Citation:

L.E.J. Brouwer, About difference quotients and differential quotients, in:
KNAW, Proceedings, 11, 1908-1909, Amsterdam, 1909, pp. 59-66

DESCRIPTION OF FIGURES.

Fig. 1. Median longitudinal section of the infundibular organ of a Branchiostoma of 52 m.M. in length, 600 : 1.
Fig. 2. Cross section through the same of a Branchiostoma of 54 m.M in length, 600 : 1.
Fig. 3. The same as Fig. 1. Neurofibrillae stained with chloride of gold.
Fig 4. Cells of the infundibular organ, *a* of a Branchiostoma of 22 m.M. in length, *b* of 50 m.M. in length, *c* cross-section of the upper ends of the cells.
Fig. 5. Median section of the brain of a Branchiostoma larva of 3,4 m.M., reconstructed from cross-sections.
Fig. 6. The same of a specimen of 10 m M. long.
Fig. 7. The same of a specimen of 21 m.M. long.
Fig. 8. Median section through the brain of a Branchiostoma of 28 m.M. in length.

Mathematics. — *"About difference quotients and differential quotients"*. By Dr. L. E. J. Brouwer (Communicated by Prof. D. J. Korteweg).

(Communicated in the meeting of May 30, 1908).

Different investigations have been made which are very completely summed up in the work of Dini: "Grundlagen fur eine Theorie der Functionen emer veranderlichen reellen Grösse" Chapt. XI and XII, on the connection between difference quotients and differential quotients, particularly on the necessary and satisfactory properties which the difference quotients must possess in order that there be a differential quotient. One however always regards in the first place these different difference quotients in one and the same point x_0 together, forming as a function of the increase of x the *derivatory function in* x_0. The existence of a differential quotient means then, that that derivatory function has a single limiting point in x_0 i.o.w. that in x_0 the right as well as the left *derivatory oscillation* is equal to zero.

Other conditions for the existence of a differential quotient are found when in the first place the difference quotient for constant x-increase Δ is regarded as a function of x and then the set of these functions for varying Δ is investigated. Let $f(x)$ be the given function which we suppose to be finite and continuous and let $\varphi_\Delta(x)$ be the difference quotient for a constant x-increase Δ. The different functions $\varphi_\Delta(x)$ form an infinite set of functions, in which each function is continuous. We shall occupy ourselves with the case that the set is *uniformly continuous*, i. e. that for any quantity ε, however small it may be, a quantity σ can be pointed out so that in any interval of the size of σ not one of the functions of the set has oscillations larger than ε. Concerning infinite uniformly continuous sets of functions there is a theorem that if they are limited (i.e., if a maximum value and a minimum value can be given between which all functions move) they possess *at least one* continuous limiting curve, to which uniform convergence takes place [1]).

We shall prove, that for the set of functions of the difference quotients of a finite continuous function, if it be uniformly continuous, follows in the first place the limitedness and furtheron for indefinite decrease of the x-increase the existence of only *one* limiting curve, so that holds:

Theorem 1. If a finite continuous function $f(x)$ has a uniformly continuous set of difference quotients, then it possesses a finite continuous differential quotient [2]).

To prove this we call ${}_{\delta}\varepsilon_{\Delta}(x)$ the size of the region of oscillation between x and $x + \Delta$ of the difference quotient for an x-increase δ. If we allow δ to assume successively all positive values, then it follows from the supposed uniform continuity, that Δ can always be chosen so small as to keep all values ${}_{\delta}\varepsilon_{\Delta}(x)$ below a certain amount as small as one cares to make it. If we thus call $\varepsilon_{\Delta}(x)$ the maximum of the values ${}_{\delta}\varepsilon_{\Delta}(x)$ for definite x and Δ, then $\varepsilon_{\Delta}(x)$ tends with Δ uniformly to 0.

