In order to find the composition of the solid phases which can be in equilibrium with definite solutions I have acted in the same manner as I did previously with ternary systems; I have applied the "residue-method".

If the solution is in equilibrium with one solid substance the conjugation line solution-residue must pass through the point indicating this solid substance; if it is in equilibrium with two solid substances the conjugation line solution-residue intersects the communication line of the two solid substances and if it is in equilibrium with three solid substances it intersects the triangle which has those three solid substances as its angular points.

These constructions are much facilitated by taking a rectangular tetrahedron instead of an equilateral one and projecting the whole on two of the side planes.

**Astronomy.** — "*The investigation of the weights in equations according to the principle of the least squares*". By J. WEEDER. (Communicated by Prof. H. G. VAN DE SANDE BAKHUYZEN).

When results of measurement deduced from different modes of measuring or originating from different observers are equated mutually, it is generally advisable to test the weights assigned to these results, before equating, with the apparent errors produced by the equation in order to be able to judge whether it is necessary to correct them and to distinguish in what direction correction is obtained. Let the material of observation break up according to its origin into groups and let out of the apparent errors of each group separately the mean error of the unity of weight be deduced, then it is a necessity for the differences of those values to be small, at least they may not overstep the limits which can be fixed taking into account the numbers of apparent errors in each group.

Already at the outset of such investigations the problem thus appears how the mean error of the unity of weight can be calculated, if one wishes to use but a part of the apparent errors.

When equating determinations of errors of division of the Leyden meridian circle I have applied the following formula:

$$\mu = \sqrt{\frac{\Sigma gf^2}{n-k}}.$$

Here

$\mu =$ the mean error of unity of weight,

$g =$ the weight of a result of observation,

$f$ = the apparent error calculated for this result,
$n$ = the number of errors out of the group,
$k$ = a number depending on the weights of the results of measurement and on the coefficients the unknown quantities, determined by the equating, are associated with in the equations expressing the connection between these unknown quantities and the results of measurement of the group.

In what way $k$ is dependent on the above-mentioned quantities will become clear by an example for which I choose the case that 3 unknown quantities $x$, $y$, $z$ are determined by $N$ equations of the form $ax + by + cz = l$, whilst to the quantities $l$ appearing in this equation and obtained by measurement the weights $g$ are due. In this case

$$k = \Sigma\, g\, (a^2\, Q_{xx} + 2ab\, Q_{xy} + b^2\, Q_{yy} + 2ac\, Q_{xz} + 2bc\, Q_{yz} + c^2\, Q_{zz})$$

where the summations in the formulae for $k$ and $\mu$ include expressions relating to the same results of measuring. In the above formula the quantities $Q$, the well-known numbers of weight, can be calculated by means of the coefficients of the normalequations.

For the deduction of this formula we have the same considerations which lead to the mean error of the unity of weight out af *all* observations. If the real errors are indicated by $h$ then $n\mu^2 = \Sigma\, gh^2$; this sum is expressed in the apparent errors that can be calculated, and in the errors $\Delta x$, $\Delta y$ and $\Delta z$ of the quantities $x$, $y$ and $z$, calculated out of the normal equations, by means of the relation $h = f + a\Delta x + b\Delta y + c\Delta z$, so that

$$n\mu^2 = \Sigma\, gf^2 + 2\,(\Delta x\; \Sigma\, gfa + \Delta y\; \Sigma\, gfb + \Delta z\; \Sigma\, gfc) +$$
$$+ (\Delta x)^2\; \Sigma\, ga^2 + 2\,(\Delta x)\,(\Delta y)\; \Sigma'\, gab + (\Delta y)^2\; \Sigma'\, gb^2 +$$
$$+ 2\,(\Delta x)\,(\Delta z)\; \Sigma\, gac + 2\,(\Delta y)\,(\Delta z)\; \Sigma\, gbc + (\Delta z)^2\; \Sigma\, gc^2$$

If we were to use the whole material of errors, then the first three of the unknown terms would fall out on account of $[gfa] = 0$, $[gfb] = 0$ and $[gfc] = 0$. (Here and for the future I make use of [ ] as sign of a summation extending over *all* observations). To take as well as possible the unknown terms in the above into account we replace them by their mean values in the supposition that the same complex of observations repeats itself manifold times so that all calculable quantities return unmodified in each repetition. In that supposition $\Delta x$, $\Delta y$ and $\Delta z$ have zero as mean values and the mean values of their squares and products are in the above order:

$$Q_{xx}\,\mu^2\;,\; Q_{xy}\,\mu^2\;,\; Q_{yy}\,\mu^2\;,\; Q_{xz}\,\mu^2\;,\; Q_{yz}\,\mu^2 \text{ and } Q_{zz}\,\mu^2$$

If we connect these mean values having $\mu^2$ as factor with the term

$n\mu^2$ in the above equation, if we put

$$\Sigma\, g\,(a^2 Qxx + 2abQxy + b^2 Qyy + 2acQxz + 2bcQyz + c^2 Qzz),$$

equal to $k$ and if we solve $\mu$ out of the equation, we obtain $\mu = \sqrt{\frac{\Sigma g f^2}{n-k}}$, i. e. the formula of which I made use to determine the mean error of the unity of weight out of a particular group of apparent errors.

