_{n_3}, \ldots$ converging uniformly to an arc $C_\omega D_\omega$ of a tangent curve u_ω in such a way, that all arcs $C_{n_k} D_{n_k}$ lie on the same side of $C_\omega D_\omega$.

If we describe round B_ω a geodetic circle with radius $\frac{1}{8}\,\beta_{\varepsilon_p}$, then it cuts from $C_\omega D_\omega$ an arc FI containing B_ω; this arc divides its inner domain into two regions, into one of which, to be called g, neither the arcs $C_{n_k} D_{n_k}$, nor any other parts of the curves u_{n_k} can penetrate, as they would get there a distance $\leqq \frac{1}{4}\,\beta_{\varepsilon_p}$ from B_{n_k}.

As further the region g cannot lie outside all curves u_{n_k}, it must lie inside all curves u_{n_k}.

So there is certainly a domain or a set of domains G, common to all the inner domains of the curves u_k, and to the boundary of G belong all points of the limit set λ of the u_k's, which are not points zero, thus also all points of λ, which are points zero, as the latter are limit points of the former ones. So the boundary of G is identical to the limit set of the u_k's, is therefore coherent and identical to its *outer circumference,* whilst abroad from the points zero it consists of tangent curves to the vector distribution, which on account of the existence of the domain g can show nowhere in a non-singular point the character mentioned in theorem 1 sub 3.

We shall now show that a tangent curve r' belonging to the boundary of G cannot have the property of r, that its pursuing or recurring branch converges spirally to the boundary of a domain or set of domains G'.

We should then namely be able to form, in the same way as was done above and in the first communication for r, also for r' a closed curve u'_k consisting of a geodetic arc $\leqq \frac{1}{4} \beta_{\varepsilon_k}$ and an arc φ' of r', joining the same two points K' and H'. And there would exist arcs of r which would converge uniformly to φ' from the same side, e.g. from the inner side of u'_k. But when pursuing such an arc ψ of r situated in sufficient vicinity of φ', we should never be able to return between ψ and φ'.

As farthermore in the case considered here, that the pursuing branch of r lies in the inner domain of u_1, it is also excluded, that r' reaches the boundary of γ, only *one* form remains possible for r', namely *that of an arc of simple curve, starting from a point zero, and stopping at a point zero.* (For the rest these two end points can very well be identical).

Of such tangent curves there can be in the boundary of G at most two, which possess the same end points, when these end points are different; but there can be an infinite number, which are closed in the same point zero. Of these however there are only a finite number, of which the extent surpasses an arbitrarily assumed finite limit. For, each of these contributes to G a domain with an area, which surpasses a certain finite value.

The curves r' whose extent surpasses a certain finite limit are run along by a u_k of sufficient high index in the same order, as they succeed each other on the outer circumference of G. From this ensues *that for all curves r' the pursuing sense belongs to the same sense of circuit of the outer circumference of G.*

If the pursuing branch of r lies in the *outer domain* of u_1, the preceding holds with slight modifications. A point of the limit set of the u_k's now necessarily bounds a region belonging to γ, and lying outside all u_k's, only then when it is not a point of the boundary of γ. The *inner circumference,* to which r *now* converges spirally *on the inner side,* consists here again of arcs of simple curve, which are tangent curves to the vector distribution, but these tangent curves can lie entirely or partially in the boundary of γ.

However they have all again a pursuing sense belonging to the same sense of circuit of the circumference.

We now agree about the following: When a pursuing branch of a tangent curve reaches a point zero, we continue it, if possible, along a pursuing branch, starting from that point zero, and not meeting the former within a certain finite distance; but if such a continuation is impossible, we stop the branch at that point zero, and so we do likewise when the branch has entered into a closed curve or has approximated spirally a circumference. Then we can resume the preceding reasonings as follows:

THEOREM 2. *A tangent curve is either a simple closed curve, or save its ends it is an arc of simple curve, of which the pursuing as well as the recurring branch shows one of the following characters:* 1st. *stopping at a point of the boundary of* γ; 2nd. *stopping at a point zero;* 3rd. *entering into a simple closed tangent curve;* 4th. *spirally converging to a circumference, consisting of one or more simple closed tangent curves.*

From this ensues in particular:

THEOREM 3. *A tangent curve cannot return into indefinite vicinity of one of its points, after having reached a finite distance from it, unless it be to close itself in that point.*

That the last theorem is not a matter of course, is evident from the fact that it does not hold for an annular surface. On this it is easy to construct tangent curves of the form pointed out by LORENTZ (Enz. der Math. Wiss. V 2, p. 120, 121).

