Huygens Institute - Royal Netherlands Academy of Arts and Sciences (KNAW)

Citation:

H.J.E. Beth, The oscillations about a position of equilibrium where a simple linear relation exists between the frequencies of the principal vibrations (second part), in:
KNAW, Proceedings, 12, 1909-1910, Amsterdam, 1910, pp. 735-750

This PDF was made on 24 September 2010, from the 'Digital Library' of the Dutch History of Science Web Center (www.dwc.knaw.nl) > 'Digital Library > Proceedings of the Royal Netherlands Academy of Arts and Sciences (KNAW), http://www.digitallibrary.nl'

Mathematics, — "*The oscillations about a position of equilibrium where a simple linear relation exists between the frequencies of the principal vibrations.*" (Second part). By H. J. E. BETH. (Communicated by Prof. D. J. KORTEWEG.)

(Communicated in the meeting of February 26, 1910).

$S = 4.$ [1])

§ 14. In this case the ordinary expansions in series hold as long as $\frac{\varrho}{n_1}$ is great with respect to $\left(\frac{h}{R_1}\right)^2$ (see page 7 of the paper by Prof. KORTEWEG, mentioned above). The difficulty arises as soon as $\frac{\varrho}{n_1}$ has fallen to the order $\left(\frac{h}{R_1}\right)^2$. The calculations not getting simpler with the absence of a residue of relation, we shall immediately assume a residue of relation of order h^2.

When the relation

$$n_2 + \varrho = 3n_1$$

exists and we proceed to investigate with a view to this which terms in (2) (page 620 of these Proceedings) become disturbing in the sense indicated in § 3, we easily see that no terms of order h^2 appear among the disturbing ones. So when determining the first approximation we may omit the terms of order h^3 in the equation of the surface, which terms agree with the just mentioned terms of order h^2. It then becomes

$$z = \frac{1}{g}\,(c_1x^2 + c_3y^2 + e_1x^4 + e_2x^3y + e_3x^2y^2 + e_4xy^3 + e_5y^4)\,;$$

for we need not take for the first approximation in the equations of movement any terms of higher order than h^3.

The abridged equations of motion, containing only terms of order h, still run as follows:

$$\left.\begin{array}{l}\ddot{x} + 2c_1\,x = 0,\\ \ddot{y} + 2c_3\,y = 0.\end{array}\right\}$$

Now

$$n_1 = \sqrt{2c_1} \quad , \quad n_2 = \sqrt{2c_3}$$

are the frequencies of the principal vibrations.

---

[1]) For the case $S = 3$ see 1st part, pages 619—635 of these Proceedings.

So

$$2c_3 = (3n_1 - \varrho)^2.$$

We change the abridged equations into:

$$\left.\begin{aligned} \ddot{x} + n_1^2\, x &= 0, \\ \ddot{y} + 9n_1^2\, y &= 0\,; \end{aligned}\right\}$$

but then we must admit into the function R a term:

$$3n_1\, \varrho\, y^2.$$

The canonical solution of the abridged equations is:

$$x = \frac{\sqrt{\alpha_1}}{n_1} \cos\,(n_1 t + 2n_1 \beta_1),$$

$$y = \frac{\sqrt{\alpha_2}}{3n_1} \cos\,(3n_1 t + 6n_1 \beta_2).$$

To find which functions the α's and β's are of t, we must investigate which form the function R now assumes.

§ 15. As the disturbing terms in the equations of motion are of order h^3 we shall find that $\dot{\alpha}_1$, $\dot{\alpha}_2$, $\dot{\beta}_1$, and $\dot{\beta}_2$ can never exceed order h. Of this we may make use to simplify the terms of order h^3 containing $\ddot{x}, \ddot{y}, \dot{x}^2$, and $\dot{y}^2$. We may namely replace in those terms:

$$\begin{aligned} \dot{x}^2 \quad &\text{by} \quad \alpha_1 - n_1^2\, x^2, \\ \dot{y}^2 \quad &\text{,,} \quad \alpha_2 - 9n_1^2\, y^2, \\ \ddot{x} \quad &\text{,,} \quad -n_1^2\, x \end{aligned}$$

and

$$\ddot{y} \quad \text{,,} \quad -9n_1^2\, y.$$

Then the equations become:

$$\left.\begin{aligned} &\ddot{x} + n_1^2\, x + 4e_1\, x^3 + 3e_2\, x^2 y + 2e_3\, xy^2 + e_4\, y^3 + \\ &\qquad + \frac{n_1^4}{g^2}(\alpha_1 + 9\alpha_2)\, x - \frac{2n_1^6}{g^2}(x^2 + 81\, y^2)\, x = 0. \\ &\ddot{y} + 9n_1^2\, y - 6n_1\, \varrho y + e_2\, x^3 + 2e_3\, x^2 y + 3e_4\, xy^2 + 4e_5\, y^3 + \\ &\qquad + \frac{9n_1^4}{g^2}(\alpha_1 + 9\alpha_2) y - \frac{18n_1^6}{g^2}(x^2 + 81\, y^2)\, y = 0. \end{aligned}\right\}$$

Now the terms of order h^3 are all disturbing except $e_4 y^3$ in the first and $3e_4\, xy^2$ in the second equation; so these may be omitted.

The terms $3e_2\, x^2 y$ in the first and $e_2\, x^3$ in the second equation owe their disturbing property to the supposed relation.

The remaining terms are always disturbing, also when no relation exists.

To transform the equations to such a form that the disturbing terms may be regarded as derivatives of one and the same function resp. to x and y, let us consider the term with xy^2 in the first and that with x^2y in the second equation. If we substitute the expressions found above as first approximation for x and y in these terms, after the development of the products and powers of the cosines among others terms will appear, differing only in coefficient from the expressions indicated for x and y; the remaining terms which appear are not disturbing. From this ensues that we may replace:

in the first equation: xy^2 by $\frac{1}{18}\frac{\alpha_2}{n_1^2}x$,

in the second equation: x^2y by $\frac{1}{2}\frac{\alpha_1}{n_1^2}y$.

