Mathematics. — *"On polar figures with respect to a plane cubic curve.* By Prof. JAN DE VRIES.

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1. If a plane cubic curve γ^3 is represented symbolically by $a_x^3=0$ then $a_x a_y a_w = 0$ represents the polar line p_{xy} of the points X and Y, i.e. the polar line of X with respect to the polar conic π_y of Y and at the same time the polar line of Y with respect to the polar conic π_x of X.

The three polar lines p_{xy}, p_{xz}, and p_{yz} will concur in one point W when the three conditions are satisfied

$$a_x a_y a_w = 0, \qquad a_x a_z a_w = 0, \qquad a_y a_z a_w = 0 \quad . \quad . \quad . \quad . \quad (1)$$

By elimination of the coordinates w_k we find out of it

$$(abc)\, a_x a_y b_x b_z c_y c_z = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

So to two given points X, Y belongs a conic γ^2_{xy} as locus of the point Z; it passes also through X and Y, for when Z and X coincide, we find

$$(abc)\, a_x a_y b_x^2 c_y c_x \equiv (cba)\, c_x c_y b_x^2 a_y a_x \equiv -\, (abc)\, a_x a_y b_x^2 c_y c_x \equiv 0.$$

As we can substitute $(abc)\, a_x a_y c_x c_z b_y b_z = 0$ for (2), thus also $(abc)\, a_x a_y b_z c_z (b_x c_y - b_y c_x) = 0$, we can also represent γ^2_{xy} by $(abc)\, a_x a_y b_z c_z (bc\xi) = 0$, where ξ_k are the coordinates of the line XY. Consequently (2) can be replaced by

$$(bc\xi)\,(bc\eta)\, b_z c_z = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From this ensues that the conic γ^2_{xy} is the *poloconica* $\pi_{\xi\eta}$ of the lines ξ and η.

So the poloconica of two lines is the locus of the points Z which with relation to the points of intersection X, Y of this conic with one of the given lines are in such a position that the polar lines p_{xz} and p_{yz} concur on the other one of the given lines, which is then at the same time polar line of X and Y.

2. If Z and W are the points of intersection of $\pi_{\xi\eta}$ with η, it follows out of the symmetry of (3) in connection with the equations (1), that the four points X, Y, Z, W form a closed group, so that each side of the quadrangle determined by them is the polar line of the vertices not lying on it, therefore *a polar quadrangle* (REYE).

Out of our considerations ensues that a polar quadrangle is determined by two of its vertices, but also by two of its opposite sides. In the last case the vertices are determined by the poloconica of the given lines; in the former case we can use the poloconica belonging to the polar line of the given points and their connecting line.

Out of $a_x a_y a_w = 0$ and $a_x a_y a_z = 0$ follows

$$a_x a_y (\lambda a_w + \mu a_z) = 0.$$

Here λ and μ can be determined in such a way that $\lambda a_w + \mu a_z = a_u$ relates to the point of intersection U of XY with WZ.

As $a_u a_x^2 = 0$ indicates the polar conic π_u of U we find that X and Y according to the relation $a_x a_y a_u = 0$ lie harmonically with respect to π_u. In an analogous way ensues from $a_w a_z a_y = 0$ and $a_w a_z a_x = 0$ the relation $a_w a_z a_u = 0$, according to which W and Z are also separated harmonically by π_u.

But then also the points $T \equiv (XZ, YW)$ and $V \equiv (XW, YZ)$ are conjugated with respect to π_u, i.e. we have $a_u a_v a_t = 0$. Now U, V, T are the *diagonal points* of the complete quadrangle $XYZW$, so that it is proved that *the diagonal triangle of a polar quadrangle is always a polar triangle* [1]).

3. When the conic γ^2_{xy} degenerates we can take for Z each point on the line XY. To trace for this the condition, we put $z_k = \lambda x_k + \mu y_k$; from (2) follows

$$(abc)\, a_x a_y b_x c_y (\lambda b_x + \mu b_y)(\lambda c_x + \mu c_y) = 0,$$

so

$$\lambda^2 (abc)\, a_x a_y b_x^2 c_x c_y + \lambda\mu\, (abc)\, a_x a_y b_x^2 c_y^2 +$$
$$+ \lambda\mu\, (abc)\, a_x a_y b_x b_y c_x c_y + \mu^2\, (abc) a_x a_y b_x b_y c_y^2 = 0.$$

