Mathematics. — "*The cubic involution of the first rank in the plane.*" By Dr. W. VAN DER WOUDE. (Communicated by Prof. P. H. SCHOUTE.)

1. If V is a plane it is in different ways possible to arrange the points of V in groups of three in such a way, that an arbitrary point forms a part of only one group. If P_1 is a point of V there must exist between the coordinates of P_1 and those of the other points of the group, to which P_1 belongs, some relations by which those other points are entirely determined. It is however possible that P_1 can be chosen is such a way that one of these relations is identically satisfied; in that case P_1 forms part of an infinite number of groups.

We now start from the following definition:

The points of a plane V form a cubic involution of the first rank, when they are conjugate to each other in groups of three in such a way that (with the exception of some definite points) each point forms a part of only o n e group.

A triangle of which the vertices belong to a selfsame group we call an *involution triangle*; each point which is a vertex of more than *one*, therefore of an infinite number of involution triangles, we call a *singular point* of the involution; each point coinciding with one of its conjugate points is called a *double point.* If one of the sides of an involution triangle rotates around a fixed point, then the third vertex of this triangle will describe a right line or a curve; *we shall restrict ourselves in this investigation to the case, that one vertex of an involution triangle describes a right line, when the opposite side rotates around a fixed point.*

2. When the points of a plane V form a cubic involution of the first rank which satisfies the just mentioned condition and which we shall furtheron indicate by (i_3), we can conjugate projectively to each point of V the connecting line of its conjugate points. Each vertex of an involution triangle and its opposite side are pole and polar line with respect to a same conic, which in future we shall always call γ_2; each involution triangle is a polar triangle of γ_2. It is clear that reversely not every polar triangle of γ_2 is an involution triangle of (i_3); for each point of V is a vertex of an infinite number of polar triangles of γ_2, but of only *one* involution triangle. If however S is a singular point of the involution, then S must be a vertex of an infinite number of involution triangles, thus each polar triangle of γ_2 having S as vertex is at the same time an

involution triangle. If we assume a point G of the conic γ_2 as a vertex of an involution triangle, then one of the other vertices must coincide with G, so G is a double point of the involution; γ_2, the locus of these double points, is the *double curve* of the involution.

Each line l whose pole with respect to γ_2 is no singular point of the involution is a side of only one involution triangle, namely of that triangle having the pole of l as vertex. On the other hand each line whose pole is a singular point is a side of an infinite number of involution triangles all having that point as vertex. From this ensues that also the lines of V form a cubic involution (i'_3) of the first rank; the polar lines of the singular points of (i_3) are the singular lines of (i'_3), the tangents of γ_2 are its double lines and γ_2 is its double curve. Both involutions are with respect to γ_2 polarly related.

The involution triangles of γ_2 are all polar triangles of a selfsame conic γ_3, which is at the same time the double curve of (i_3). The lines of V form an involution (i'_3) which is with respect to γ_3 the polar figure of (i_3). Each polar triangle of γ_2 having a singular point of the involution as vertex is at the same time an involution triangle.

3. We make a point describe a line a_1 and we ask after the locus of its conjugate points. If we draw through A_1, the pole of a_1 with respect to γ_2, an arbitrary line p_1, then P_1, the pole of p_1, lies on a_1, whilst the two points conjugate to P_1 lie on p_1; these two points lie also on the locus under discussion. Moreover A_1 itself is conjugated to two points of a_1, so that A_1 is a double point of this curve and each line through A_1 cuts this curve in a double point and two points more. Hence we find:

If one of the vertices of an involution triangle describes a line a_1, then the two others describe a curve α^4 of order four with a node in A_1, the pole of a_1 with respect to γ_2. As a_1 cuts all singular lines, all singular points lie on α^4.

A few properties of this curve α^4 may still be given here:

1. Let A_2 and A_3 be the points conjugated to A_1, then the polar line of A_3 with respect to γ_2 — that is the line A_1A_2 — must cut α^4 in A_1 and in the points forming with A_3 an involution triangle. These two points are A_1 and A_2. So will α^4 be touched in A_1 by the lines A_1A_2 and A_1A_3; A_2 and A_3 are points of intersection of a_1 and α^4.