Accordingly the equations may be written:

$$\left.\begin{aligned} &\ddot{x} + n_1^2 x + \left(4e_1 - \frac{2n_1^6}{g^2}\right)x^3 + 3e_2 x^2 y + \left(\frac{e_3}{9n_1^2}\alpha_2 + \frac{n_1^4}{g^2}\alpha_1\right)x = 0, \\ &\ddot{y} + 9n_1^2 y + \left(4e_5 - \frac{1458n_1^6}{g^2}\right)y^3 + e_2 x^3 + \\ &\qquad + \left(-6n_1\varrho + \frac{e_3}{n_1^2}\alpha_1 + \frac{81n_1^4}{g^2}\alpha_2\right)y = 0. \end{aligned}\right\}$$

We thus see that they take the form of:

$$\left.\begin{aligned} &\ddot{x} + n_1^2 x - \frac{\partial R}{\partial x} = 0, \\ &\ddot{y} + 9n_1^2 y - \frac{\partial R}{\partial y} = 0; \end{aligned}\right\}$$

where:

$$-R \equiv \left(e_1 - \frac{n_1^6}{2g^2}\right)x^4 + \left(e_5 - \frac{729n_1^6}{2g^2}\right)y^4 + \frac{1}{2}\left(\frac{e_3}{9n_1^2}\alpha_2 + \frac{n_1^4}{g^2}\alpha_1\right)x^2 + $$
$$+ \frac{1}{2}\left(-6n_1\varrho + \frac{e_3}{n_1^2}\alpha_1 + \frac{81n_1^4}{g^2}\alpha_2\right)y^2 + e_2 x^3 y.$$

§ 16. We must now write R as function of the α's and β's by substituting for x and y, in the expressions obtained, the expressions by which they are represented at first approximation, and by retaining only those terms in which t does not appear explicitly. Thus we arrive at:

$$-R = \frac{1}{2}a\alpha_1^2 + b\alpha_1\alpha_2 + \frac{1}{2}c\alpha_2^2 + \varrho' h^2 \alpha_2 + m_1 \alpha_1^{\frac{3}{2}} \alpha_2^{\frac{1}{2}} \cos\varphi,$$

where:

$$a = \frac{3e_1}{4n_1^{\,4}} + \frac{n_1^{\,2}}{8g^2},$$

$$b = \frac{e_3}{18n_1^{\,4}},$$

$$c = \frac{e_5}{108n_1^{\,4}} + \frac{9n_1^{\,2}}{8g^2},$$

$$m_1 = \frac{e_2}{24n_1^{\,4}},$$

$$\varrho = -\frac{\varrho}{6n_1 h^2},$$

$$\varphi = 6n_1(\beta_1 - \beta_2).$$

The system of equations giving the time-variability of the α's and β's, is now:

$$\left.\begin{aligned} \frac{d\alpha_1}{dt} &= 2Nm_1\,\alpha_1^{\frac{3}{2}}\,\alpha_2^{\frac{1}{2}}\,\sin\varphi, \\ \frac{d\alpha_2}{dt} &= -\,2Nm_1\,\alpha_1^{\frac{3}{2}}\,\alpha_2^{\frac{1}{2}}\,\sin\varphi, \\ \frac{d\beta_1}{dt} &= a\alpha_1 + b\alpha_2 + \frac{3}{2}\,m_1\,\alpha_1^{\frac{1}{2}}\,\alpha_2^{\frac{1}{2}}\cos\varphi, \\ \frac{d\beta_2}{dt} &= b\alpha_1 + c\alpha_2 + \varrho' h^2 + \frac{1}{2}\,m_1\,\alpha_1^{\frac{3}{2}}\,\alpha_2^{-\frac{1}{2}}\cos\varphi; \end{aligned}\right\} \quad . \quad . \quad (17)$$

where N is put instead of $3n_1$.

From this system it appears at once that:

$$\frac{d\alpha_1}{dt} + \frac{d\alpha_2}{dt} = 0,$$

therefore

$$\alpha_1 + \alpha_2 = constant.$$

So we put:

$$\alpha_1 = R_0^{\,2}\,n_1^{\,2}\,h^2\,\zeta \quad , \quad \alpha_2 = R_0^{\,2}\,n_1^{\,2}\,h^2\,(1-\zeta).$$

Furthermore according to § 4:

$$\frac{1}{2}\,a\alpha_1^{\,2} + b\alpha_1\,\alpha_2 + \frac{1}{2}\,c\alpha_2^{\,2} + \varrho' h^2\,\alpha_2 + m_1\,\alpha_1^{\frac{3}{2}}\,\alpha_2^{\frac{1}{2}}\cos\varphi = constant$$

is an integral of the system.

By introduction of ζ this integral takes the form of:

$$\zeta\sqrt{\zeta(1-\zeta)}\,\cos\varphi = p\zeta^2 + q\zeta + r, \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where:

$$p = \frac{1}{m_1}\left(-\frac{1}{2}a + b - \frac{1}{2}c\right),$$

$$q = \frac{1}{m_1}\left(-b + c + \frac{\varrho'}{n_1^2 R_0^2}\right),$$

$$r = \frac{1}{m_1}\left(-\frac{1}{2}c - \frac{\varrho'}{n_1^2 R_0^2} + \frac{C}{n_1^4 R_0^4}\right),$$

where C represents a constant, dependent on the initial state.