I arrived at *about* the same result by another consideration putting to myself the problem to determine the mean value $M_{fi}$ of a definite apparent error $f_i$. In the relation:

$$f_i = l_i - a_i\, x - b_i\, y - c_i\, z$$

I substituted for $x$, $y$, $z$ respectively $[\alpha l]$, $[\beta l]$, $[\gamma l]$ to obtain $f_i$ in the form of a linear expression of the results of measuring $l$ which are supposed to be quite independent of each other.

Then:

$$M^2{}_{fi} = \left\{1 - 2\,(a_i\,\alpha_i + b_i\,\beta_i + c_i\,\gamma_i\,)\right\}\frac{\mu^2}{g_i} + \left[(a_i\,\alpha + b_i\,\beta + c_i\,\gamma)^2\frac{\mu^2}{g}\right].$$

It would now be the only way of reduction of this equation to make use of the well-known relations existing between the coefficients $\alpha$, $\beta$, $\gamma$ and $a$, $b$, $c$ and the numbers of weight $Q$, namely:

$$\alpha = g\,(a\, Qxx + b\, Qxy + c\, Qxz)\,,$$
$$\beta = g\,(a\, Qxy + b\, Qyy + c\, Qyz)\,,$$
$$\gamma = g\,(a\, Qxz + b\, Qyz + c\, Qzz)\,,$$

in order to prove that

$$\left[a_i\,\alpha + b_i\,\beta + c_i\,\gamma)^2\,\frac{1}{g}\right] = \frac{a_i\,\alpha_i + b_i\,\beta_i + c_i\,\gamma_i}{g_i}\,.$$

I propose however to deduce this equation directly from the minimum condition:

$$[g\,(l - ax - by - cz)^2] = \text{minimum}.$$

If here too $x$, $y$, $z$ are replaced by $[\alpha l]$, $[\beta l]$ and $[\gamma l]$, then after calculation and combination of the equal powers and products of the quantities $l$ an expression appears of the form $\Sigma\,\Sigma\, C^{\nu}_{\mu}\, l_\mu\, l_\nu$, having for the right set of coefficients $\alpha$, $\beta$, $\gamma$ a minimum value. I observe here that the coefficients $\alpha$, $\beta$, $\gamma$ have to satisfy the minimum condition independently of the particular values which the measurements furnished for the quantities $l$. Out of this observation ensues that the partial derivatives of $C^{\nu}_{\mu}$ with respect to each of the coefficients $\alpha$, $\beta$, $\gamma$ furnish zero by substitution of the right values of these coefficients.

By calculation and arrangement of the terms of the minimum condition we arrive at

$$C_\mu^\nu = 2\,[g\,(a\alpha_\mu + b\beta_\mu + c\gamma_\mu)(a\alpha_\nu + b\beta_\nu + c\gamma_\nu)] -$$
$$- 2\,g_\mu\,(a_\mu\alpha_\nu + b_\mu\beta_\nu + c_\mu\gamma_\nu) - 2\,g_\nu\,(a_\nu\alpha_\mu + b_\nu\beta_\mu + c_\nu\gamma_\mu)$$

The expression $[g\,(a\,\alpha_\mu + b\,\beta_\mu + c\,\gamma_\mu)(a\,\alpha_\nu + b\,\beta_\nu + c\,\gamma_\nu)]$ being put equal to $F$, the conditions for the minimum furnish the following equations:

$$\frac{\partial F}{\partial \alpha_\mu} = g_\nu a_\nu \quad \frac{\partial F}{\partial \beta_\mu} = g_\nu b_\nu \quad \frac{\partial F}{\partial \gamma_\mu} = g_\nu c_\nu,$$
$$\frac{\partial F}{\partial \alpha_\nu} = g_\mu a_\mu \quad \frac{\partial F}{\partial \beta_\nu} = g_\mu b_\mu \quad \frac{\partial F}{\partial \gamma_\nu} = g_\mu c_\mu.$$