By exchanging two of the symbolic factors a, b, c, we see that three of these terms are identically zero; so we have

$$\lambda\mu\, (abc)\, a_x a_y b_x^2 c_y^2 = 0.$$

For an arbitrary choice of X and Y this equation furnishes only $\lambda = 0$ and $\mu = 0$, thus the points X and Y. It furnishes *each* point of XY, as soon as

$$(abc)\, a_x a_y b_x^2 c_y^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

When X, Y, and Z are collinear, the polar line p_x of X and the polar lines p_{xy}, p_{xz} concur in *one* point; for these three lines are the polar lines of X, Y, Z with respect to the polar conic π_x. If now (4) is satisfied, then also p_{yz} passes through that point, hence, the six polar lines $p_x, p_y, p_z, p_{xy}, p_{yz}, p_{zx}$ concur in a point W. But when p_x, p_y and p_z are concurrent, the poloconica of $\xi \equiv XYZ$, degenerates and ξ is tangent of the *Cayleyana*.

From this ensues that for given Y the equation (4) will represent

1) Mentioned without proof by CAPORALI (Transunti d. R. A. dei Lincei 1877, p. 236).

three right lines, namely the three tangents which we can draw out of Y to the *Cayleyana*.

This can be confirmed as follows. Let Z be a point of the locus of X, which is determined by 4) and X a second point of that locus lying on YZ, so that we have $a_x = \lambda a_y + \mu a_z$. Out of (4) then follows

$$(abc)\, a_y c_y^2 (\lambda a_y + \mu a_z)(\lambda b_y + \mu b_z)^2 = 0.$$

By exchanging a and c we see at once that

$$(abc)\, a_y^2 c_y^2 (\lambda b_y + \mu b_z)^2$$

vanishes identically. Analogously we find that $(abc)\, a_y c_y^2 a_z b_y^2$ and $(abc)\, a_y c_y^2 a_z b_y b_z$ vanish identically. As finally the form $(abc)\, a_y c_y^2 a_z b_z^2$ is zero because Z lies on the locus indicated by (4) the above relation is satisfied by all points of YZ, so the locus consists of three lines through Y.

4. That the line $\xi \equiv XY$ is tangent to the *Cayleyana* as soon as (4) is satisfied, can be confirmed by reducing (4) to the tangential equation of that curve. In the first place we find out of

$$(abc)\, a_x a_y b_x^2 c_y^2 = 0 \quad \text{and} \quad (acb)\, a_x a_y c_x^2 b_y^2 = 0$$

the relation

$$(abc)\, a_x a_y (b_x c_y + b_y c_x)(b_x c_y - b_y c_x) = 0.$$

The last factor can be replaced by $(bc\xi)$ where ξ_k indicate the coordinates of XY. After that the equation can be broken up into two terms, which pass into each other when b and c are exchanged. So we can replace it by

$$(abc)\, a_x a_y b_x c_y (bc\xi) = 0 \quad \ldots\ldots\ldots \quad (5)$$

farthermore it is evident from

$$(abc)\, a_x a_y b_x^2 c_y^2 = 0 \quad \text{and} \quad (cba)\, c_x c_y b_x^2 a_y^2 = 0,$$

that at the same time is satisfied

$$(abc)\, b_x^2 a_y c_y (ac\xi) = 0,$$

so also

$$(bac)\, a_x^2 b_y c_y (bc\xi) = 0. \quad \ldots\ldots\ldots \quad (6)$$

By combining (5) and (6) we find

$$(abc)\, a_x c_y (bc\xi)(ab\xi) = 0.$$

So

$$(abc)\, c_x a_y (bc\xi)(ab\xi) = 0.$$

Out of the last two relations follows finally

$$(abc)\,(ac\xi)\,(bc\xi)\,(ab\xi) = 0 \quad . \quad . \quad . \quad . \quad (7)$$

This tangential equation really represents the *Cayleyana* [1]).

So we have found that *the six polar lines $p_x, p_y, p_z, p_{xy}, p_{yz}, p_{zx}$ concur in one point when the points X, Y, Z lie on a tangent of the Cayleyana.*

5. When p_x, p_y, p_z are concurrent we have

$$(abc)\,a_x^2\,b_y^2\,c_z^2 = 0 \quad . \quad . \quad . \quad . \quad (8)$$

This equation gives thus the relation between the coordinates of three points lying on one and the same polar conic.