The first equation of (17) becomes by the introduction of ζ:

$$\frac{d\zeta}{dt} = \frac{2}{9} m_1 R_0^2 N^3 h^2 . \zeta \sqrt{\zeta(1-\zeta)} . \sin\varphi. \quad . \quad . \quad . \quad . \quad (19)$$

By eliminating φ between (18) and (19) we arrive at:

$$\frac{d\zeta}{\sqrt{\zeta^3(1-\zeta) - (p\zeta^2 + q\zeta + r)^2}} = \pm \frac{2}{9} m_1 R_0^2 N^3 h^2 . dt.$$

Let

$$f(\zeta) \equiv \zeta^3(1-\zeta) - (p\zeta^2 + q\zeta + r)^2,$$

then $f(\zeta) > 0$ for the initial value of ζ, but $f(\zeta) < 0$ for $\zeta = 0$ and $\zeta = 1$; so $f(\zeta)$ becomes zero for two values ζ_1 and ζ_2 lying between 0 and 1.

So ζ will generally vary periodically between two limits. It may be expressed in the time with the aid of elliptic functions, after which β_1, β_2, x, and y are also known as functions of the time.

For the extreme values *zero* and *one* of the modulus $\varkappa$ of the elliptic functions ($\varkappa = \sqrt{\frac{(\beta-\alpha)(\zeta_2-\zeta_1)}{(\beta-\zeta_2)(\alpha-\zeta_1)}}$, when the equation $f(\zeta) = 0$ has two real roots α and β besides ζ_1 and ζ_2) we get special cases.

Osculating curves.

§ 17. At first approximation we have found:

$$x = \frac{\sqrt{\alpha_1}}{n_1} \cos(n_1 t + 2n_1 \beta_1),$$

$$y = \frac{\sqrt{\alpha_2}}{3n_1} \cos(3n_1 t + 6n_1 \beta_2),$$

where the α's and β's slowly vary with the time.

By introduction of ζ and φ and by change of the origin of time we find that we may determine the equation of an osculating curve by eliminating t between

$$x = R_0\, h \sqrt{\zeta}\, cos\, n_1\, t$$

and

$$y = \frac{1}{3} R_0\, h \sqrt{1-\zeta}\, cos\, (3 n_1\, t - \varphi).$$

For ζ and φ we must substitute the values, which these quantities have at the moment for which we wish to know the osculating curve.

The osculating curves are Lissajous curves answering to the value $\frac{1}{3}$ for the ratio of the periods of the vibrations. They are described in the rectangles having as sides $2\ R_0\, h \sqrt{\zeta}$ and $\frac{2}{3}\, R_0\, h \sqrt{1-\zeta}$. As ζ varies between two limits the rectangles in which the curves are described lie between two extremes. The vertices lie on the circumference of an ellipse having $2\, R_0\, h$ and $\frac{2}{3}\, R_0\, h$ as lengths of axes.

The shape of the curve described in a definite rectangle is still dependent on the value of φ, i. e. on the value of the difference in phase at the moment of the greatest deviation to the right.

To an arbitrary value of φ the wellknown Lissajous curve with two nodes of fig. 8 answers. For $\varphi = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ the curve is symmetrical in respect to the axes; the nodes lie in the X-axis on either side of O at distances $\frac{1}{2}\, R_0\, h$ (fig. 9). For $\varphi = 0$ or π we get a curve, which is described in both directions alternately and which passes through O (fig. 10).

In fig. 11 we find some of those osculating curves represented for a definite case of motion: *two* belonging to $\varphi = \pi$; *two* for $\varphi = \frac{\pi}{2}$, and *one* for an arbitrary value of $\varphi \left(> \frac{\pi}{2}\right)$.

Out of (19) follows that $\frac{d\zeta}{dt} = 0$ for $sin\, \varphi = 0$. In the extreme rectangles the curves are described which we have for $\varphi = 0$ or π. Now a number of different cases are possible, of which we get a clear representation by representing equation (18) in polar coordinates. In fig. 12 some of the curves obtained in this way are represented, where φ is taken as polar angle, $\sqrt{1-\zeta}$ as radius vector. The different shapes of the curves correspond to the roots of the equation:

$$\zeta^3 (1 - \zeta) - (p\zeta^2 + q\zeta + r)^2 = 0. \quad . \quad . \quad . \quad . \quad (20)$$

The cases are:

1. The curve indicated in the figure by - - - keeps to the right or to the left of O_1; φ changes between two limits; the limits are equal and opposite; the positive is smaller than $\frac{\pi}{2}$. For the extreme values of ζ we find φ either both times 0 or both times π.

2. The curve — — — intersects the straight line $\varphi = \frac{\pi}{2}$ at two points above O_1 and at 2 points below O_1. For the extreme values of ζ we again find φ either both times 0 or both times π.

3. The curve consists of two closed parts (a continuous line in the figure), which surround O_1. Now φ assumes all values. For the extreme values of ζ $\varphi = 0$ *one* time $\varphi = 0$ and $\varphi = \pi$ the other.

The transition case between 2 and 3 is represented by — . — . —.

Fig. 11 relates to the 2nd case; for the two extreme values of ζ we find $\varphi = \pi$.

Special cases.

§ 18. These occur for the extreme values of the modulus $\varkappa$ of the elliptic functions; two roots of equation (20) have coincided.

1. $\varkappa = 1$. The elliptic functions pass into hyperbolic ones. The geometrical representation just now discussed of the relation between ζ and φ and already mentioned as transition case between the second and third cases has a node situated on the axis of the angles. The form of motion approaches asymptotically to a form of motion belonging to $\varphi = 0$ or $\varphi = \pi$.

2. $\varkappa = 0$. The elliptic functions pass into goniometrical ones. The curve of fig. 12 becomes an isolated point C (special case belonging to the 1st case of § 17 as limiting case) or it consists of an isolated point and a closed curve (special case belonging to the 3rd case of § 17 as limiting case). If the initial value of ζ coincides with the twofold root of (20) we find that ζ remains constant; φ is continually 0 or π. Thus the same curve is continually described.