We finally notice that the vector distribution considered in this §, does not possess of necessity a singular point (as is the case on the sphere). This is proved directly, by considering in the inner domain of a circle, situated in a Euclidean plane, a vector everywhere constant.

§ 2.

The structure of the field in the vicinity of a non-singular point.

To classify the singular points we shall surround each of them with a domain which we shall cover entirely with tangent curves not crossing each other and we shall investigate the different ways in which that covering takes place in different cases. For the sake of more completeness and as an introduction we first do the same for a non-singular point.

Let P be the point under consideration, RS an arc of tangent curve r containing P, UV an arc containing P of an orthogonal curve of the vector distribution. We draw through U and V tangent curves α_0 and α_1, and through R and S orthogonal curves γ and δ, and we let the four points R, S, U, and V converge together to P. Before they have reached P, a moment comes when α_0, α_1, γ, and δ form a curvilinear rectangle, inside which lies P, and inside which lies no point zero of the vector distribution, thus inside which on account of the first communication no closed tangent curve can be drawn.

We shall cover this curvilinear rectangle with tangent curves not crossing each other.

We number α_0 with 0, r with $\frac{1}{2}$, α_1 with 1. Let $Q_{\frac{1}{4}}$ be a point inside or on the rectangle $A_0\,B_0\,S\,R$ (fig. 2) having from α_0 and r

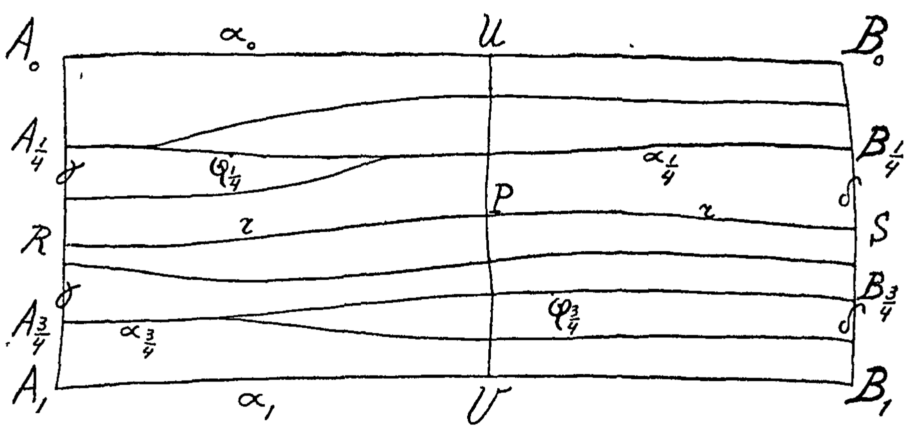

Fig. 2. Non-singular point.

a distance as large as possible. We draw through $Q_{\frac{1}{4}}$ a tangent curve $\alpha_{\frac{1}{4}}$, about which we agree, that, if it meets α_0 or r, we shall continue it, by pursuing or recurring α_0, or r, until we come upon γ or δ.

Then $\alpha_{\frac{1}{4}}$ is a tangent curve joining two points $A_{\frac{1}{4}}$ and $B_{\frac{1}{4}}$ of γ and δ between α_0 and r. In the same way we construct inside the rectangle $A_1 B_1 S R$ a tangent curve $\alpha_{\frac{3}{4}}$, joining two points $A_{\frac{3}{4}}$ and $B_{\frac{3}{4}}$ of γ and δ between r and α_1. The rectangle $A_0 B_0 B_1 A_1$ is then divided into four regions. In these we choose in the way described above successively the points $Q_{\frac{1}{8}}$, $Q_{\frac{3}{8}}$, $Q_{\frac{5}{8}}$, $Q_{\frac{7}{8}}$, draw through $Q_{\frac{1}{8}}$ a tangent curve $\alpha_{\frac{1}{8}}$ joining two points $A_{\frac{1}{8}}$ and $B_{\frac{1}{8}}$ of γ and δ, and we deal analogously with the other three points.