2. Besides in A_2 and A_3 the curve α^4 will be intersected in two points more by a_1; these points are at the same time the points of intersection of a_1 and γ_2.

3. Besides in these last points α^4 will still be cut by γ_2 in 6 points more, the tangents in these 6 points to α^4 must pass through A_1. From this ensues that α^4 is of the tenth class, by which the PLÜCKER numbers of α^4 are entirely determined ($n = 4, m = 10, d = 1$). This holds, for it is easy to investigate that α^4 cannot possess a double point differing from A_1.

4. If a vertex of an involution triangle describes a line, on which lies a singular point, the curve described by the two other vertices degenerates into the polar line of that singular point and a curve which must be of order three. If a vertex of an involution triangle describes a singular line s, then one of the other two vertices will be a fixed point, namely the pole of s and the other point will describe s itself and as many other lines as there are singular points on s. As both points together must describe a curve of order four, three singular points will lie on s. In like manner each singular point is point of intersection of three singular lines.

If now again a_1 is an arbitrary line and if α^4 has the same signification as above, then the curve α^4 will cut a line b_1 four times; from this ensues that four times a point of a_1 and a point of b_1 are vertices of a selfsame involution triangle. These vertices we call P_1, Q_1, R_1, S_1 and P_2, Q_2, R_2, S_2, whilst the third vertices of these triangles may be represented by P_3, Q_3, R_3, S_3; farthermore T_1 is the point of intersection of a_1 and b_1 and T_2 and T_3 are the two points forming with T_1 an involution triangle.

If now a point describes the line b_1, then its conjugate points describe a curve β^4 of order four; α^4 and β^4 have 16 points of intersection. These are:

1. the two points T_2 and T_3;
2. the four points P_3, Q_3, R_3, S_3;
3. ten points more having the property that to each of them two, so an infinite number of pairs of points, are conjugated and which are thus the singular points. Therefore:

The involution (i_3) *has* 10 *singular points; their polar lines are the* 10 *singular lines of* (i'_3).

These singular elements have such a position that on each of these lines three of these points lie and that in each of the points three of the lines intersect each other; so they form a configuration $(10_3, 10_3)$.

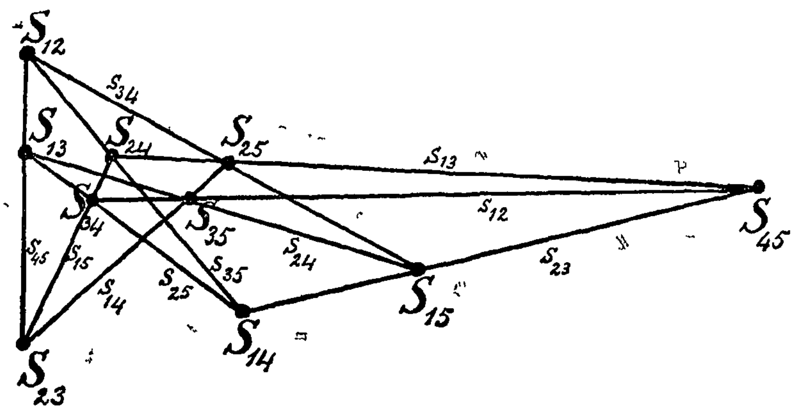
If s_{12} is a singular line and S_{12} its pole with respect to γ_2, then there are besides S_{12} still 6 singular points not lying on s_{12}. If S_{13} is one of these points and s_{13} the polar line of S_{13}, then the point of intersection of s_{12} and s_{13} is at the same time the pole of $S_{12}S_{13}$.

This point forms an involution triangle with S_{12} and with another point of s_{12} and an *other* one with S_{13} and with a point of s_{13} (an "other one", as S_{12} and S_{13} which do not lie on each other's polar line cannot be vertices of a selfsame involution triangle); so the point of intersection of s_{12} and s_{13} is also a singular point and $S_{12}S_{13}$ a singular line.

Each line connecting two singular points not lying on each other's polar line is a singular line; each point which is the point of intersection of two singular lines not passing through each other's pole is a singular point.