Arbitrary mechanism with 2 degrees of freedom for which $S = 4$.

§ 19. In the case that $n_2 = 3n_1 + \varrho$ the terms of order h^2 can give no disturbing terms in the equations of motion.

So we may write:

$$U = \tfrac{1}{2} n_1^2 q_1^2 + \tfrac{1}{2} n_2^2 q_2^2 + U_4,$$

where U_4 represents a homogeneous function of degree 4 in q_1 and q_2. Furthermore we find:

$$T = \tfrac{1}{2}\dot{q}_1^2 + \tfrac{1}{2}\dot{q}_2^2 + \tfrac{1}{2}P_1\dot{q}_1^2 + P_2\dot{q}_1\dot{q}_2 + \tfrac{1}{2}P_3\dot{q}_2^2;$$

where:

$$P_1 = a_1 q_1^2 + a_2 q_1 q_2 + a_3 q_2^2,$$
$$P_2 = b_1 q_1^2 + b_2 q_1 q_2 + b_3 q_2^2,$$
$$P_3 = c_1 q_1^2 + c_2 q_1 q_2 + c_3 q_2^2,$$

the a's, b's, and c's being constants.

The equations of Lagrange become:

$$\left.\begin{aligned} \ddot{q}_1 + n_1^2 q_1 &= -P_1\ddot{q}_1 - P_2\ddot{q}_2 - \frac{1}{2}\frac{\partial P_1}{\partial q_1}\dot{q}_1^2 - \frac{\partial P_1}{\partial q_2}\dot{q}_1\dot{q}_2 + \\ &\qquad + \left(\frac{1}{2}\frac{\partial P_3}{\partial q_1} - \frac{\partial P_2}{\partial q_2}\right)\dot{q}_2^2 - \frac{\partial U_4}{\partial q_1}. \\ \ddot{q}_2 + n_2^2 q_2 &= -P_2\ddot{q}_1 - P_3\ddot{q}_2 + \\ &\qquad + \left(\frac{1}{2}\frac{\partial P_1}{\partial q_2} - \frac{\partial P_2}{\partial q_1}\right)\dot{q}_1^2 - \frac{\partial P_3}{\partial q_1}\dot{q}_1\dot{q}_2 - \frac{1}{2}\frac{\partial P_3}{\partial q_2}\dot{q}_2^2 - \frac{\partial U_4}{\partial q_2}. \end{aligned}\right\}$$

In the same way as was done in § 15 we may replace $\ddot{q}_1$, $\ddot{q}_2$, $\dot{q}_1^2$, and $\dot{q}_2^2$ in the terms of order h^3 by others.

Now in the first equation a term $-a_2 q_1 \dot{q}_1 \dot{q}_2$ appears which we must consider separately (in the second equation also there are terms containing $q_1 \dot{q}_2$, but these are not disturbing).

We introduce for this a new variable q'_1 in such a way that:

$$q'_1 = q_1 + \frac{1}{4} a_2 q_1^2 q_2.$$

Then we find:

$$\ddot{q}'_1 = \ddot{q}_1 + \frac{1}{4} a_2 q_1^2 \ddot{q}_2 + \frac{1}{2} a_2 q_2 (q_1 \ddot{q}_1 + \dot{q}_1^2) + a_2 q_1 \dot{q}_1 \dot{q}_2,$$

where $\ddot{q}_1$ and $\ddot{q}_2$ in the terms of order h^3 may again be simplified.

Of the terms now appearing in the equations of motion the following are disturbing: in the first equation those with $h^2 q_1$, q_1^3, $q_1^2 q_2$ and $q_1 q_2^2$, in the second those with $h^2 q_2$, q_1^3, q_2^3 and $q_1^2 q_2$. Now just as in § 15 the terms with $q_1 q_2^2$ in the first equation, those with $q_1^2 q_2$ in the second equation may still be simplified.

If we perform these calculations the result proves that the terms of order h^3 to be inserted in the equations may be put in this form:

$Ph^2 q_1 + e q_1^2 q_2 + c q_1^3$ in the first equation.

$Qh^2 q_2 + f q_1^3 + d q_2^3$ „ „ second „

Here P and Q are homogeneous quadratic functions of $\sqrt{a_1}$ and $\sqrt{a_2}$; and

$$e = \frac{1}{4}(3n_1^2 - n_2^2)\,a_2 + b_1 n_2^2 - 3l,$$

$$f = -\frac{1}{2}n_1^2 a_2 + 3b_1 n_1^2 - l.$$

(The terms $-3l$ in e and $-l$ in f originate from the term $lq_1^3 q_2$ appearing in U_4).

In the terms of higher order we may substitute $3n_1$ for n_2 in the coefficients. We then find:

$$e = 3\left\{\left(-\frac{1}{2}a_2 + 3b_1\right)n_1^2 - l\right\},$$

$$f = \left(-\frac{1}{2}a_2 + 3b_1\right)n_1^2 - l.$$

So we find that

$$e = 3f.$$

We may now write the equations of motion:

$$\left.\begin{aligned} \ddot{q}_1 + n_1^2 q_1 &= \frac{\partial R}{\partial q_1}, \\ \ddot{q}_2 + n_2^2 q_2 &= \frac{\partial R}{\partial q_2}; \end{aligned}\right\}$$

where

$$R = \frac{1}{2}Ph^2 q_1^2 + \frac{1}{2}Qh^2 q_2^2 + fq_1^3 q_2 + \frac{1}{4}cq_1^4 + \frac{1}{4}dq_2^4.$$

So they get the same form as for the simple mechanism so that in case $S=4$ also the horizontal projection of the point moving over the surface may be regarded as representative point for an arbitrary mechanism with 2 degrees of freedom.