Such an expression $F$ which, as far as the coefficients $\alpha$, $\beta$, $\gamma$ appear in it, contains only products of one of the $\alpha_\mu\,\beta_\mu\,\gamma_\mu$ with *one* of the $\alpha_\nu\,\beta_\nu\,\gamma_\nu$ can be written as linear expression in each of those sets of 3 coefficients in the following way:

$$F = \alpha_\mu \frac{\partial F}{\partial \alpha_\mu} + \beta_\mu \frac{\partial F}{\partial \beta_\mu} + \gamma_\mu \frac{\partial F}{\partial \gamma_\mu} = \alpha_\nu \frac{\partial F}{\partial \alpha_\nu} + \beta_\nu \frac{\partial F}{\partial \beta_\nu} + \gamma_\nu \frac{\partial F}{\partial \gamma_\nu}.$$

So that by substituting the equations resulting from the minimum we arrive at the following relations:

$$F = g_\nu\,(a_\nu\alpha_\mu + b_\nu\beta_\mu + c_\nu\gamma_\mu) = g_\mu\,(a_\mu\alpha_\nu + b_\mu\beta_\nu + c_\mu\gamma_\nu).$$

In words this relation runs: with equal weights an error in $l_\mu$ has equal influence on the apparent error $f_\nu$ as an equally large error in $l_\nu$ has on the apparent error $f_\mu$. If the weights of the two results of measuring are unequal, errors in these which are in inverse ratio with their weights will cause each other's apparent errors to deviate to the same amount from the true ones.

Let us put in the condition

$$[g\,\{l - a\,[\alpha l] - b\,[\beta l] - c\,[\gamma l]\,\}^2] = \text{minimum}$$

$l_i = 1$ and all other quantities $l = 0$, then from this arises

$$g_i\,(1 - 2a_i\alpha_i - 2b_i\beta_i - 2c_i\gamma_i) + [g\,(a\alpha_i + b\beta_i + c\gamma_i)^2] = \text{min.}$$

from which we deduce putting $[g\,(a\alpha_i + b\beta_i + c\gamma_i)^2] = G$:

$$\frac{\partial G}{\partial \alpha_i} = 2g_i a_i \quad \frac{\partial G}{\partial \beta_i} = 2g_i b_i \quad \frac{\partial G}{\partial \gamma_i} = 2g_i c_i$$

and from this ensues again:

$$G = \frac{1}{2}\alpha_i \frac{\partial G}{\partial \alpha_i} + \frac{1}{2}\beta_i \frac{\partial G}{\partial \beta_i} + \frac{1}{2}\gamma_i \frac{\partial G}{\partial \gamma_i} = g_i\,(a_i\alpha_i + b_i\beta_i + c_i\gamma_i).$$

With the aid of the above deduced theorem each term of the summation in the expression $[g\,(a\alpha_i + b\,\beta_i + c\,\gamma_i)^2]$ can be replaced by a corresponding one in which the constant index $i$ is given to the coefficients $a, b, c$, so we have:

$$G = \left[\frac{g_i^2}{g}(a_i\,\alpha + b_i\,\beta + c_i\,\gamma)^2\right] = g_i\,(a_i\,\alpha_i + b_i\,\beta_i + c_i\,\gamma_i)$$

from which results after division by $g_i^2$ the relation I was to prove.

Using this relation I find:

$$M_h^2 = (1 - a_i\,\alpha_i - b_i\,\beta_i - c_i\,\gamma_i)\frac{\mu^2}{g_i}.$$

If we call $a\,\alpha + b\,\beta + c\gamma = \varkappa$, then $f\sqrt{\frac{g}{1-\varkappa}}$ can be calculated out of each apparent error and the mean value of this system of errors is equal to $\mu$, as that of the system of unknown errors is $h\sqrt{g}$. It therefore seems to me not only permissible, but for a test of the weights even useful, to make use of that system of errors which allows the mean error of the unity of weight to be deduced out of each definite part of these errors. The connection between the quantities $\varkappa$ and the number $k$ of the above formula applied by me can be indicated by the relation $\Sigma\,\varkappa = k$.

**Physics.** - *"Contribution to the theory of binary mixtures".* VII. By Prof. J. D. VAN DER WAALS.

ON THE RELATION BETWEEN THE QUANTITIES $a_{12}$ AND $a_1$ AND $a_2$, WHICH OCCUR IN THE THEORY OF A BINARY MIXTURE.

I have already frequently traced the course of the thermodynamic curves for the case that for a binary system minimum plaitpoint temperature occurs, and so also the quantity $\frac{a_x}{b_x}$ has a minimum value for certain value of $x$. Both the course of the isobars and the course of the lines $\left(\frac{dp}{dx}\right)_v = 0$ and $\left(\frac{dp}{dv}\right)_x = 0$ may be assumed as known for that case. And experiment has shown that the shape of these lines predicted by theory it at least qualitatively accurate.

I purpose to demonstrate in these pages that in the case mentioned the course of these lines (see among others fig. 1 page 626 Vol. IX of these Proceedings 1907) is not compatible with the supposition $a_{12}^2 = a_1\,a_2$.