Going on in this manner we construct for each fraction $\frac{a}{2^n} < 1$ a tangent curve $\alpha_{\frac{a}{2^n}}$ joining two points of γ and δ; two of these curves chosen arbitrarily can coincide *partially*, but they cannot cross each other.

All these tangent curves must now cover everywhere densely the inner domain of the rectangle $A_0 B_0 B_1 A_1$. For, if they left there open a domain G, then a domain G'_n bounded by two tangent curves with indices $\frac{a}{2^n}$ and $\frac{a+1}{2^n}$ would converge to G. For n sufficiently great however the point $Q_{\frac{2a+1}{2^{n+1}}}$ would then lie inside G, thus in contradiction to the supposition also a tangent curve $\alpha_{\frac{2a+1}{2^{n+1}}}$ would pass through G.

From this ensues, that, if we add the limit elements of the tangent curves $\alpha_{\frac{a}{2^n}}$, which are likewise tangent curves, the inner domain of the rectangle $A_0 B_0 B_1 A_1$ is entirely covered, and further there is for each real number between 0 and 1 *one* and not more than *one* of these tangent curves having that number as its index.

§ 3.

The structure of the field in the vicinity of an isolated singular point. First principal case.

We surround the point zero P, supposed isolated, with a simple closed curve c, inside which lies no further point zero. And we assume as a first principal case that c can be chosen in such a way that inside c no simple closed tangent curve exists, inside which P lies. On account of the first communication there can exist inside c neither a simple closed tangent curve, outside which P lies. We now distinguish 2 cases:

a. There exists inside c a simple closed tangent curve ϱ through P. We can then choose c smaller, so that it meets ϱ, thus containing in its inner domain a tangent curve ϱ_1 which (in its pursuing direction) runs from P to c, and another ϱ_2 running from c to P, and we further look for such tangent curves inside c which cross neither ϱ_1 nor ϱ_2. Of the possible kinds of tangent curves mentioned at the conclusion of § 1 we shall agree about those, which enter into a closed tangent curve, to continue them along that tangent curve until they reach either P or c, and to stop there. Spirally converging to an inner circumference cannot appear, as the other end of such a tangent curve would be separated from P as well as from c, and so would determine a closed tangent curve, outside which P would be lying, which is impossible. Neither can appear spirally converging to an outer circumference, as P would have to lie in that outer circumference and the spiral would necessarily have to cross ϱ_1 and ϱ_2.

b. There exists inside c no simple closed tangent curve through P. Then inside c there exists no simple closed tangent curve at all, so that again spirally converging is excluded.

In any case, if we agree not to continue a tangent curve, when it reaches P or c, we can distinguish the tangent curves inside c, and not crossing ϱ_1 and ϱ_2 if the latter exist, into three categories:

1st. *Closed curves, containing P but not reaching c.*

2nd. *Arcs of curve, joining two points of c, but not containing P.*

3rd. *Arcs of curve which run from P to a point of c* (positive curves of the third kind) *or from a point of c to P* (negative curves of the third kind).

Of this third kind there must certainly exist tangent curves. For otherwise the closed sets determined by the curves of the first, and by those of the second kind would cover the whole inner domain of c, thus would certainly possess a point in common; through this point however a curve of the third kind would pass.

So we can commence by constructing one curve of the third kind and we choose eventually ϱ_1 for it. *If possible,* we then draw a second curve of the third kind not crossing the first and we choose eventually ϱ_2 for it. Into each of the two sectors, determined in this way inside c, we introduce if possible again a curve of the third kind, not crossing the already existing ones, and chosen in such a way that it reaches a distance as great as possible from the two curves of the third kind, which bound the sector, whilst, if the new curve terminates somewhere on one of the curves bounding the sector, we further follow the latter curve. In each of the sectors, determined after that in the inner domain of c, we repeat if possible this insertion, and we continue this process as often as possible, eventually to an indefinite number of insertions.

If in this manner we have obtained an infinite number of tangent curves of the third kind, they determine limit elements which each are either again a tangent curve of the third kind, or contain such a curve as a part. And in particular a fundamental series of positive respectively negative curves of the third kind determines in its limit elements again positive respectively negative curves of the third kind.

After addition of these limit curves of the third kind we are, however, quite sure that no new curves of the third kind not crossing the existing ones can be inserted. This is evident from a reasoning analogous to that followed in § 2. The whole of the curves of the third kind, obtained now, we shall call *a system of base curves of the vicinity of P*.