$S = 2$.

§ 20. So we suppose that the relation exists:

$$n_1 = n_2 + \varrho,$$

where $\frac{\varrho}{n_1}$ is of order $\left(\frac{h}{R_1}\right)^2$. However, as we have already seen in the cases $S=3$ and $S=4$ in which way such a residue of relation may be taken into account by inserting in the function R a term with ϱa_2, we restrict ourselves here to the case that the residue of relation is zero, therefore:

$$n_2 = n_1 \equiv n.$$

For the surface the lowest point is an umbilical point. To this belongs as special case the surface of revolution with the Z-axis as axis of revolution, which case is treated by Prof. KORTEWEG at the close of his treatise quoted before.

Omitting the terms of higher order than h^4, because in the equations of motion we admit no terms of higher order than h^3, and omitting the terms of order h^3, because in the equations of motion no terms of order h^2 can be disturbing, we may write the equation of the surface:

$$z = \frac{1}{g}(c_1 x^2 + c_3 y^2 + e_1 x^4 + e_2 x^3 y + e_3 x^2 y^2 + e_2\, xy^3 + e_5 y^4),$$

where we avail ourselves of the fact, that by means of a rotation of the system of coordinates round the Z-axis the coefficients of xy^3 and x^3y may be rendered equal.

The solution at first approximation is:

$$x = \frac{\sqrt{\alpha_1}}{n} \cos(nt + 2n\beta_1),$$

$$y = \frac{\sqrt{\alpha_2}}{n} \cos(nt + 2n\beta_2);$$

where $n = \sqrt{2c_1} = \sqrt{2c_3}$.

§ 21. Let us now pass to the simplification of the equations of motion. Corresponding to what was said in § 15 for the case $S=4$ we may here replace in the terms of order h^3 of the equations of motion:

$$\begin{array}{lll} \dot{x}^2 & \text{by} & \alpha_1 - n^2x^2, \\ \dot{y}^2 & ,, & \alpha_2 - n^2y^2, \\ \ddot{x} & ,, & -n^2x, \\ \ddot{y} & ,, & -n^2y. \end{array}$$

The equations become:

$$\left.\begin{array}{l} \ddot{x} + n^2x + 4e_1x^3 + 3e_2x^2y + 2e_3xy^2 + e_2y^3 + \frac{n^4}{g^2}(\alpha_1+\alpha_2)x - \frac{2n^6}{g^2}(x^2+y^2)\,x = 0. \\ \ddot{y} + n^2y + e_2x^3 + 2e_3x^2y + 3e_2xy^2 + 4e_5y^3 + \frac{n^4}{g^2}(\alpha_1+\alpha_2)\,y - \frac{2n^6}{g^2}(x^2+y^2)\,y = 0. \end{array}\right\}$$

Here we may omit no terms, for all the terms of order h^3 are disturbing. The equations may be written as follows:

$$\left.\begin{array}{l} \ddot{x} + n^2x - \frac{\partial R}{\partial x} = 0, \\ \ddot{y} + n^2y - \frac{\partial R}{\partial y} = 0; \end{array}\right\}$$

where we must take

$$-R \equiv e_1 x^4 + e_2 x^3 y + e_3 x^2 y^2 + e_2 x y^3 + e_5 y^4 +$$
$$+ \frac{n^4}{2g^2}(\alpha_1 + \alpha_2)(x^2 + y^2) - \frac{n^6}{2g^2}(x^2 + y^2)^2.$$

§ 22. If we substitute in the function R for x and y the expressions assumed at first approximation and if we retain only those terms not containing t explicitly, we arrive at

$$-R \equiv \frac{1}{2} a\alpha_1^2 + b\alpha_1\alpha_2 + \frac{1}{2} c\alpha_2^2 + f\alpha_1\alpha_2 \sin^2\varphi + f_1(\alpha_1 + \alpha_2)\sqrt{\alpha_1\alpha_2}\cos\varphi,$$

where

$$a = \frac{3e_1}{4n^4} + \frac{n^2}{8g^2}$$
$$b = \frac{3e_3}{8n^4} + \frac{n^2}{8g^2},$$
$$c = \frac{3e_5}{4n^4} + \frac{n^2}{8g^2},$$
$$f = -\frac{e_3}{4n^4} + \frac{n^2}{4g^2},$$
$$f_1 = \frac{3e_2}{8n^4},$$
$$\varphi = 2n(\beta_1 - \beta_2).$$

The system of differential equations indicating the time-variability of the α's and β's becomes:

$$\left.\begin{aligned}
\frac{d\alpha_1}{dt} &= -4nf\alpha_1\alpha_2 \sin\varphi\cos\varphi + 2nf_1(\alpha_1 + \alpha_2)\sqrt{\alpha_1\alpha_2}\,.\sin\varphi \\
\frac{d\alpha_2}{dt} &= +4nf\alpha_1\alpha_2 \sin\varphi\cos\varphi - 2nf_1(\alpha_1 + \alpha_2)\sqrt{\alpha_1\alpha_2}\,.\sin\varphi, \\
\frac{d\beta_1}{dt} &= a\alpha_1 + b\alpha_2 + f\alpha_2\sin^2\varphi + \frac{1}{2}f_1\left(3\sqrt{\alpha_1\alpha_2} + \alpha_2\sqrt{\frac{\alpha_2}{\alpha_1}}\right)\cos\varphi, \\
\frac{d\beta_2}{dt} &= b\alpha_1 + c\alpha_2 + f\alpha_1\sin^2\varphi + \frac{1}{2}f_1\left(\alpha_1\sqrt{\frac{\alpha_1}{\alpha_2}} + 3\sqrt{\alpha_1\alpha_2}\right)\cos\varphi.
\end{aligned}\right\} \quad (21)$$