We have fartheron if $\frac{p}{n}$ is a proper fraction:

$$\varphi_{\Delta}(x) = \frac{1}{n}\varphi_{\frac{\Delta}{n}}(x) + \frac{1}{n}\varphi_{\frac{\Delta}{n}}\left(x + \frac{1}{n}\Delta\right) + \dots + \frac{1}{n}\varphi_{\frac{\Delta}{n}}\left(x + \frac{n-1}{n}\Delta\right) \quad (1)$$

$$\varphi_{\frac{p}{n}\Delta}(x) = \frac{1}{p}\varphi_{\frac{\Delta}{n}}(x) + \frac{1}{p}\varphi_{\frac{\Delta}{n}}\left(x + \frac{1}{n}\Delta\right) + \dots + \frac{1}{p}\varphi_{\frac{\Delta}{n}}\left(x + \frac{p-1}{n}\Delta\right). \quad (2)$$

If we break up each of the n terms of the second member of (1) into p equal parts and each of the p terms of the second member of (2) into n equal parts, then the difference of those two second members can be divided into pn terms, each remaining in absolute value smaller than $\frac{1}{pn} \cdot \varepsilon_{\Delta}(x)$, so that the difference of $\varphi_{\Delta}(x)$ and $\varphi_{\frac{p}{n}\Delta}(x)$ remains smaller than $\varepsilon_{\Delta}(x)$ in absolute value.

1) Compare Arzelà, "*Sulle funzioni di linee*", Memorie della Accademia di Bologna, serie 5, V, page 225.

2) We suppose the function to be given in a certain interval of values of the independent variable x.

So if we regard for any definite x all difference quotients the x-increases of which are equal to proper fractions of Δ, then the amount $\tau_\Delta(x)$ of their region of oscillation is smaller than $2\varepsilon_\Delta(x)$. The same holds for the region of oscillation of *all* difference quotients for definite x with x-increases smaller than Δ, because these can be approximated by the preceding on account of the continuity of f.

So if we allow Δ to decrease indefinitely, then also $\tau_\Delta(x)$ decreases indefinitely; as furthermore when Δ becomes smaller, each following region of oscillation is a part of the preceding, the limit of the region of oscillation is for each x a single definite value, to which uniform convergence takes place, which is the limit of the difference quotients, the *differential quotient.*

That this (forward) differential quotient cannot show any discontinuities, is evident as follows: If there were a discontinuity, then there would be a quantity σ which could be overstepped there for any interval by the oscillations of the differential quotient; but if the value of the differential quotient differs in two points more than σ, then there is also a difference quotient the values of which in those two points differ more than σ; so there would be for each interval, which contains the indicated discontinuity, a difference quotient with a region of oscillation larger than σ, i. o. w. the functional set of the difference quotients would not be uniformly continuous.

Out of the continuity of the forward differential quotient follows at the same time that the forward and the backward differential quotient are equal.

Analogously it is evident that also a point at infinity in the differential quotient would disturb the uniform continuity of the difference quotients; in this is at the same time included the limitedness of the difference quotients, for they would otherwise on account of the finiteness of f be able to tend to infinity only for indefinitely decreasing x-increase, but that would furnish an infinity point in the differential quotient.

Theorem 2. Of a function with finite continuous differential quotient the difference quotients are uniformly continuous.

Let namely ε be a definite quantity, to be taken as small as one likes. Now we may have each x included by an interval i in such a way, that the oscillations of the differential quotient within each of those intervals remain smaller than $\frac{1}{2}\varepsilon$. On account of the uniform convergence, evident from the formula $\varphi_\Delta(x) = f'(x + \vartheta\Delta)$, a Δ' can be pointed out in such a way that all φ_Δ for which $\Delta < \Delta'$ differ from the differential quotient less than $\frac{1}{4}\varepsilon$ for any x, thus have their oscillations below ε in the intervals mentioned. On account of the uniform continuity of f we may furthermore have each x included by an interval i' chosen in such a way that for all $\Delta \geqq \Delta'$ the corresponding φ_Δ have within those intervals oscillations below ε only; to that end we have but to choose i' in such a way that the oscillations of f remain within the intervals below $^1/_2\, \varepsilon\, \Delta'$. If thus i'' is the smaller of the two quantities i and i', each x can be included by an interval i'' in such a way, that the oscillations of *all* difference quotients within it remain below ε, with which we have proved the uniform continuity of the difference quotients.

Theorem 3. If there is among the difference quotients of a finite continuous function a uniform continuous fundamental series with indefinitely decreasing x-increases, there exists a finite continuous differential quotient.