An arbitrary positive base curve and an arbitrary negative one enclose inside c a sector, of which the area cannot fall below a certain finite limit. For otherwise we should have a fundamental series of positive base curves, and a fundamental series of negative ones, possessing the same base curve as a limit element, which is impossible, as that limit base curve would have to be positive as well as negative.

So the inner domain of c is divided into a finite number of sectors which can be brought under the two following categories:

First category. Sectors bounded by a positive and a negative base curve, between which lie no further base curves. The areas of these sectors surpass a certain finite limit.

Second category. Sectors bounded by two positive (respectively two negative) base curves and containing only positive (respectively negative) base curves. A sector of this category can reduce itself in special cases to a single base curve.

We shall first treat a sector of the *first category* and to that end we first notice that *outside* a curve of the second kind lying in it (i. e. between that curve and c) lie only curves of the second kind, and *inside* a curve of the first kind lying in it only curves of the first kind.

If we draw in the sector a well-ordered series, continued as far as possible, of curves of the second kind enclosing each other, then it converges either to a curve of the second kind, or to two curves of the third kind and between them a finite or denumerable set of curves of the first kind, *not* enclosing each other, and *not* approaching c indefinitely.

If we can construct an infinite number of such series not enclosing each other, then there are among them which cut from the sector an area as small as one likes, and at the same time the maximum distance, which such a series reaches from c, decreases under each finite limit.

And analogously, if we draw in the sector a well-ordered series, continued as far as possible, of curves of the first kind enclosing each other, it converges either to a curve of the first kind, or to two curves of the third kind and between them a finite or denumerable set of curves of the second kind, *not* enclosing each other, and *not* approaching P indefinitely.

If we can construct an infinite number of such series not enclosing each other, then there are among them which enclose an area as small as one likes, and at the same time the maximum distance, which such a series reaches from P, decreases under each finite limit.

From this ensues that for the sectors of the first category we have to distinguish two cases:

First case. There are curves of the second kind in indefinite vicinity of P. Then the domain of the curves of the second kind is bounded by the two base curves which bound the sector, and a finite or denumerable number of curves of the first kind, *not* enclosing each other, and *not* approaching c indefinitely, in whose inner domains, which we call the *leaves* of the sector, can lie only curves of the first kind.

The region outside the leaves can be covered as follows with curves of the second kind not crossing each other: we first construct one which reaches a distance as great as possible from c and the boundary of the leaves; in this way two new regions are determined, in each of which we repeat this insertion. This process we continue indefinitely, and finally we add the limit curves. That then the region outside the leaves is entirely covered, is evident from the reasoning followed in § 2.

And in the same way we fill each of the leaves with curves of the first kind not crossing each other. The whole of the tangent curves filling the sector finally gets the form indicated in fig. 3. The sectors being in the discussed first case we shall call *hyperbolic sectors*.

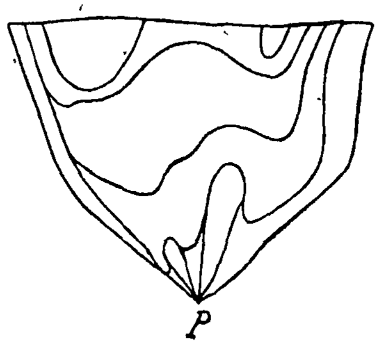

Fig. 3. Hyperbolic sector.

Second case. There are no curves of the second kind in indefinite vicinity of P. Then the domains covered by these curves are cut off from the sector by a finite or denumerable number of curves of the second kind, *not* enclosing each other, and *not* approaching P indefinitely. These domains we take from the sector (consequently modify an arc of c), and there remains a new sector, bounded by the same base curves as the old one, but consisting of one *leaf* inside which lie only curves of the first kind. This leaf we can fill with curves of the first kind not crossing each other (see fig. 4).

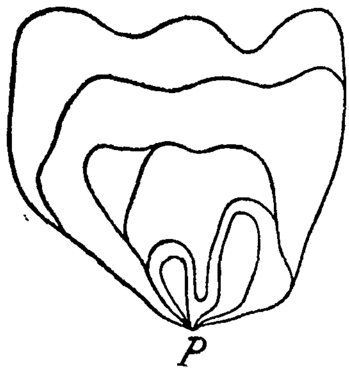

Fig. 4. Elliptic sector.