It appears at once from the system that:

$$\frac{d\alpha_1}{dt} + \frac{d\alpha_2}{dt} = 0,$$

so

$$\alpha_1 + \alpha_2 = \textit{constant}.$$

50*

Another integral is according to § 4:

$$\frac{1}{2} a\alpha_1^2 + b\alpha_1\alpha_2 + \frac{1}{2} c\alpha_2^2 + f\alpha_1\alpha_2 \sin^2\varphi + f_1(\alpha_1+\alpha_2)\sqrt{\alpha_1\alpha_2}\cos\varphi = const.$$

§ 25. The results become very intricate for the general case. This is evidently a consequence of the circumstance, that in the function R appear $\cos\varphi$ and $\sin^2\varphi$ or in other words $\cos\varphi$ and $\cos 2\varphi$. The problem is considerably simplified if we suppose $f_1 = 0$, thus $e_2 = 0$, which means, that we suppose the planes XZ and YZ to be planes of symmetry for the surface.

Let us again introduce ζ, so that

$$\alpha_1 = R_0^2 h^2 \zeta, \qquad \alpha_2 = R_0^2 h^2 (1-\zeta),$$

then the last integral may be written in the form:

$$\sqrt{\zeta(1-\zeta)}\cos\varphi = p\zeta^2 + q\zeta + r,$$

so that we can perform again all integrations in finite form, and x and y may then be found as functions of the time.

Osculating curves.

§ 24. We return to the general case and shall proceed to investigate what becomes of the osculating curves. They are ellipses whose equations are found by eliminating t between

$$x = \frac{\sqrt{\alpha_1}}{n}\cos(nt + 2n\beta_1)$$

and

$$y = \frac{\sqrt{\alpha_2}}{n}\cos(nt + 2n\beta_2).$$

By changing the origin of time we see that for a definite osculating curve we can also find the equation by elimination of t between

$$x = \frac{\sqrt{\alpha_1}}{n}\cos nt$$

and

$$y = \frac{\sqrt{\alpha_2}}{n}\cos(nt - \varphi),$$

so φ represents the difference in phase.

When φ has an arbitrary value, the ellipse has an arbitrary shape and position.

If $\varphi = 0$ or π a straight line is described passing through O.

If $\varphi = \frac{\pi}{2}$ the axes of the ellipse lie along the axes of coordinates.

The ellipses are described in rectangles having their sides parallel to the axes and whose vertices, as is evident from

$$\alpha_1 + \alpha_2 = constant,$$

lie on the circumference of a circle.

To investigate the change in shape and position we may write down the well-known relations which may serve for the calculation of the axes of the ellipse and the angle of inclination of the long axis with the X-axis. If Ah and Bh are half the larger and half the smaller axis and if θ is the angle in view, then these relations become:

$$\frac{1}{A^2h^2} + \frac{1}{B^2h^2} = \frac{n^2 R_0{}^2 h^2}{\alpha_1 \alpha_2 \sin^2 \varphi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{1}{A^2B^2h^4} = \frac{n^4}{\alpha_1 \alpha_2 \sin^2 \varphi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$tg\, 2\theta = \frac{2\sqrt{\alpha_1 \alpha_2}}{\alpha_2 - \alpha_1} \cdot \cos \varphi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From (1) and (2) we now deduce at once: The sum of the squares of the axes of the ellipse is constant.

§ 25. From what we have just found we can easily prove that in case the surface is a surface of revolution the osculating ellipse has an invariable shape.

Then namely we find:

$$-R = \tfrac{1}{2}\, a\, (\alpha_1 + \alpha_2)^2 + f\, \alpha_1\, \alpha_2 \sin^2 \varphi,$$

where:

$$a = \frac{3e_1}{4n^4} + \frac{n^2}{8g^2},$$

$$f = -\frac{e_1}{2n^4} + \frac{n^2}{4g^2}.$$

As

$$\tfrac{1}{2}\, a\, (\alpha_1 + \alpha_2)^2 + f\, \alpha_1\, \alpha_2 \sin^2 \varphi = constant,$$

and also

$$\alpha_1 + \alpha_2 = constant,$$

we find

$$\alpha_1\, \alpha_2 \sin^2 \varphi = constant.$$

From (2) it then follows that

$$ABh^2 = constant,$$

from which in connection with the close of § 24 we may conclude that

$$Ah = const. \quad , \quad Bh = const.,$$

and so our proposition is proved.

If in further consideration of the case of the surface of revolution we wish to see in what way θ varies, we have to write down the differential equations giving the variability of the α's and β's. They now become:

$$\left.\begin{aligned}
\frac{d\alpha_1}{dt} &= -4\,nf\,\alpha_1\,\alpha_2 \sin\varphi \cos\varphi\,,\\
\frac{d\alpha_2}{dt} &= 4\,nf\,\alpha_1\,\alpha_2 \sin\varphi \cos\varphi\,,\\
\frac{d\beta_1}{dt} &= a(\alpha_1+\alpha_2) + f\,\alpha_2 \sin^2\varphi\,,\\
\frac{d\beta_2}{dt} &= a(\alpha_1+\alpha_2) + f\,\alpha_1 \sin^2\varphi\,.
\end{aligned}\right\}$$

We see that in $\frac{d\beta_1}{dt}$ and $\frac{d\beta_2}{dt}$ an equal constant term $a(\alpha_1+\alpha_2) = a R_0^2 h^2$ appears. This means that the frequency n is modified by an amount of $2na R_0^2 h^2$.

When we now differentiate according to t the relation

$$tg\,2\theta = \frac{2\sqrt{\alpha_1\alpha_2}}{\alpha_2-\alpha_1}\cos\varphi$$

we may arrive after some reduction at:

$$\frac{d\theta}{dt} = -2\,fn^3\,ABh^2\,,$$

from which it is evident, that the ellipse revolves with a constant angular velocity.