On s_{12}, the polar line of S_{12}, lie 3 singular points; the remaining 6 are connected with S_{12} by 3 singular lines. So each line connecting S_{12} with one of the singular points on s_{12} is not a singular line, as only 3 of these lines pass through S_{12}.

We can indicate the position of the singular points by the following diagram, where the indices have been chosen in such a way that always the points S_{ik}, S_{kl} and S_{il} lie on a selfsame line, that the lines s_{ik}, s_{kl} and s_{il} intersect each other in a selfsame point, and that the point S_{ik} and the line s_{ik} are each other's pole and polar line with respect to γ_2.



5. We make a point describe a conic α_2 and an other point a line b_1 the two points which are conjugated to the former describe a curve α^n, those which are conjugated to the latter a curve β^4. As β^4 and α_2 intersect each other in 8 points, b_1 and α^n must have 8 points in common, so α^n is a curve of order eight; we shall call it in future α^8. As α_2 intersects all singular lines twice, α^8 will have in each of the 10 singular points a node.

If α_2 is described around an involution triangle, then α^8 has also double points in the vertices of this triangle. As all involution triangles are at the same time polar triangles of a selfsame conic γ_2, we can describe a conic around each pair of involution triangles; if a conic β_2 is described around two of these triangles, then the curve β^8 conjugate to it will have 6 nodes in its circumference. Also the remaining points of intersection of β_2 and β^8 are easily indicated; they are the four points of intersection of β_2 and γ_2.

We know moreover that a conic described around an involution triangle and through two of the vertices of an other involution triangle must also contain the third vertex of the latter.

6. It is also clear, that we can easily construct conics described around three involution triangles; to that end we make a conic, pass through the vertices of an arbitrary involution triangle and through two singular points not lying on each other's polar line; for this we choose S_{12} and S_{13}. As α_2 is described around a polar triangle of γ_2, it is described around an infinite number of these triangles; further *each* polar triangle of γ_2 having one of the singular points as vertex is at the same time an involution triangle, so that α_2 is described around three involution triangles.

Now the curve α^8 will have in the circumference of α_2 nine nodes; so it must degenerate and α_2 must be one of the parts into which it breaks up. If P_1 is an arbitrary point of α_2 then always one of the two points P_2 and P_3 forming with P_1 an involution triangle will also lie on α_2, so also the third vertex lies on α_2 (5). If now we let P_1 describe the conic α_2, then P_2 and P_3 will describe the same curve; every time however that P_1 coincides with one of the singular points on α_2, P_2 and P_3 will be bound to no other condition, than that they must lie on the polar line of that point and must form with P_1 a polar triangle of γ_2. So the parts into which α^8 degenerates are:

1. the conic α_2 to be counted double;

2. as many lines as there are singular points lying on α_2.

From this ensues that besides S_{12} and S_{13} 2 more singular points lie on α_2.

This last we can prove still in another way; we construct a second conic β_2, described around an involution triangle $Q_1\, Q_2\, Q_3$ and through S_{12} and S_{13}; it will cut α_2 in two points more, which being both the vertices of two, i.e. of an infinite number of involution triangles, are therefore singular points. If we construct another conic δ_2 described around a triangle of involution $R_1\, R_2\, R_3$ and through S_{12} and S_{13}, then this must still cut α_2 in two singular points; these must be the same as the points of intersection of α_2 and β_2, because on α_2 no more than four singular points can lie.

So all conics passing through S_{12} and S_{13} and farthermore described around *one*, hence around an infinite number of involution triangles will form a pencil; the two other base points of this pencil are also singular points. We determine these first: if we choose as β_2 the pair of lines S_{24} and S_{35} and as σ_2 the pair S_{34} and S_{25}, it is evident that S_{14} and S_{15} are the discussed base points. Therefore: *If the* 10 *singular points, hence also the double curve γ_2, of the involution are known, we can generate the involution triangles in this way:*

We can construct five different pencils of conics of which each conic is described around an infinite number of polartriangles of γ_2, which are then at the same time the involution triangles in view; the base points of these pencils consist of the sets of points (S_{12}, S_{13}, S_{14}, S_{15}), (S_{12}, S_{23}, S_{24}, S_{25}), (S_{13}, S_{23}, S_{34}, S_{35}), (S_{14}, S_{24}, S_{34}, S_{45}) and (S_{15}, S_{25}, S_{35}, S_{45}).