These sectors of the second case, which are reduced to a single leaf, we shall call *elliptic sectors.*

We now pass to the discussion of a sector of the *second category,* of which, to fix our ideas, we assume, that it is bounded by two positive base curves.

Let us consider the set of points lying in the sector or on its boundary, through which curves of the second kind not crossing the base curves can be drawn. This set of points cannot approach P indefinitely, as otherwise it would give rise to a negative curve of the third kind not crossing the base curves, which is excluded.

In the same way as for the elliptic sectors we destroy the regions covered by this set of points, and there remains a sector of the second category bounded by a modified arc of c, inside which no curves of the second kind not crossing the base curves can be drawn.

In the modified sector we now consider the set of points, through which curves of the first kind not crossing the base curves can be drawn, and it is clear that this set of points cannot indefinitely approach the just now modified curve c. The regions covered by it are therefore bounded by a finite or denumerable number of curves of the first kind, *not* enclosing each other, *not* indefinitely approaching c, and each enclosing a domain which forms a *leaf,* not differing from those appearing in the hyperbolic sectors.

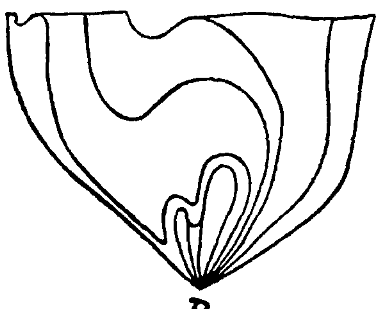

Fig. 5. Parabolic sector.

By the method applied above already several times the region outside the leaves can be filled with curves of the third kind (for instance we can choose for them the system of base curves present already in the sector), and finally each of the leaves with curves of the first kind (see fig. 5).

The sectors of the second category we shall call *positive* (resp. *negative*) *parabolic sectors.*

In special cases the whole inner domain of c can reduce itself to a single positive (resp. negative) parabolic sector. A point zero where this occurs we shall call a *source point* resp. *vanishing point.*

§ 4.

The structure of the field in the vicinity of an isolated singular point. Second principal case.

In this case any vicinity of P contains a simple closed tangent curve inside which P lies. We can then construct a fundamental series c, c', c'', ... of simple closed tangent curves converging to P, of which each following one lies inside each preceding one, and we can fill in the following way the inner domain of c with tangent curves not crossing each other.

In each annular domain between two curves $c^{(n)}$ and $c^{(n+1)}$ we choose a point having from the boundary of that domain a distance as great as possible and we lay through it a tangent curve situated in the annular domain. According to § 1 it is either closed or it gives rise to two closed curves, situated in the annular domain with its boundary, into which it terminates or to which it converges spirally, and which we draw likewise. (These closed tangent curves can entirely or partially coincide with $c^{(n)}$ or $c^{(n+1)}$). So the annular domain is either made singly connected or it is divided into two or three (annular or singly connected) new domains.

In each of these we again choose a point having from the boundary a distance as great as possible and we lay through it again a tangent curve. A singly connected domain is certainly divided by it into two singly connected domains; on an annular domain it has the effect just now mentioned.

We repeat this process indefinitely. For each domain it can

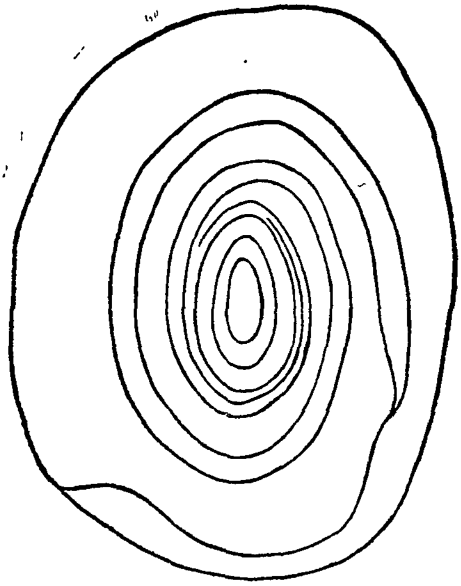

Fig. 6. Rotation point,

happen only once that it undergoes no division; after that namely it becomes singly connected, so is divided at each new insertion of a tangent curve (see fig. 6).