These results agree quantitatively with those found by Prof. KORTEWEG.

§ 26. The change in shape and position of the osculating curve does not seem to become simple for the general case, $n_2 = n_1$.

Let us therefore restrict ourselves to the case $e_2 = 0$; then the XZ-plane and the YZ-plane are planes of symmetry for the surface.

The first equation of (21) now becomes

$$\frac{d\alpha_1}{dt} = -4\,nf\,\alpha_1\,\alpha_2 \sin\varphi \cos\varphi.$$

Or by introduction of ζ:

$$\frac{d\zeta}{dt} = -4\,nf\,R_0^2\,h^2\,\zeta\,(1-\zeta) \sin\varphi \,.\, \cos\varphi.$$

The relation between ζ and φ becomes:

$$\cos^2 \varphi = \frac{p\zeta^2 + q\zeta + r}{\zeta(1-\zeta)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Here ζ again varies periodically between a greater and a smaller value. Now however $\frac{d\zeta}{dt}$ may become equal to 0 for $\sin \varphi = 0$ and for $\cos \varphi = 0$. Thus barring special cases there are 3 general cases:

1st. For the extreme values of ζ $\cos \varphi = 0$. Then in the extreme rectangles ellipses are described with the axes along the X-axis and Y-axis (fig. 13).

2nd. For the extreme values of ζ $\sin \varphi = 0$. In the extreme rectangles straight lines are described (fig. 14).

3rd. For *one* of the extreme values of ζ $\sin \varphi = 0$, for the other $\cos \varphi = 0$. (fig. 15).

Special cases.

§ 27. These we have again for the extreme values of the modulus $\varkappa$ ($\varkappa$ has the same form as in § 16) of the elliptic functions; which occurs when 2 roots of the equation:

$$f(\zeta) \equiv (p\zeta^2 + q\zeta + r)\{\zeta(1-\zeta) - (p\zeta^2 + q\zeta + r)\} = 0$$

have coincided.

The special case corresponding to B of § 9 and the second of § 18 occurs here in two ways. We refer to the cases in which the same straight line is continually described (continually $\sin \varphi = 0$; when the surface is surface of revolution, this form of motion is possible in every meridian) and that continually the same ellipse is described ($\cos \varphi = 0$; this becomes for the surface of revolution the uniform motion in a parallel circle).

The special case corresponding to A of § 9 and to the first of § 18 exists here too. The form of motion approaches asymptotically the motion in a definite ellipse.

Envelope of the osculating curves.

§ 28. Two cases may be indicated, in which the envelope assumes a simple shape.

1. For $p = -1$, $q = 1$ in (22) (the case of a surface of revolution), the envelope has degenerated into two concentric circles.

2. For $p = 0$ and $q = 0$ in (22) the envelope has degenerated into two pairs of parallel lines, enclosing a rectangle.

Arbitrary mechanism with 2 degrees of freedom for which $S = 2$.

§ 29. The equations of LAGRANGE get quite the same form here, as for $S=4$. In the terms of order h^3 we may in the same way substitute other terms for the terms $\ddot{q}_1$, $\ddot{q}_2$, $\dot{q}_1^2$, and $\dot{q}_2^2$.

Then we have to reduce the terms $-\frac{\partial P_1}{\partial q_2}\dot{q}_1\dot{q}_2$ and $-\frac{\partial P_3}{\partial q_1}\dot{q}_1\dot{q}_2$.

To this end we introduce q'_1 and q'_2 in such a way, that

$$q'_1 = q_1 + \frac{1}{4} a_2 q_1^2 q_2 + \frac{1}{2} a_3 q_1 q_2^2 .$$

$$q'_2 = q_2 + \frac{1}{4} c_1 q_1 q_2^2 + \frac{1}{2} c_2 q_1^2 q_2 .$$

After these reductions it is evident that the terms of order h^3 in the first equation assume the form:

$$\left\{- a_1 \frac{a_1}{n^2} + (a_3 - b_2 + c_1)\frac{a_2}{n^2}\right\} q_1 + \left(\frac{1}{2} c_2 - 2b_3\right)\frac{a_2}{n^2} q_2 +$$
$$+ 2a_1 q_1^3 + \left(\frac{1}{2} a_2 + b_1\right) q_1^2 q_2 - (a_3 - 2b_2 + c_1) q_1 q_2^2 - \left(\frac{1}{2} c_2 - 3b_3\right) q_2^3 .$$

We now substitute $\frac{4}{3} q_2^3$ for $B^2h^2q_2$. This is allowed, because substituting $q_2 = Bh\cos(nt+\lambda)$ in $\frac{4}{3} q_2^3$, we obtain besides a term $B^2h^2q_2$ terms which are non-disturbing.

We wish to investigate whether the disturbing terms in the two equations are again derivatives of the same function. For this we need not consider the terms with q_1 and q_1^3, in the first equation and those with q_2 and q_2^3 in the second. The remaining terms become in the first equation:

$$\left(\frac{1}{2} a_2 + b_1\right) q_1^2 q_2 - (a_3 \quad 2b_2 + c_1) q_1 q_2^2 + \frac{1}{3}\left(b_3 + \frac{1}{2} c_2\right) q_2^3 .$$

In the second:

$$\frac{1}{3}\left(\frac{1}{2} a_2 + b_1\right) q_1^3 - (a_3 - 2b_2 + c_1) q_1^2 q_2 + \left(b_3 + \frac{1}{2} c_2\right) q_1 q_2^2$$

So finally we find that the disturbing terms are derivatives of the same function R; so the equations become:

$$\ddot{q}_1 + n^2 q_1 - \frac{\partial R}{\partial q_1} = 0 ,$$

$$\ddot{q}_2 + n^2 q_2 - \frac{\partial R}{\partial q_2} = 0 ,$$

where $$R = Ph^2 q_1^2 + Qh^2 q_2^2 + U_4 ,$$

when P and Q are homogeneous quadratic functions of $\sqrt{a_1}$ and $\sqrt{a_2}$ and when U_4 is a homogeneous function of order four of q_1 and q_2. The results found for the simple mechanism hold therefore for an arbitrary mechanism with two degrees of freedom.