Let namely $\varphi_{\Delta'}(x), \varphi_{\Delta''}(x), \ldots$ be the fundamental series of functions under consideration, then for any quantity ε we can point out a quantity σ in such a way that $\varphi_{\Delta^{(\nu)}}(x+h) - \varphi_{\Delta^{(\nu)}}(x) < \varepsilon$ for any x, any $h < \sigma$ and any ν. If now the set of *all* difference quotients were not uniformly continuous, it would have to occur that for a certain Δ° not belonging to the fundamental series we should have $\varphi_{\Delta^\circ}(x+h) - \varphi_{\Delta^\circ}(x) > \varepsilon$. If we now approximate Δ° by a series $\alpha_1 \Delta', \alpha_2 \Delta'', \ldots$, where the α's represent integers, in such a way that $\alpha_p \Delta^{(p)} < \Delta^\circ < (\alpha_p + 1)\Delta^{(p)}$, then also $\varphi_{\Delta^\circ}(x+h) - \varphi_{\Delta^\circ}(x)$ is approximated by $\varphi_{\alpha_p \Delta^{(p)}}(x+h) - \varphi_{\alpha_p \Delta^{(p)}}(x)$, which last expression always remains $< \varepsilon$ however large p may become, so that $\varphi_{\Delta^\circ}(x+h) - \varphi_{\Delta^\circ}(x)$ cannot be $> \varepsilon$, so the set of *all* difference quotients is uniformly continuous, and there is a finite continuous differential quotient.

Theorem 1 is applied when building up the theory of continuous groups out of the theory of sets, (where one remains independent of LIE's postulates), in a certain region finite and continuous functions of one or more variables occurring there, whose difference quotients are in a certain system of coordinates linear functions of the original functions.[1]) As on account of the finiteness of the original functions there cannot be a region within which any quantity could be overstepped everywhere by one and the same difference quotient, the

[1]) Comp. L. E. J. BROUWER, "*Die Theorie der endlichen continuierlichen Gruppen unabhängig von den Axiomen von* LIE", Atti del IV^0 Congresso Internazionale dei Matematici. It is the differentiability in one and the same system of coordinates of all the functions, which express the different infinitesimal transformations of a group, which is proved in this way.

coefficients of the above mentioned linear functions remain within finite limits, the system of the difference quotients is uniformly continuous, and the differential quotients exist.

Theorem 4. If the conditions of theorem 1 are satisfied and if the system of all second difference quotients (of which each is determined by two independent x-increases) forms a uniformly continuous system, then there exists a finite continuous "second differential quotient" which at the same time is the only limit of the above set of functions when both x-increases decrease indefinitely, and the differential quotient of the (first) differential quotient.

To prove this we call $\varepsilon'_\Delta(x)$ the maximum size of the regions of oscillation of the different second difference quotients between x and $x+\Delta$; then again $\varepsilon'_\Delta(x)$ tends with Δ uniformly to zero.

If we represent the difference quotient of $\varphi_{\Delta_1}(x)$ for an x-increase Δ_2 by $\varphi_{\Delta_1\Delta_2}(x)$ and if $\frac{p_1}{n_1}$ and $\frac{p_2}{n_2}$ are proper fractions then we have:

$$\varphi_{\Delta_1\Delta_2}(x)=\frac{1}{n_1n_2}\sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1}\varphi_{\frac{\Delta_1}{n_1}\frac{\Delta_2}{n_2}}\left(x+k_1\frac{\Delta_1}{n_1}+k_2\frac{\Delta_2}{n_2}\right). \quad . \quad . \quad (1)$$

$$\varphi_{\frac{p_1\Delta_1}{n_1}\frac{p_2\Delta_2}{n_2}}(x)=\frac{1}{p_1\,p_2}\sum_{k_1=0}^{p_1-1}\sum_{k_2=0}^{p_2-1}\varphi_{\frac{\Delta_1}{n_1}\frac{\Delta_2}{n_2}}\left(x+k_1\frac{\Delta_1}{n_1}+k_2\frac{\Delta_2}{n_2}\right). \quad . \quad (2)$$

If we break up each of the $n_1\,n_2$ terms of the second member of (1) into $p_1\,p_2$ equal parts and each of the $p_1\,p_2$ terms of the second member of (2) into $n_1\,n_2$ equal parts, then the difference of those two members breaks up into $p_1\,p_2\;n_1\,n_1$ terms, each of which remaining in absolute value smaller than $\frac{1}{p_1p_2\;n_1n_2}\,\varepsilon'_{\Delta_1+\Delta_2}(x)$, so that the difference of $\varphi_{\Delta_1\Delta_2}(x)$ and $\varphi_{\frac{p_1\Delta_1}{n_1}\frac{p_2\Delta_2}{n_2}}(x)$ remains in absolute value smaller than $\varepsilon'_{\Delta_1+\Delta_2}(x)$.