We finally add the limit curves, and we prove in the same way as in § 2 that then through each point of the inner domain of c passes a tangent curve.

A point zero being in the second principal case we shall call a *rotation point.*

So we can say:

THEOREM 4. *An isolated singular point is either a rotation point, or a vicinity of it can be divided into a finite number of hyperbolic, elliptic, and parabolic sectors.*

The filling of a vicinity of a non-singular point in § 2 furnishes in this terminology two hyperbolic and two parabolic sectors.

We must add the observation that in the most general case, where neither in a singular, nor in a non-singular point the tangent curve is determined, sometimes by a modified method of construction, the structure of the first principal case can be given to a vicinity of a point zero being in the second principal case.

Even the form of the sector division of the first principal case is then not necessarily unequivocally determined. Out of the reasonings of the following § we can, however, deduce that, if modifications are possible in the form of the sector division, the difference of the number of elliptic sectors and the number of hyperbolic sectors always remains the same.

§ 5.

The reduction of an isolated singular point.

For what follows it is desirable to represent the domain γ on a Euclidean plane, and farther to substitute for the curve c a simple closed curve c' emerging nowhere from c, containing likewise P in its inner domain, and consisting of arcs of tangent curves and of orthogonal curves. In the second principal case this is already attained, and in the first principal case we have to modify in a suitable way only those arcs of c which bound the hyperbolic and the parabolic sectors.

In a hyperbolic sector we effect this by choosing a point on each of the two bounding base curves, and by drawing from those points H and K inside into the sector orthogonal arcs not intersecting one another. Then there is certainly an arc of a curve of the second kind joining a point B of one of these orthogonal arcs with a point C of the other, and we bound the modified sector by the orthogonal arcs HB and CK and the tangent arc BC.

If a parabolic sector is bounded by the base curves k and k', it is always possible to choose between them a finite number of base curves $k_1, k_2, \ldots k_n$ in such a way, that each k_p and k_{p+1} can be connected, inside the sector but outside the leaves lying in it, by an orthogonal arc. By those orthogonal arcs and the arcs of base curves joining their endpoints we bound the modified sector. The simple closed curve c' obtained in this way has a direction of tangents varying everywhere continuously, with the exception of a finite number of rectangular bends. To a definite sense of circuit of c', which we shall call the positive one, corresponds in each point of c' a definite tangent vector, and for a full circuit of c' that tangent vector describes a positive angle 2π.

We shall now consider two successive parabolic sectors, π_1 and π_2, of which (for the positive sense of circuit) the first is positive, therefore the second negative, and we suppose them to be separated by a hyperbolic sector F. On the orthogonal arcs belonging to the boundary of π_1 the given vector then forms with the tangent vector an angle $\left(2n - \frac{1}{2}\right)\pi$ (measured in the positive sense), on the orthogonal arcs belonging to the boundary of π_2 an angle $\left(2n + \frac{1}{2}\right)\pi$.

The transition takes place along the tangent arc belonging to the boundary of F, by a negative rotation over an angle π of the given vector with respect to the tangent vector.

The same remains the case if we suppose π_1 to be negative, π_2 to be positive.

But if we suppose F to be an elliptic sector, then the transition under discussion takes place along the tangent arc bounding F, by a positive rotation over an angle π of the given vector with respect to the tangent vector.

As now the total angle, which the given vector describes for a full circuit of c', is equal to the total angle which the tangent vector describes plus the total angle which the given vector describes with respect to the tangent vector, the former angle is equal to $\pi\,(2 + n_1 - n_2)$, where n_1 represents the number of elliptic sectors, n_2 the number of hyperbolic ones.

Let further j be an arbitrary simple closed curve enveloping P, but enveloping no other singular point, then we can transform c' into j by continuous modification in such a way, that at every moment

P, but no other singular point, is enveloped by the modified curve. If we consider for each of the intermediary curves the total angle which the given vector describes by a positive circuit, then on one hand it can only have continuous modifications and on the other hand it must remain a multiple of 2π. Thus it remains unchanged, and we can formulate:

THEOREM 5. *The total angle which, by a circuit of a simple closed curve enveloping only one point zero, the vector describes in the sense of that circuit, is equal to* $\pi(2+n_1-n_2)$, *where* n_1 *represents the number of elliptic sectors,* n_2 *the number of hyperbolic ones, which appear when a vicinity of the point zero is covered with tangent curves not crossing each other.*

In particular for source points, vanishing points and rotation points this angle is equal to $+2\pi$.