H. J. E. BETH. "The oscillations about a position of equilibrium where a simple linear relation exists between the frequencies of the principal vibrations." (Second part).

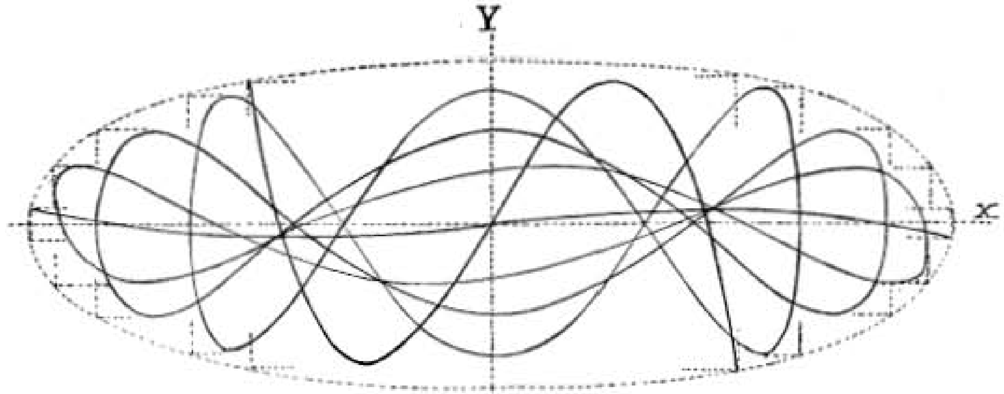


Fig. 11.

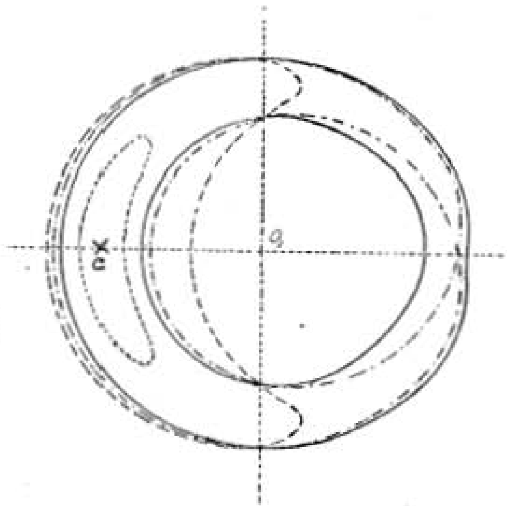


Fig. 12.

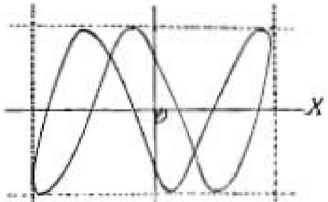
Fig. 8.


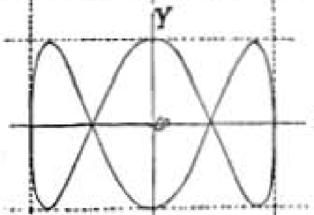
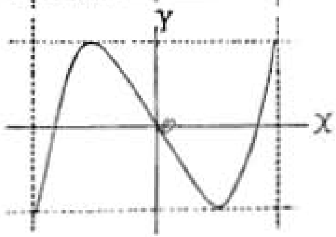
Fig. 9.


Fig. 10.


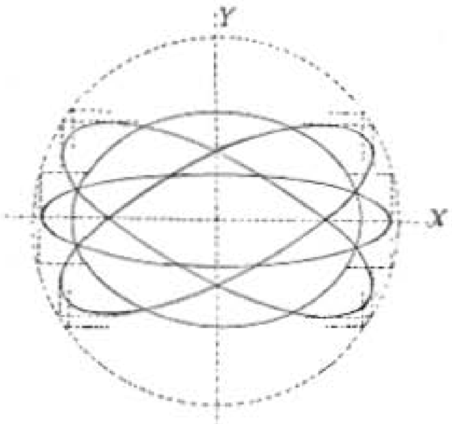


Fig. 13.

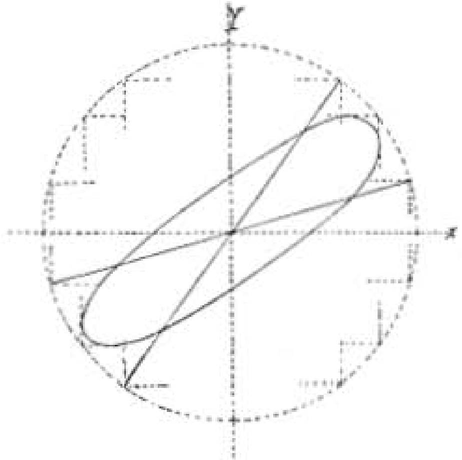


Fig. 14.

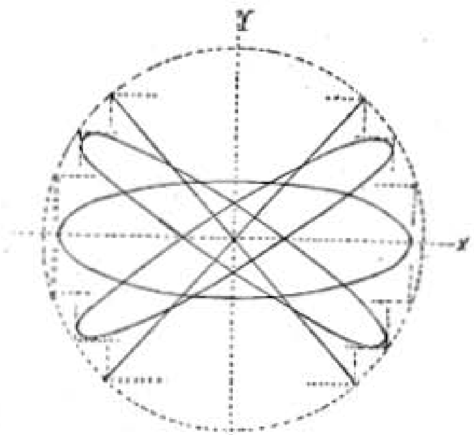


Fig. 15.