So if we consider for any definite x all difference quotients whose x-increases are equal to proper fractions of Δ_1 and Δ_2, then the size $\tau_{\Delta_1\Delta_2}(x)$ of their region of oscillation is smaller than $2\varepsilon'_{\Delta_1+\Delta_2}(x)$, from which we deduce as above in the proof of theorem 1 the existence of one single limit, to which the convergence is uniform and which is finite and continuous.

If we now regard the difference quotient with x-increase Δ_2, on one hand for all φ_Δ's, whose Δ is smaller than Δ_1, and on the other hand for the (first) differential quotient, then the former all differ less than $\varepsilon'_{\Delta_1+\Delta_2}(x)$ from the limiting function just deduced, so also the latter, which can be approximated by them. This holds independently of Δ_1;

the difference for x-increase Δ_2 of the (first) differential quotient can therefore not differ more than $\varepsilon'_{\Delta_2}(x)$ from the just deduced limiting function which is thus differential quotient of the (first) differential quotient i.e. second differential quotient. –

Theorem 5. If a function possesses a finite continuous second differential quotient, then the system of the first and second difference quotients is uniformly continuous.

To find namely an interval size i'' which keeps the oscillations of all second difference quotients everywhere $< \varepsilon$, we first take the interval size i, which keeps the oscillations of the second differential quotient everywhere $< \frac{1}{2}\varepsilon$; then a Δ'_1 and a Δ'_2 in such a way, that all $\varphi_{\Delta_1 \Delta_2}$, for which $\Delta_1 < \Delta'_1$ and $\Delta_2 < \Delta'_2$, differ along the whole course less than $^1/_4\varepsilon$ from the second differential quotient [1]); finally an interval size i' which keeps the oscillations of the function f everywhere $< {}^1/_4\, \varepsilon\, \Delta'_1 \Delta'_2$. For i'' we take the smaller of the two quantities i and i'.

Theorem 6. If there is among the second difference quotients of a finite continuous function with finite continuous differential quotient a uniformly continuous fundamental series, in which the two x-increases decrease indefinitely, then there exists a finite continuous second differential quotient.

Let namely $\varphi_{\Delta_1'\Delta_2'}(x)$, $\varphi_{\Delta_1''\Delta_2''}(x)$... be the indicated fundamental series, then for any quantity ε a quantity σ can be pointed out in such a way, that $\varphi_{\Delta_1^{(\nu)}\Delta_2^{(\nu)}}(x+h) - \varphi_{\Delta_1^{(\nu)}\Delta_2^{(\nu)}}(x) < \varepsilon$ for any x, any $h < \sigma$ and any ν. If now the set of *all* second difference quotients were not uniformly continuous, then it would be possible for a certain Δ_1° and Δ_2° not belonging to the fundamental series, that $\varphi_{\Delta_1^\circ \Delta_2^\circ}(x+h) - \varphi_{\Delta_1^\circ \Delta_2^\circ}(x) > \varepsilon$. Let us now approximate Δ_1° by means of a series $\alpha_1\Delta_1', \alpha_2\Delta_1'', \ldots$ and Δ_2° by means of a series $\beta_1\Delta_2', \beta_2\Delta_2'', \ldots$, where the α's and β's represent integers, in such a way that

$$\alpha_p\Delta_1^{(p)} < \Delta_1^\circ < (\alpha_p+1)\,\Delta_1^{(p)} \text{ and } \beta_p\Delta_2^{(p)} < \Delta_2^\circ < (\beta_p+1)\,\Delta_2^{(p)},$$

[1]) The uniform convergence of all difference quotients is evident from that of the difference quotients, for which $\Delta_1 = \Delta_2$ (out of these the other can be approximated in the manner indicated in the proof of theorem 6); the latter is evident by developing the terms of $f(x+2\Delta) - 2f(x+\Delta) + f(x)$ according to TAYLOR's series, in which we make the second differential quotient form the restterm; the terms preceding this restterm then destroy each other.

then also $\varphi_{\Delta_1^\circ\,\Delta}(x+h) - \varphi_{\Delta_1^\circ\,\Delta_2^\circ}(x)$ is approximated by

$$\varphi_{\alpha_p \Delta_1^{(p)}, \beta_p \Delta_2^{(p)}}(x+h) - \varphi_{\alpha_p \Delta_1^{(p)}, \beta_p \Delta_2^{(p)}}(x),$$

which last expression remains $< \varepsilon$, however great p may become, so that the first can neither be $> \varepsilon_1$; so the set of *all* second difference quotients is uniformly continuous and there is a finite continuous second differential quotient.