We now surround P with a simple closed curve $\varkappa$ which can be supposed as small as one likes, and we leave the vector distribution outside $\varkappa$ and on $\varkappa$ unchanged, but inside $\varkappa$ we construct a modified distribution in the following way:

Let us first suppose that for a positive circuit of $\varkappa$ the vector describes a positive angle $2n\pi$. From an arbitrary point Q inside $\varkappa$ we draw to $\varkappa$ n arcs of simple curve $\beta_1, \beta_2, \ldots \beta_n$, not cutting each other and determining in this order a positive sense of circuit. Let us call ${}_p\varkappa$ the arc of $\varkappa$ lying between β_p and β_{p+1}, and G_p the domain bounded by β_p, ${}_p\varkappa$ and β_{p+1}. Along β_1 we bring an arbitrary continuous vector distribution becoming nowhere zero and passing on $\varkappa$ into the original one. Then along β_2 such a one passing on $\varkappa$ and in Q into the already existing vectors, that along the boundary of G_1 positively described the vector turns a positive angle 2π. Then along β_3 such a one passing on $\varkappa$ and in Q into the existing vectors, that along the boundary of G_2 positively described the vector turns a positive angle 2π, etc.

As the angle described by the vector in a positive circuit of $\varkappa$ is equal to the sum of the angles described in positive circuits of the boundaries of the domains $G_1, G_2, \ldots . G_n$, it is finally evident, that also for a positive circuit of G_n the vector describes a positive angle 2π.

In each of the domains G_p with boundary $\varkappa_p$ we choose a simple closed curve c_p not meeting $\varkappa_p$, of which in a suitable system of coordinates the equation can be written in the form $x_p^2+y_p^2=r^2$.

Inside and on c_p we introduce a finite continuous vector distribution vanishing only in the point $(o,o)_p$, which is directed along the lines

$\frac{y_p}{x_p} = a$ and from the point $(o,o)_p$. This vector describes along c_p a positive angle 2π, just as the existing one along $\varkappa_p$. If then according to SCHOENFLIES we fill the annular domain between $\varkappa_p$ and c_p with simple closed curves enveloping each other and as functions of a cyclic parameter passing continuously into each other, then we can thereby at the same time make the vector distribution along $\varkappa_p$ pass continuously into that along c_p, and in this way give to the annular domain between $\varkappa_p$ and c_p a finite continuous vector distribution vanishing nowhere. Inside $\varkappa_p$ we have now obtained a finite continuous vector distribution, having but *one* point zero, namely the point $(o,o)_p$, and that a source point of very simple structure, which we shall call a *radiating point*.

And the inner domain of $\varkappa$ is covered with a finite continuous vector distribution passing on $\varkappa$ into the original one and possessing inside $\varkappa$, instead of the original point zero P, n radiating points.

Let us furthermore suppose that for a positive circuit of $\varkappa$ the vector describes a negative angle $2n\pi$. In an analogous way as above we then divide the inner domain of $\varkappa$ into n regions G_p with boundaries $\varkappa_p$, and we bring along each of these boundaries such a vector distribution, that for a positive circuit of $\varkappa_p$ the vector describes a negative angle 2π.

The curves c_p are introduced again as above, but inside and on c_p we introduce a finite continuous vector distribution vanishing only in the point $(0,0)_p$, which is directed along the lines $x_p\, y_p = a$. For a positive circuit this vector describes along c_p a negative angle 2π, just as the existing vector along $\varkappa_p$.

So the annular domain between $\varkappa_p$ and c_p can be filled up in an analogous way as just now with a finite continuous vector distribution vanishing nowhere, and the whole distribution inside $\varkappa_p$ possesses then only *one* point zero, namely the point $(0,0)_p$, having four hyperbolic sectors of very simple form (the four separating parabolic sectors are each reduced to a single line), which structure we characterize by the name of *reflexion point*.

After this the inner domain of $\varkappa$ is covered with a finite continuous vector distribution passing on $\varkappa$ into the original one and possessing inside $\varkappa$, instead of the original point zero P, n reflexion points.