For an arbitrary choice of X and Y this equation is satisfied except by X and Y by no point of the line XY. If it is to be satisfied by $z_k = \lambda x_k + \mu y_k$ we must have

$$(abc)\,a_x^2\,b_y^2\,(\lambda c_x + \mu c_y)^2 = 0,$$

therefore

$$\lambda\mu\,(abc)\,a_x^2\,b_y^2\,c_x\,c_y = 0.$$

This is satisfied for each value of $\lambda:\mu$ when the relation (4) is satisfied, so when X, Y, Z lie on a tangent of the *Cayleyana.*

Now in general the polar lines p_{xy}, p_{xz}, p_{yz} form a triangle inscribed in the triangle $p_x\,p_y\,p_z$ (see § 3). If (4) is satisfied then p_{xy}, p_{xz}, p_{yz} are concurrent; but then their point of intersection must be at the same time point of intersection of p_x, p_y, p_z.

If X, Y, Z are three collinear points of the cubic, then p_{yz}, p_{zx} and p_{xy} pass successively through X, Y, and Z.

For, from $a_x^3 = 0$, $a_y^3 = 0$ and $(\lambda a_x + \mu a_y)^3 = 0$ follows that the point Z is indicated by $a_x a_y\,(\lambda a_x + \mu a_y) = 0$. So we have $a_x a_y a_z = 0$, so Z lies on the polar line p_{xy}.

If moreover X, Y, Z lie on a tangent of the *Cayleyana*, then p_{yz}, p_{zx}, p_{xy} must coincide with the tangents p_x, p_y, p_z in X, Y, Z.

6. For p_x, p_y, and p_{xy} to be concurrent, there must be a point W for which we have $a_x^2 a_w = 0$, $b_y^2 b_w = 0$, and $c_x c_y c_w = 0$.

But then $(abc)\,a_x^2 b_y^2 c_x c_y = 0$.

For arbitrarily chosen Y the locus of X becomes a figure of the third order, passing through Y, because we have $(abc)\,a_y^2 b_y^2 c_y^2 \equiv 0$. But by taking notice of (4) we see that this figure consists of three tangents of the *Cayleyana.* Out of

$$a_x^2 a_w = 0,\; a_y^2 a_w = 0 \text{ and } a_x a_y a_w = 0$$

[1]) See e.g. Clebsch, Leçons sur la géométrie, II, p. 284.

follows indeed

$$(\lambda a_x + \mu a_y)^2 a_w = 0\,;$$

i.e. if Z lies on XY, then p_z will pass through the point of intersection W of p_x, p_y and p_{xy}, which bears then at the same time p_{yz} and p_{xz}.

So the three lines p_x, p_y, and p_{xy} concur only then in one point when X and Y are united by a tangent of the *Cayleyana*. Their point of intersection bears then also all the polar lines and mixed polar lines belonging to the points of those lines.

The lines p_x, p_{xy}, and p_{xz} will be concurring, when

$$(abc)\, a_x^2 b_x b_y c_x c_z = 0$$

is satisfied, thus also

$$(abc)\, a_x^2 c_x c_y b_x b_z = 0,$$

hence also

$$(abc)\, a_x^2 b_x c_x\, (b_y c_z - b_z c_y) = 0.$$

If we put

$$y_k z_l - y_l z_k = \xi_m\,,$$

we have the condition

$$(abc)\, a_x^2 b_x c_x\, (bc\xi) = 0.$$

As this can also be written in the forms

$$(abc)\, a_x b_x^2 c_x\, (ac\xi) = 0 \quad \text{and} \quad (abc)\, a_x b_x c_x^2\, (ab\xi) = 0$$

and as out of

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_x \\ b_1 & b_2 & b_3 & b_x \\ c_1 & c_2 & c_3 & c_x \\ \xi_1 & \xi_2 & \xi_3 & \xi_x \end{vmatrix} = 0$$

follows the relation

$$(abc)\, \xi_x = a_x\, (bc\xi) + b_x\, (ca\xi) + c_x\, (ab\xi)$$

the above condition can be replaced by

$$(abc)^2\, a_x b_x c_x \xi_x = 0.$$

With arbitrary position of X this is satisfied by $\xi_x = 0$, i.e. when X, Y, and Z are collinear (see § 3).

If however

$$(abc)^2\, a_x b_x c_x = 0,$$

so that X lies on the *Hessian*, then X, Y, and Z are quite arbitrary. This was to be foreseen, now namely π_x is a pair of lines, so that the lines p_x, p_{xy}, and p_{xz} concur in the node of π_x.