These pencils we shall call in future respectively (B_1), (B_2), (B_3), (B_4) and (B_5).

If α_1 and α_2 are two conics, the first taken arbitrarily out of (B_1), the second arbitrarily out of (B_2), these two will have four points of intersection, viz. S_{12} and the vertices of an involution triangle. Now it can happen in two different ways that 2 of these points of intersection coincide: 1. S_{12} can be at the same time a vertex of the involution triangle 2. one of these vertices can lie on the double curve γ_2. In each of these two cases α_1 and α_2 will have only three different points in common, but they will touch each other moreover in one of these points.

7. Out of these 5 pencils we choose one — e.g. (B_1) — arbitrarily; an arbitrary conic σ_2 out of (B_1) is described around an infinite number of involution triangles whose vertices form in its circumference an involution of order three. The latter has four double points in the points of intersection of σ_2 with γ_2, the double curve of the involution (i_3). Inversely the conics of the pencil (B_1) determine an involution of order four on γ_2; the latter has 6 double points in the points in which γ_2 is touched by a conic out of (B_1). In each of these points three points have thus coincided, forming together a group of (i_3).

The involution (i_3) has 6 triple points; in each of the points γ_2 is touched by a conic out of each of the pencils (B_1), (B_2), (B_3), (B_4), and (B_5).

8. A point whose conjugate points coincide we call a *branch point*, the locus of these points the *branch curve*. If we let a point G describe the conic γ_2, then the curve of order eight, generated by the points conjugate to G, must degenerate into 2 parts, of which one is γ_2 itself and the other the branch curve. From this ensues that the latter is of order six and possesses nodes in the 10 singular points; so it is rational as it should be, as it corresponds point for point to a conic.

Also in an other way we can easily deduce the order of the branch curve; if a point describes a line a_1, then the conjugate points describe a curve a^4 having with γ_2 eight points of intersection, of which two coincide with the points of intersection of a_1 and γ_2, whilst the others point to 6 points of intersection of a_1 with the branch curve.

If G_{12} is a point of the double curve γ_2 and g the tangent in that point to γ_2, then g will intersect the branch curve in 6 points of which one G_3 forms with the double point G_{12} a group of conjugate points; so in the triple points of the involution γ_2 and the branch curve will have to touch each other.

The branch curve is a rational curve of order six, having double points in the singular points and touching the double curve in the triple points of the involution.

Observation. A rational curve of order six has 10 double points; of which however only 8 can be taken arbitrarily [1]; *from the preceding follows however that* 10 *points determining a* Cf $(10_3, 10_3)$ *can always be double points of a rational curve of order six.*

In an other form C. F. GEISER (see his paper quoted in the following number) makes the same observation.

9. We shall now apply the preceding to some problems out of Threedimensional Geometry. To that end we regard the pencil (B) of twisted cubics which can be brought through 5 fixed points P_1, P_2, P_3, P_4, and P_5. These determine on an arbitrary plane V a cubic involution of rank one; the lines $P_i\ P_j$ cut V in the singular points S_{ij}, the planes $P_k\ P_l\ P_m$ cut V in the singular lines s_{ij} of the involution. Through an arbitrary point of V passes only one curve out of this pencil, through a singular point S_{ij} however pass an infinite number of curves, which have all degenerated into the fixed line $P_i P_j$ and a variable conic; these conics form a pencil with P_k, P, P_m and the point of intersection of $P_i P_j$ with the plane $P_k P_l P_m$ as base points. Each double point of the involution in V is now a point, in which a twisted curve out of the pencil (B)

[1]) SALMON-FIEDLER: Höhere ebene Kurven, Zweite Auflage, p. 42.

touches the plane V; the third point of intersection of this curve with V is a point of the branch curve forming with the point of contact a group of mutually conjugate points of the involution. A triple point of the involution is a point, in which a twisted curve out of (B) is osculated by V. From this ensues:

1. *All twisted cubics passing through 5 given points and touching a given plane V form a surface F^{10} of order ten, which touches V in a conic and cuts V moreover according to a rational curve of order six.*

2. *There are 6 twisted cubics passing through five given points and having a given plane as osculating plane.*

As a special case of this last theorem we have still. *through five given points pass six twisted parabolae.*

Through the pencil (B) of twisted cubics with P_1, P_2, P_3, P_4 and P_5 as base points a plane V is cut according to a cubic involution of the first rank. If α is a curve out of this pencil cutting V in A_1, A_2 and A_3, then α is projected out of A_1 by a cone cutting V according to the lines $A_1 A_2$ and $A_1 A_3$. If however a curve γ out of (B) touches a plane V in a point G_{12} and if moreover it cuts V in a point G_3, then γ is projected out of G_{12} by a cone cutting V according to $G_{12} G_3$ and the tangent in G_{12} to γ; γ is projected out of G_3 by a cone touching V according to $G_3 G_{12}$. We have seen that G_{12} must lie on the double curve and G_3 on the branch curve of the involution, whilst $G_3 G_{12}$ touches the former; if therefore a quadratic cone is to pass through the base points of the pencil (B) and to touch V moreover, then its vertex must lie on the branch curve and the tangent with V must touch the double curve.

The number of quadratic cones passing through five given points and touching a given plane is singly infinite; the tangents envelope a conic. The vertices of the cones form a rational curve of order six.[1])

The tangential planes of all these cones whose number is ∞^2 envelope a surface of which we wish to determine the class and which for the present we will call Φ_n. If K_2 is one of these cones and G_3 its vertex, then through a line l drawn in V through G_3 one more tangential plane to K_2 will pass; as l has with the branch curve 6 points of intersection, it lies still in 6 tangential planes of Φ_n except in V. Farthermore V is a trope of Φ_n (that is a tangential plane touching (γ_2) in the points of a conic) to be counted double; the surface Φ_n is therefore of class eight.

The tangential planes of these cones envelope a surface of class eight[1])

[1]) C. F. GEISER: "Uber Systeme von Kegeln zweiten Grades".

We finally put the question how many twisted circles can be brought through five points where we understand by a twisted circle a twisted cubic cutting the isotropic circle in two points. All twisted cubics through these five points describe on the plane at infinity an involution, if now a point describes the isotropic circle, its conjugate points will describe a curve of order eight having with this circle sixteen points in common; four of these points are at the same time double points of the involution, whilst the other lie two by two on a same twisted circle.

So through five given points pass ten twisted circles, of which four touch the plane at infinity.

Mathematics. — *"On the surfaces the asymptotic lines of which can be determined by quadratures".* By J. Bruin. (Communicated by Prof. Hk. de Vries).

In a paper entitled as above A. Buhl (*Nouv. Ann. de Math.*, 4e série, vol. 8, page 433, vol. 9, page 337, *Rev. sem.* XVII 2, page 62, XVIII 1, page 58) discusses the surfaces given by the parameter representation

$$x = r\cos\theta,$$
$$y = r\sin\theta,$$
$$\phi(z) = a\,\theta + F(r),$$

in which x, y, z refer to a rectangular system of coordinates, so that z, θ, and r are the so-called cylindric coordinates; these are the only ones which are used in the course of the investigation.

Buhl now gives the differential equation of the asymptotic lines of $\phi(z) = a\,\theta + F(r)$ with θ and r as independent variables as well as with z and θ. It is then evident that this equation embraces many special cases, where the determination of the asymptotic lines comes to quadratures.

We can put the question more in general: which are the surfaces of one of the forms $z = \varphi(r, \theta)$, or $\theta = f(r, z)$, or $r = f(z, \theta)$, whose asymptotic lines can be determined by quadratures?

Starting from the differential equation of the asymptotic lines

$$D\,du^2 + 2\,D'\,du\,dv + D''\,dv^2 = 0$$

(Bianchi-Lukat, "Vorlesungen über Differentialgeometrie", page 109), where D, D' and D'' have the values, to be found on page 87 of the quoted work, we find for the differential equation in r and θ of the asymptotic lines of $z = \varphi(r, \theta)$:

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