Let us finally suppose that for a circuit of $\varkappa$ the total angle described by the vector is zero. We can then choose inside $\varkappa$ such a simple closed curve c, that in a suitable system of coordinates its equation can be written in the form $x^2 + y^2 = r^2$. Inside and on c we introduce a finite continuous vector distribution vanishing nowhere, which is directed along the lines $y = a$. The total angle described by this vector along c is zero, just as the one described by the existing vector along $\varkappa$. The annular domain between $\varkappa$ and c can thus be filled up as in the two preceding cases with such a finite continuous vector distribution, that the whole distribution inside $\varkappa$ is now free of points zero.

So we can formulate:

THEOREM 6. *A finite continuous vector distribution with a finite number of points zero can be transformed, by modifications as small as one likes inside vicinities of the points zero which can be chosen as small as one likes, into a new finite continuous vector distribution which has as points zero only a finite number of radiating points, and a finite number of reflexion points.*

In particular those points zero about which the angle, described by the vector for a positive circuit, is positive, are broken up into radiating points; those about which this angle is negative, are broken up into reflexion points; whilst those for which it is zero, vanish.

In a following communication we shall extend this theorem to distributions with an infinite (denumerable or continuous) number of points zero.

§ 6.

Remarks on the tangent curves and singular points on a sphere.

If we have on a sphere a finite continuous vector distribution with a finite number of singular points, then the reasonings of § 1 lead with small modifications to:

THEOREM 7. *A tangent curve to a finite continuous vector distribution with a finite number of singular points on a sphere is either a simple closed curve, or save its ends it is an arc of simple curve, of which the pursuing as well as the recurring branch either stops at a point zero, or enters into a simple closed tangent curve, or converges spirally to a circumference consisting of one or more simple closed tangent curves.*

From this ensues that also on a sphere a tangent curve cannot return into indefinite vicinity of one of its points, after having reached a finite distance from it, unless it be, to close itself in that point.

Out of the reasoning of § 1 we can deduce farthermore without difficulty that a fundamental series of closed tangent curves with the property that of the two domains determined by one of them, one contains no points of the preceding, the other no points of the following curves, converges either to a single singular point, or to the outer circumference, consisting of simple closed tangent curves, of a domain or set of domains.

Let now an arbitrary finite continuous vector distributionon a sphere be given. On account of § 5 we reduce it by means of indefinitely small modifications to a "reduced distribution", possessing as singular points only radiating points and reflexion points, and we investigate the tangent curves of that reduced distribution.

A closed tangent curve can possess no radiating points, but reflexion points it can possess (its tangent direction shows there a rectangular bend).

On the other hand a tangent curve can only stop at a radiating point.

We now consider an arbitrary tangent curve: according to theorem 7 *it is either an arc of simple curve joining two radiating points*, or it gives rise to a simple closed tangent curve j_0, which divides the sphere into two domains G and G'.

Then on j_0 no radiating point can lie, but we shall prove, that in G as well as in G' there must lie one.

If namely there were no radiating point in G, we could consider within G a new tangent curve, and as this would not be able to stop in G, it would on account of theorem 2 give rise to a new simple closed tangent curve j_1 enclosing a domain G_1 being a part of G. Within G_1 we could again consider an arbitrary tangent curve, and in this way we should arrive at a simple closed tangent curve j_2 enclosing a domain G_2 being a part of G_1.

Continuing this process indefinitely we construct a fundamental series of closed tangent curves $j_0, j_1, j_2, j_3, \ldots$, which cannot converge to a single singular point, as neither a radiating point nor a reflexion point contains closed tangent curves in an indefinitely small vicinity. On account of the remark made at the beginning of this § there must thus be at least *one* domain G_ω, bounded by a simple closed tangent curve j_ω, and contained in each of the domains $G_1, G_2, G_3, \ldots$

Within G_ω we could again construct a closed tangent curve $j_{\omega+1}$ bounding a domain $G_{\omega+1}$ being a part of G_ω, and we could continue this process to *any* index of the second class of numbers, which on the other hand is impossible, as the set of domains $G - G_1$, $G_1 - G_2, \ldots G_\omega - G_{\omega+1}, \ldots G_\alpha - G_{\alpha+1}, \ldots$ must remain denumerable.

So we finally formulate:

THEOREM 8. *A reduced distribution on a sphere possesses at least two radiating points.*