Theorem 7. If there is among the second difference quotients of a finite continuous function a uniformly continuous fundamental series, in which both x-increases decrease indefinitely, the function possesses finite and continuous first and second differential quotients.

For, according to the above given proof of theorem 6 the whole system of the second difference quotients proves to be uniformly continuous, and out of the above given proof of theorem 4 this system proves to possess for indefinite decrease of the two x-increases one single finite continuous limiting function $f''(x)$ to which they converge uniformly. Let τ' be the maximum deviation from this limiting function of the second difference quotients, whose x-increases are smaller than Δ'_1 and Δ'_2, and let us regard the system ζ of all $\varphi_\Delta(x)$ whose $\Delta < \Delta'_1$, then all difference quotients with x-increase $< \Delta'_2$ of the system ζ lie between $f''(x) + \tau'$ and $f''(x) - \tau'$, from which may be deduced easily, that the system ζ is uniformly continuous, so that now first according to the proof of theorem 1 a finite continuous first differential quotient exists and then according to the proof of theorem 4 a finite continuous second differential quotient.

Analogous to the preceding are the proofs of the following more general theorems:

Theorem 8. If there is among the n^{th} difference quotients of a finite continuous function a uniformly continuous fundamental series, in which all x-increases decrease indefinitely, then the function possesses finite and continuous first, second, up to the n^{th} differential quotients; each p^{th} differential quotient is here first the only limit for indefinitely decreasing x-increases of the p^{th} difference quotients to which limit a uniform convergence takes place, and then differential quotient of the $(p-1)^{st}$ differential quotient.

Theorem 9. If a function possesses a finite continuous n^{th} differential quotient, then the system of the first, second, up to the n^{th} difference quotients is uniformly continuous [1]).

Theorem 10. If $n_1, n_2, n_3 \ldots$ is an infinite series of increasing integers and if of a finite continuous function the systems of the n_1^{st}, of the $n_2^{nd} \ldots$ difference quotients are uniformly continuous, then the function has all differential quotients and these are all finite and continuous.

Theorem 11. A finite continuous function of several variables, among whose difference quotients of the n^{th} order there is for each kind a uniformly continuous fundamental series in which the increases of the independent variables decrease indefinitely, possesses all differential quotients up to the n^{th} order; these finite and continuous differential quotients are first each other's differential quotients in the manner expressed by their form, where the order of succession of the differentiations proves to be irrelevant, and then each differential quotient is the only limit of the corresponding difference quotients for indefinitely decreasing increases of the independent variables, to which limit a uniform convergence takes place.

Theorem 12. If a function of several variables possesses all kinds of n^{th} differential quotients and if these are finite and continuous, then the system of the 1^{st}, 2^{nd} up to the n^{th} difference quotients is uniformly continuous.[2])

Finally the observation, that what was treated here leads infinite differentiability back to continuity in a more extensive sense, and in this way may somewhat explain, that for so long all finite continuous functions were supposed to be infinitely differentiable, and may somewhat justify that so many wish to limit themselves in natural science to infinitely differentiable functions.

1) To prove the uniform convergence of the n^{th} difference quotients for equal independent x-increases we break off just as for theorem 5 the Taylor development at the n^{th} term and we apply the formula:

$$n^a - \binom{1}{n}(n-1)^a + \binom{2}{n}(n-2)^a \ldots\ldots = 0$$

(n and a integers; $a < n$).

2) To prove the uniform convergence of the n^{th} difference quotients with equal independent increases Δ (these Δ indefinitely decreasing) we develop the elements of such a difference quotient according to TAYLOR's series, in which we make the n^{th} differential quotients form the restterms. The terms preceding the restterms then fall out and of the restterms one kind converges uniformly to the differential quotient corresponding to the difference quotient considered and the other converge uniformly to zero.